MATH 1220 — Exam 1 — Spring 2008

Solutions

1. (5 points) You drive your car 60 mph for an hour. Compare the distance you travel in the first five minutes to the distance you travel in the last five minutes, ie. is the distance you travel in the first five minutes less than, equal to, or greater then the distance you travel in the last five minutes.

Solution:

You travel the same distance in the first five minutes as in the last five minutes, since your speed is constant.

2. (5 points) A warm object is placed in cold water. You observe it for an hour. The temperature of the object changes according to the differential equation

$$\frac{dT(t)}{dt} = -k\left(T(t) - T_A\right)$$

where $T(t)$ is the temperature in degrees, t is time in minutes, T_A is the temperature of the water, and $k > 0$ is constant. Compare the change in the temperature of the object in the first five minutes to the change in temperature in the last five minutes.

Solution:

The temperature of the object changes more in the first five minutes. The rate of temperature change is proportional to the difference between the current temperature and the ambient temperature, and this difference decreases with time.

3. (5 points) One bank offers a savings account paying 4% interest compounded annually. As a marketing gimick a competing bank wants to offer an account with continuously compounding interest, but they don't actually want to pay more interest. What continuous growth rate should they offer to match the return of the 4% annually compounding account?

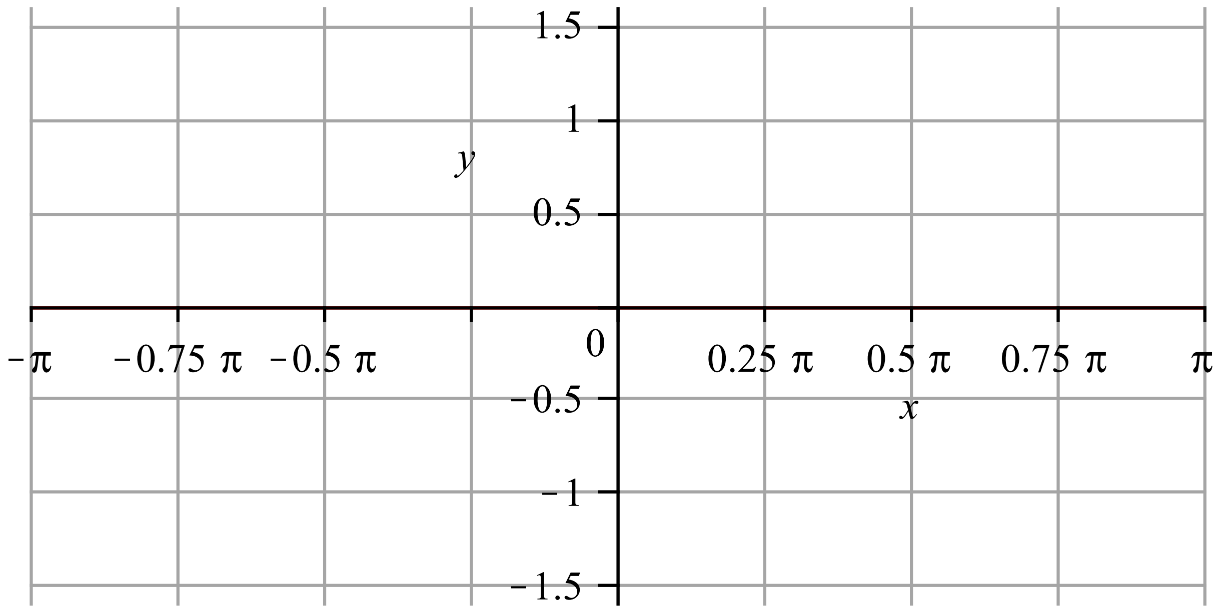
Solution:

Continuous growth rate of r would mean the balance after t years is $B(t) = B_0 e^{rt}$. We want this to be equal to annually compounding interest at 4%, so

$$\begin{aligned} B_0 e^{rt} &= B_0(1+.04)^t \\ e^{rt} &= 1.04^t \\ e^r &= 1.04 \\ r &= \ln(1.04) \approx .039 \end{aligned}$$

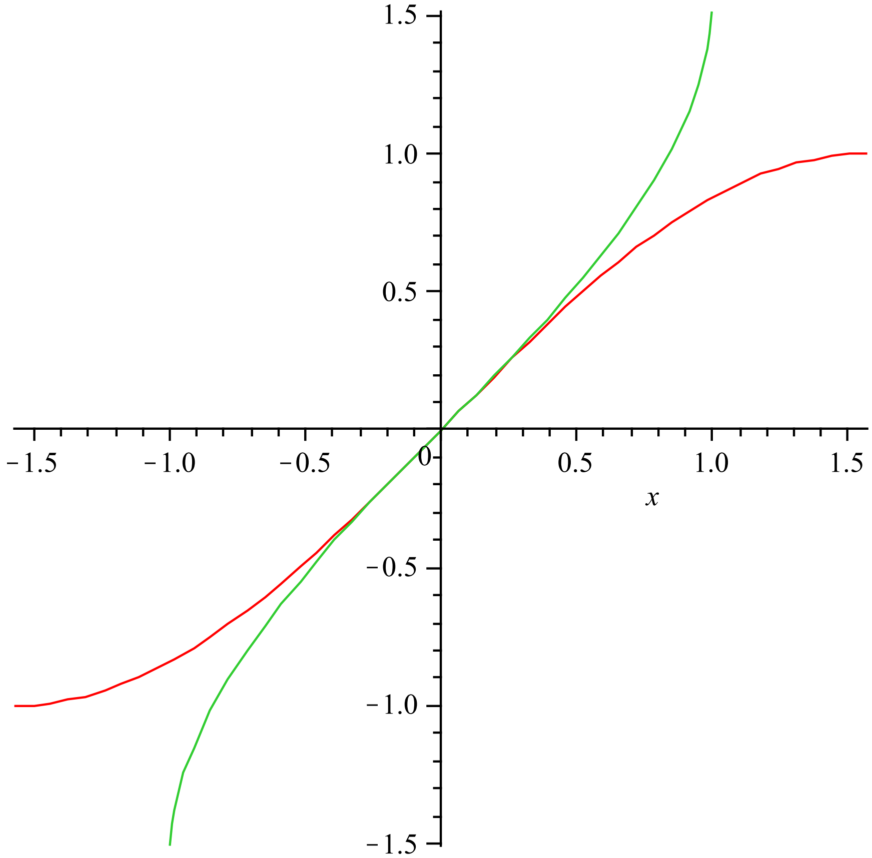
So the bank should offer about 3.9% continuously compounding interest.

4. (5 points) Sketch a graph of $\sin(x)$ and $\arcsin(x)$. Make sure you indicate the domain that you are restricting sin to in order to make it invertible.



Solution:

Restrict sin to the interval $= -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



5. (5 points) Express $\ln(3^3 \cdot 2^2 \cdot 13)$ in terms of $\ln(2)$, $\ln(3)$, and $\ln(13)$.

Solution:

$$\ln(3^3 \cdot 2^2 \cdot 13) = 3\ln(3) + 2\ln(2) + \ln(13)$$

6. (5 points) If $g(x)$ is pictured in Figure 1 and

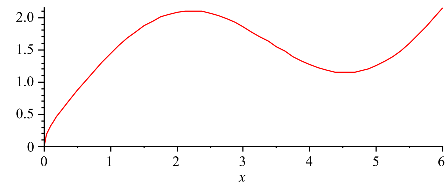


Figure 1: $g(x)$

$$f(x) = \int_0^x g(t)dt \quad ; \quad x \geq 0,$$

is $f(x)$ invertible?

Solution:

Yes. $f(x)$ is an accumulation function and $g(x)$ is strictly positive, so $f(x)$ is strictly increasing. Monotonic functions have inverses.

7. (5 points) What is the derivative of $e^{x+\sinh x}$?

Solution:

Let $w = x + \sinh(x)$.

$$\begin{aligned} \frac{d}{dx}\left(e^{x+\sinh(x)}\right) &= \frac{d}{dx}\left(e^{w}\right) \\ &= \frac{d(e^w)}{dw} \cdot \frac{dw}{dx} \\ &= e^w \cdot \frac{dw}{dx} \\ &= e^{x+\sinh(x)} \cdot (1 + \cosh(x)) \end{aligned}$$

8. (10 points) $\frac{dy}{dx} = ky$, $y(0) = 2$ and $y'(0) = 4$. Find $y(x)$.

Solution:

$$4 = y'(0) = \frac{dy}{dx}(0) = k \cdot y(0) = 2k$$

So $k = 2$.

$$\begin{aligned} \frac{dy}{dx} &= 2y \\ \frac{1}{y}dy &= 2dx \\ \int \frac{1}{y}dy &= \int 2dx \\ \ln(y) &= 2x + C \\ y &= e^{-2x+C} = C_2 e^{2x} \end{aligned}$$

$$y(0) = 2 = C_2$$

So

$$y = 2e^{2x}$$

9. (15 points) Compute:

$$\int \frac{x^2}{(x-1)(x-2)}dx$$

Solution:

$$\begin{aligned}\frac{x^2}{(x-1)(x-2)} &= \frac{x^2-3x+2}{(x-1)(x-2)} + \frac{3x-2}{(x-1)(x-2)} \\ &= 1 + \frac{3x-2}{(x-1)(x-2)}\end{aligned}$$

$$\begin{aligned}\frac{3x-2}{(x-1)(x-2)} &= \frac{A}{x-1} + \frac{B}{x-2} \\ &= \frac{A}{x-1}\cdot\frac{x-2}{x-2} + \frac{B}{x-2}\cdot\frac{x-1}{x-1}\end{aligned}$$

so

$$3x-2 = A(x-2) + B(x-1)$$

if $x=1$ then

$$1 = 3(1)-2 = A(1-2) \implies A = -1$$

if $x=2$ then

$$4 = 3(2)-2 = B(2-1) = B$$

$$\begin{aligned}\int \frac{x^2}{(x-1)(x-2)}dx &= \int 1 + \frac{-1}{x-1} + \frac{4}{x-2}dx \\ &= \int 1dx + \int \frac{-1}{x-1}dx + \int \frac{4}{x-2}dx \\ &= x - \ln|x-1| + 4\ln|x-2| + C\end{aligned}$$

10. (10 points) $f(x) = x^2 - 3x + 1$. Restrict the domain of $f(x)$ so that it has an inverse, while keeping its range as large as possible. Then find $f^{-1}(x)$.

Solution:

Consider the derivative $f'(x) = 2x - 3$. This is negative for $x < \frac{3}{2}$ and positive for $x > \frac{3}{2}$. Thus, $x = \frac{3}{2}$ is the global minimum for $f(x)$, and if we restrict the domain to either $(-\infty, \frac{3}{2}]$ or $[\frac{3}{2}, \infty)$ then $f(x)$ will be strictly monotonic, hence, invertible.

We will choose to restrict the domain to $[\frac{3}{2}, \infty)$. The range of f is $y \geq -\frac{5}{4}$.

$$\begin{aligned}
f(x) = y &= x^2 - 3x + 1 \\
0 &= x^2 - 3x + (1 - y) \\
x &= \frac{3 \pm \sqrt{9 - 4(1-y)}}{2} \qquad \text{(by quadratic formula)} \\
x &= \frac{3 + \sqrt{9 - 4(1-y)}}{2} \qquad \text{(since } x \geq \frac{3}{2}\text{)} \\
x &= \frac{3 + \sqrt{5 + 4y}}{2} \\
f^{-1}(x) &= \frac{3 + \sqrt{5 + 4x}}{2}; \quad x \geq -\frac{5}{4}
\end{aligned}$$

11. (15 points) A tank initially contains 50 gallons of fresh water. Salt water containing 2 lbs. of salt per gallon runs into the tank at 3 gallons per minute, and the well stirred solution runs out at 3 gallons per minute. How long will it take until there are 25 lbs of salt in the tank?

Solution:

Let $A(t)$ be the amount of salt in pounds in the tank after t minutes. Let $V(t)$ be the volume of water in the tank. $A(0) = 0$. $V(0) = 50$.

The volume in the tank is constant, 50 gal.

$$
\begin{aligned}
\frac{dA(t)}{dt} &= \text{rate in} - \text{rate out} \\
&= (3 \text{ gal/min})(2 \text{ lbs/gal}) - (2 \text{ gal/min})\frac{A(t) \text{ lbs}}{V(t) \text{ gal}} \\
\frac{dA(t)}{dt} &= 6 - 3\frac{A(t)}{50} \\
\frac{dA(t)}{dt} + 3\frac{A(t)}{50} &= 6
\end{aligned}
$$

Let

$$
\mu(t) = e^{\int \frac{3}{50} dt} = e^{\frac{3t}{50}}
$$

$$
\begin{aligned}
\mu(t)\left(\frac{dA(t)}{dt} + 3\frac{A(t)}{50}\right) &= 6\mu(t) \\
\frac{d(\mu(t)A(t))}{dt} &= 6\mu(t) \\
A(t) &= \frac{1}{\mu(t)} \int 6\mu(t)dt \\
&= e^{-\frac{3t}{50}} \int 6e^{\frac{3t}{50}} dt \\
&= e^{-\frac{3t}{50}} \cdot \left(6\frac{e^{\frac{3t}{50}}}{\frac{3}{50}} + C\right) \\
&= 100 + Ce^{-\frac{3t}{50}}
\end{aligned}
$$

$$
A(0) = 0 = 100 + C \implies C = -100
$$

Now if $A(t) = 25$ then

$$
\begin{aligned}
A(t) = 25 &= 100 - 100e^{-\frac{3t}{50}} \\
75 &= 100e^{-\frac{3t}{50}} \\
.75 &= e^{-\frac{3t}{50}} \\
\ln(.75) &= -\frac{3t}{50} \\
t &= -\frac{50}{3}\ln(.75) \approx 4.8 \,\text{min}
\end{aligned}
$$

12. (15 points) The formula for the upper half of a circle of radius r centered at the origin is $y = \sqrt{r^2 - x^2}$. Use this to show that the area of a circle of radius r is πr^2.

Solution:

$$\text{Area of circle} = 2 \cdot \text{area of upper half of circle} = 2\int_{-r}^{r} \sqrt{r^2 - x^2}dx$$

We will do this integral with a sine substitution. If you remember the formula you could immediately substitute $x = r\sin\theta$. However, we will accomplish the same thing in two substitutions.

$$\begin{aligned}
\text{Area} = 2\int_{-r}^{r} \sqrt{r^2 - x^2}dx &= 2\int_{-r}^{r} \sqrt{r^2(1 + \frac{x^2}{r^2})}dx \\
&= 2\int_{-r}^{r} r\sqrt{1 + \frac{x^2}{r^2}}dx \\
&= 2r\int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2}}dx
\end{aligned}$$

Substitute $w = \frac{x}{r}$, so $dw = \frac{1}{r}dx$, change limits of integration

$$\begin{aligned}
\text{Area} &= 2r\int_{-1}^{1} \sqrt{1 + w^2}(rdw) \\
&= 2r^2\int_{-1}^{1} \sqrt{1 + w^2}dw
\end{aligned}$$

Now substitute $w = \sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so $dw = \cos\theta d\theta$, change limits of integration

Notice that $\frac{x}{r} = w = \sin\theta$, so we have accompished the same thing as if

we had originally substituted $x = r\sin\theta$

$$\begin{aligned}
\text{Area} &= 2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + \sin^2\theta}\cos\theta d\theta \\
&= 2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\cos\theta d\theta \\
&= 2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta, \text{ since } \cos\theta \ge 0 \text{ on } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}
\end{aligned}$$

Now we need to integrate $\cos^2\theta$. Use integration by parts with $u = \cos\theta$, $dv = \cos\theta d\theta$.

$$\begin{aligned}
\int \cos^2\theta d\theta &= \int u dv = uv - \int v du \\
&= \sin\theta\cos\theta - \int -\sin^2\theta d\theta \\
&= \sin\theta\cos\theta + \int 1 - \cos^2\theta d\theta \\
&= \sin\theta\cos\theta + \int 1 d\theta - \int \cos^2\theta d\theta \\
\int \cos^2\theta d\theta &= \sin\theta\cos\theta + \theta - \int \cos^2\theta d\theta \\
2\int \cos^2\theta d\theta &= \sin\theta\cos\theta + \theta + C \\
\int \cos^2\theta d\theta &= \frac{\sin\theta\cos\theta + \theta}{2} + C
\end{aligned}$$

Now use this in the Area equation:

$$\begin{aligned}
\text{Area} &= 2r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta \\
&= 2r^2 \left(\frac{\sin\theta\cos\theta + \theta}{2}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
\cos\theta = 0 \text{ for both } \theta &= -\frac{\pi}{2}, \frac{\pi}{2} \\
\text{Area} &= 2r^2 \left(\frac{\frac{\pi}{2}}{2} - \frac{-\frac{\pi}{2}}{2}\right) = \pi r^2
\end{aligned}$$

No new questions beyond this point.