# Research Statement

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## 1. Background

My research interests are in Geometry and Topology, particularly Geometric Group Theory. The primary goal in this field is the study of finitely generated, infinite discrete groups. It is possible to treat these groups as geometric objects by considering large scale, or coarse, geometry. A natural equivalence relation in this setting is the notion of quasi-isometry. Gromov proposed a program to classify finitely generated infinite discrete groups up to quasi-isometry [15].

Groups can be constructed from simpler pieces as *graphs of groups*. A graph of groups consists of a finite graph, and *local groups* for each vertex and edge, as well as injections of each edge group into the two vertex groups corresponding to ends of the edge. This construction gives rise to a new group, the *fundamental group of the graph of groups*. Each of the vertex groups injects into the fundamental group. The injections of edge groups into vertex groups produce relations in the fundamental group. This is essentially a gluing construction. The vertex groups give the main pieces, and the injections of edge groups tells you the gluing maps.

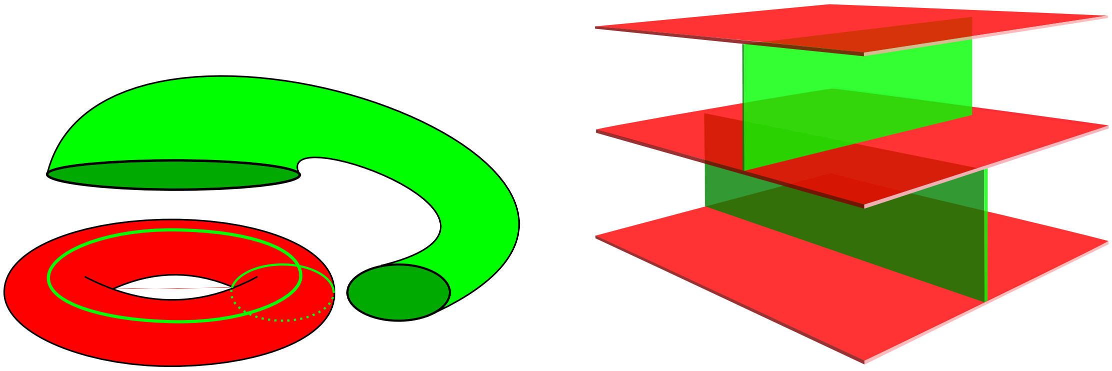
There has been much interest in decomposing or *splitting* groups into graphs of groups with simpler pieces. Stallings [22] and Dunwoody [10] addressed questions about splittings over finite groups. More recently there have been several generalizations of the JSJ-decomposition of three dimensional manifolds [21, 19, 11, 14]. Here one seeks to split a group over some class of groups including groups like $\mathbb{Z}$ or $\mathbb{Z}^2$. Papasoglu showed that one version of the JSJ decomposition is invariant under quasi-isometry [18]. More generally, Mosher, Sageev, and Whyte prove quasi-isometry invariance of graph of groups splittings in a variety of cases [17, 16].

Surprisingly little is known about questions in the other direction. Instead of splitting a group into pieces, what happens when one assembles simple pieces into a larger group? If I know something about the local groups, how much can I say about the fundamental group? When does the fundamental group inherit properties of the local groups? What different fundamental groups can be built from the same local groups? Is it possible to engineer a fundamental group with a desired property by combining local groups with some related property?

Even when the local groups are very simple, the answers to these questions can be very rich. Work of Farb and Mosher [12, 13] and Whyte [23] combine to give a complete quasi-isometry classification of graphs of groups where all the local groups are virtually $\mathbb{Z}$, sometimes called "generalized Baumslag-Solitar groups".

This work has also been generalized to graphs of groups where all the local groups are virtually abelian and of the same rank. Little is known, however, about graphs of abelian groups where the ranks differ.

I have recently worked on quasi-isometries of *tubular groups*. These are essentially graphs of groups where the vertex groups are $\mathbb{Z}^2$ and the edge groups are $\mathbb{Z}$. These groups can be realized as fundamental groups of a collection of tori glued together by annuli. The geometric model for such a group consists of vertex spaces which are Euclidean planes covering the tori, and strips $\mathbb{R} \times [0, 1]$ covering the annuli. Examples of these groups have appeared in work of several different authors, [3, 4, 1, 9, 24]



(a) An annulus gluing on to a torus. (b) A portion of the universal cover.

Mosher, Sageev, and Whyte [16] give a quasi-isometry invariant for these groups. The geometric model has Euclidean planes with infinite strips glued on along families of parallel lines. It turns out that the affine equivalence class of these line patterns is a quasi-isometry invariant of the group.

It was believed that a combination of line pattern considerations and techniques from the Baumslag-Solitar group classification would yield a classification for tubular groups.

In [5] I give an algorithm for determining if two tubular groups are quasi-isometric or not. The classification turns out to be much more delicate than anticipated. The idea is that there is a parameter, height change, that needs to be controlled. Techniques of [23] allow one to control this parameter in any particular direction. However, when one attempts to control the parameter in one direction, the line pattern considerations force new errors in other directions.

This algorithm generalizes to graphs of groups with virtually abelian vertex groups and virtually cyclic edge groups.

In the course of proving the algorithm, I find a quasi-isometry invariant called the *maximum slope*. In [6] I show how to compute this invariant from the graph of groups data, and give examples of two tubular groups which are distinguished by having different maximum slopes, but have the same line patterns and the same Dehn function.

## 2. CURRENT RESEARCH

I am currently working on quasi-isometries of graphs of groups where the vertex and edge groups are free groups. To some extent the methods of Mosher, Sageev and Whyte break down when the local groups are free groups.

A first case to consider is when the edge groups are cyclic. In this case it will be important to understand the line patterns that the edge spaces make in the vertex spaces. It turns out that such line patterns are also of use in some cases when the edge groups are free groups of higher rank.

### 2.1. Line Patterns in Free Groups.

In joint work with Natasa Macura [7] we consider quasi-isometric equivalence of line patterns in free groups.

Given a finitely generated free group $F$ of rank greater than one and a word $w \in F$, the $w$-line at $g \in F$ is the set of points of the form:

$$\{gw^n \mid n \in \mathbb{Z}\}$$

The *line pattern generated by* $w$ is the collection of distinct $w$-lines. Similarly, if we take finitely many words $w$, as above, the line pattern generated by the collection is the union of the patterns generated by the individual words. We will denote the line pattern $\mathcal{L}$.

The main question is:

**Question 1.** *Take free groups $F$ and $F'$, possibly of different rank, and words $\{w_1, \dots, w_m\} \subset F$, $\{w'_1, \dots, w'_n\} \subset F'$. Let $\mathcal{L}$ be the line pattern in $F$ generated by $\{w_1, \dots, w_m\}$, and let $\mathcal{L}'$ be defined similarly for $F'$.*

*Is there a quasi-isometry $\phi \colon F \to F'$ that preserves the patterns, in the sense that there is some constant $C$ so that for every line $l \in \mathcal{L}$ there is an $l' \in \mathcal{L}'$ such that the Hausdorff distance between $\phi(l)$ and $l'$ is at most $C$, and vice versa?*

A related question is:

**Question 2.** *Let $F$ be a free group and $\mathcal{L}$ a line pattern in $F$. What is the group $\mathcal{QI}(F, \mathcal{L})$ of quasi-isometries of $F$ that preserve the line pattern $\mathcal{L}$?*

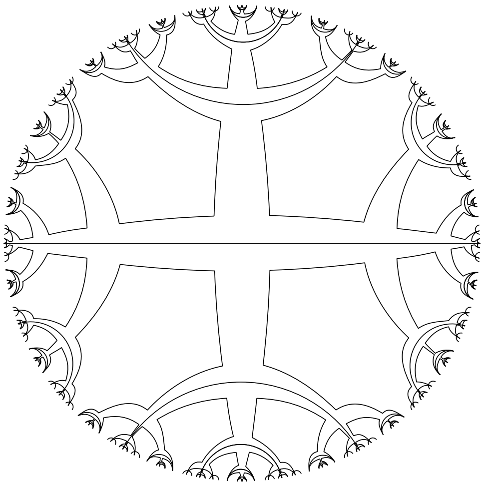
Richard Schwartz studied a similar situation for line patterns in $\mathbb{H}^n$ for $n \geq 3$ and found that the only quasi-isometries that preserve a line pattern are hyperbolic isometries.[20]

We have a result similar to that of Schwartz, at least for some line patterns.

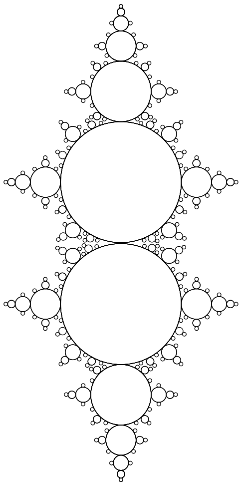
We call a line pattern $\mathcal{L}$ in $F$ *rigid* if $\mathcal{QI}(F,\mathcal{L})$ is conjugate to an isometry group in the following sense: There is a tree $T$ with $\phi\colon F \to T$ a quasi-isometry so that

$$\phi\, \mathcal{QI}(F,\mathcal{L})\phi^{-1} = \mathrm{Isom}(T, \phi(\mathcal{L})) \subset \mathcal{QI}(T)$$

This is surprising. $\mathbb{H}^n$ is somewhat rigid to begin with. Free groups, however, have the geometry of trees, which are quasi-isometrically very flexible. However, for our rigid patterns all of this flexibility disappears, and it is very hard for two line patterns to be equivalent.



(c) The line pattern in $F_2 = \langle a, b\rangle$ generated by the words $a$ and $ab\bar{a}\bar{b}$ (shown embedded in the disc model of $\mathbb{H}^2$).



(d) The corresponding decomposition space.

It is easy to find some examples of line patterns that are not rigid. Define the *decomposition space* associated to a line pattern to be the quotient of the boundary of the tree obtained by identifying the two endpoints of each line in the pattern. We show that a line pattern is not rigid when the decomposition space is disconnected, has cut points, or has certain bad cut pairs. However, we conjecture that these are the only situations in which line patterns fail to be rigid.

Cut sets in the decomposition space are related to cut sets in the *Whitehead graph* of the line patterns. We can prove that a line pattern is rigid if the Whitehead graph satisfies some connectivity hypotheses. We do this by constructing the tree $T$ in terms of the topology of the decomposition space. In particular we use the finite cut set structure of the decomposition space. We find a canonical collection of cut sets in the decomposition space and let the tree $T$ be the dual tree, which is quasi-isometric to $F$. Pattern preserving quasi-isometries of $F$ give homeomorphisms of the decomposition space, so these give isometries of the tree $T$.

We believe that these extra hypotheses on the Whitehead graph are a technical convenience. We are currently trying to adapt the argument to show that any line pattern is rigid, so long as the decomposition space is connected without cut points or bad cut pairs.

2.2. **Mapping Tori of Linearly Growing Free Group Automorphisms.** In joint work in progress with Natasa Macura [8], we study quasi-isometries of mapping tori of linearly growing free group automorphisms. Linearly growing free group automorphisms are analogous to Dehn twist homeomorphisms of two dimensional surfaces. We can realize such a mapping torus as a graph of groups with a single vertex and a single edge. The edge group and the vertex group are each a free group, and in one direction the edge group is identified with the vertex group by the identity map, and in the other direction by the automorphism. However, quasi-isometries do not preserve this decomposition. Instead we have to look at certain subgroups that are invariant under the automorphism. In the universal cover these correspond to a topological product of a tree with a line. We show that there is quasi-isometrically stable decomposition of the universal cover of the mapping torus into pieces of this form, tree×line, and that these intersect along lines.

This gives us a line pattern in each tree×line. However, the line direction is also preserved by quasi-isometries, so we can project this pattern to the tree factor to get a line pattern in a tree. We then use our previous work concerning line patterns in free groups [7] to decide when these patterns are equivalent and under what conditions an equivalence of line patterns in trees can be lifted to give equivalence of line patterns in tree×line.

We do not expect much quasi-isometric flexibility except in the simplest cases. It appears that even in the non-rigid cases, the quasi-isometric flexibility of a line pattern in a free group is used up in the process of lifting to a line pattern in tree×line.

## 3. Directions for Future Research

3.1. **Graphs of Free Groups with Cyclic Edge Groups.** Another application of line patterns in free groups will be to quasi-isometries of graphs of free groups with cyclic edge groups. When the edge pattern in each vertex is a rigid line pattern this graph of groups decomposition is the JSJ decomposition, so is preserved by quasi-isometries [18]. The assumption that the line patterns are rigid puts severe restrictions on how quasi-isometries behave on the vertex spaces, so it should be possible to give a quasi-isometric classification of these groups.

3.2. **Mapping Tori of Polynomially Growing Free Group Automorphisms.** Bestvina-Handel train track theory [2] tells us that mapping tori of polynomially growing free group automorphisms are built up as extensions of mapping tori of automorphisms of lower growth. Thus, once we have classified mapping tori of linearly growing free group automorphisms it may be possible to classify the polynomially growing cases inductively by cutting along the fastest growing strata and reducing the degree of growth.

3.3. **Mapping Tori of Fully Irreducible Free Group Automorphisms.** A fully irreducible free group automorphism is analogous to a pseudo-Anosov surface homeomorphism. The mapping torus of a pseudo-Anosov surface homeomorphism is always quasi-isometric to $\mathbb{H}^3$, three dimensional hyperbolic space. It is not known if there is more than one quasi-isometry class of mapping torus of fully irreducible free group automorphism. This would be an interesting question to consider, but it will probably require a different approach than the polynomial growing cases. In the polynomial growing case there are words in the free group that are invariant under the automorphism. These are important in our argument, they give us a quasi-isometrically invariant splitting of the group, which we use to get line patterns in free subgroups. In the fully irreducible case we do not have this kind of splitting, but we do have something like a line pattern. There is an invariant lamination instead of a pattern of periodic lines. It might be possible to upgrade our line pattern techniques to distinguish between these laminations.

## REFERENCES

[1] Jason A. Behrstock and Walter D. Neumann, *Quasi-isometric classification of graph manifold groups*, Duke Math. J. **141** (2008), 217–240, `arXiv:math.GT/0604042`.

[2] Mladen Bestvina and Michael Handel, *Train tracks and automorphisms of free groups*, The Annals of Mathematics **135** (1992), no. 1, 1–51.

[3] N. Brady and M. R. Bridson, *There is only one gap in the isoperimetric spectrum*, Geom. Funct. Anal. **10** (2000), no. 5, 1053–1070.

[4] Noel Brady, Martin R. Bridson, Max Forester, and Krishnan Shankar, *Snowflake groups, Perron-Frobenius eigenvalues, and isoperimetric spectra*, Geom. Topol. **13** (2009), no. 1, `arXiv:math.GR/0608155`.

[5] Christopher H. Cashen, *Quasi-isometries between tubular groups*, Groups, Geometry and Dynamics, to appear, `arXiv:0707.1502`.

[6] ______, *Computing the maximum slope invariant in tubular groups*, preprint, 2009, `Slope.pdf`.

[7] Christopher H. Cashen and Natasa Macura, *Line patterns in free groups*, Preliminary Version, 2009, `FreeGpPatterns.pdf`.

[8] ______, *Quasi-isometries of mapping tori of linearly growing free group automorphisms*, in preparation, 2009.

[9] Christopher B. Croke and Bruce Kleiner, *Spaces with nonpositive curvature and their ideal boundaries*, Topology **39** (2000), 549–556.

[10] Martin J Dunwoody, *The accessibility of finetly presented groups*, Invent. Math. **81** (1985), no. 3, 449–457.

[11] Martin J Dunwoody and Michah Sageev, *JSJ-splittings for finitely presented groups*, Invent. Math. **135** (1999), 25–44.

[12] Benson Farb and Lee Mosher, *A rigidity theorem for the solvable Baumslag-Solitar groups*, (1996).

[13] ______, *Quasi-isometric rigidity for the solvable Baumslag-Solitar groups, ii*, December 1997.

[14] Koji Fujiwara and Panos Papasoglu, *JSJ-decompositions of finitely presented groups and complexes of groups*, Geom. Funct. Anal. **16** (2006), 70–125.

[15] Mikhael Gromov, *Infinite groups as geometric objects*, Proceedings of the International Congress of Mathematicians (Warsaw), vol. 1,2, PWN, 1984, pp. 385–392.

[16] Lee Mosher, Michah Sageev, and Kevin Whyte, *Quasi-actions on trees II: Finite depth Bass-Serre trees*, Mem. Amer. Math. Soc., to appear, `arXiv:math.GR/0405237`.

[17] ______, *Quasi-actions on trees I: Bounded valence*, Ann. of Math. **158** (2003), no. 1, 115–164, `arXiv:math.GR/0010136`.

[18] Panos Papasoglu, *Quasi-isometry invariance of group splittings*, Annals of Mathematics **161** (2005), 1–72.

[19] Eliyahu Rips and Zlil Sela, *Cyclic splittings of finitely presented groups and the canonical JSJ-decomposition*, Ann. of Math. **146** (1995), 53–109.

[20] Richard Evans Schwartz, *Symmetric patterns of geodesics and automorphisms of surface groups*, Invent. Math. **128** (1997), 177–199.

[21] Zlil Sela, *Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 lie groups ii*, Geom. Funct. Anal. **7** (1997), no. 3, 561–594.

[22] John R. Stallings, *On torsion free groups with infinitely many ends*, Ann. of Math. **88** (1968), 312–334.

[23] Kevin Whyte, *The large scale geometry of the higher Baumslag-Solitar groups*, Geom. Funct. Anal. **11** (2001), 1327–1343, `arXiv:math.GT/0405272`.

[24] Daniel T. Wise, *A non-Hopfian automatic group*, Journal of Algebra **180** (1996), no. 3, 845–847.