Computing the Maximum Slope Invariant in Tubular Groups

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Abstract. We show that the maximum slope invariant for tubular groups is easy to calculate, and give an example of two tubular groups which are distinguishable by their maximum slopes but not by local considerations or isoperimetric function.

1. Introduction

The main examples of tubular groups are constructed from $\mathbb{Z}^2$ by amalgamating along cyclic subgroups.

Tubular groups have been used as examples of various phenomena. Brady and Bridson computed isoperimetric functions for the groups

$$BB_{p,r} = \left\langle a, b, x, y \mid [a, b] = 1,\, x^{-1}a^p x = a^r b,\, y^{-1}a^p y = a^r b^{-1} \right\rangle$$

with $0 < p < r$ and found that the degrees of the isoperimetric functions form a dense set in $[2, \infty)$. The Dehn function of $BB(p, r)$ is x^n with $n = 2 \log_2 \frac{2r}{p}$ [1].

Later Brady, Bridson, Forester and Shankar, [2], used more complicated examples of tubular groups, called "snowflake groups", to show that all the rationals in that interval appear.

Other examples include the examples of Croke and Kleiner [4] and Wise's group [7].

There are two quasi-isometry invariants readily available among these groups. For the Brady-Bridson and snowflake groups we have the isoperimetric exponent. Mosher, Sageev and Whyte give another: the collection of affine equivalence classes of edge patterns [5].

In a previous paper [3] we gave an algorithm to decide whether or not two tubular groups are quasi-isometric. In the course of the proof we constructed a tree, the "tree of P–sets", related to the Bass-Serre tree of a particular graph of groups decomposition of the group. The (directed) edges of this tree come with a parameter, the "height change across the edge". The maximum slope invariant for the tubular group is then the maximum coarse slope of any ray in the tree, where coarse slope is average height change per unit length, in an appropriate sense. It follows from the quasi-isometry algorithm that the maximum coarse slope is a quasi-isometry invariant of the group (at least in certain cases).

Among the groups $BB(p, r)$ the edge patterns are all equivalent and it turns out that the isoperimetric exponent is a complete quasi-isometry invariant [3]. One might wonder whether the equivalence classes of edge patterns and the isoperimetric exponent always determine the quasi-isometry class of a tubular group.

In this note we give an example of two tubular groups with the same equivalence classes of edge patterns and the same isoperimetric exponent but different maximum

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slopes. The method of computing the maximum slopes is easy and generalizes to many classes of tubular groups.

The trick that makes the maximum slope easy to compute is that we have a commutative diagram:

$$\begin{tikzcd}
\text{Bass-Serre tree} \arrow[r] \arrow[d, "\text{quotient by group action}"'] & \text{tree of P–sets} \arrow[d, "\text{quotient by group action}"] \\
\text{graph of groups} \arrow[r] & \text{graph of P–sets}
\end{tikzcd}$$

The map from the Bass-Serre tree to the tree of P–sets is given in [3]. The corresponding map from the graph of groups to the graph of P–sets is given in Section 3. One can arrange that the group action on the tree of P–sets is height preserving, so that height changes across an edge descend to the graph of P–sets. We can compute slopes of rays in the tree of P–sets by considering the image of the ray in the graph of P–sets. When the maximum slope is finite and non-zero, the maximum slope can always be realized as the slope of an embedded loop in the graph of P–sets, and there are only finitely many of these to consider.

2. PRELIMINARIES

Definition 2.1. A *tubular group* is the fundamental group of a finite, connected graph of groups satisfying the following conditions:

1. Every edge group is finitely generated and quasi-isometric to $\mathbb{Z}$.
2. Every vertex group is finitely generated and quasi-isometric to either $\mathbb{Z}$ or $\mathbb{Z}^2$.
3. There is at least one vertex group quasi-isometric to $\mathbb{Z}^2$ and at least one edge.
4. For every vertex whose local group is quasi-isometric to $\mathbb{Z}^2$ there are incident edges whose edge groups inject to subgroups of the vertex group which are not coarsely equivalent.

The definition of tubular groups is chosen this way to make it a quasi-isometrically closed class; any finitely generated group quasi-isometric to a tubular group is itself a tubular group.

We simplify the exposition by considering only graphs of groups where the vertex groups are $\mathbb{Z}^2$, the edge groups are $\mathbb{Z}$, and the edge groups incident to a vertex inject into exactly three distinct maximal cyclic subgroups.

For such a group there is a convenient way to choose a metric. Consider one of the vertex $\mathbb{Z}^2$ groups. Suppose $\mathbb{Z}^2 = \langle a, b \mid [a, b] = 1\rangle$. The "usual" metric would be the metric where the word $a^x b^y$ corresponds to the point (x, y) in the Euclidean plane. Suppose the incident edge groups inject into the maximal cyclic subgroups containing a, $a^r b^s$, and $a^t b^u$, where 0, $\frac{s}{r}$ and $\frac{u}{t}$ are distinct. This can be arranged by picking a new generating set, if necessary, since we have assumed that there are three distinct maximal cyclic subgroups.

Choose the metric on the vertex group to be the one where the word $a^x b^y$ corresponds to the point $A\binom{x}{y}$ in the Euclidean plane, where A is the matrix:

$$A = \begin{pmatrix} 1 & -\frac{1}{2}\frac{ru+st}{su} \\ 0 & \frac{\sqrt{3}}{2}\frac{ru-st}{su} \end{pmatrix}$$

Choosing a metric like this does not affect the quasi-isometry class of the group. It is a convenient choice because it makes the line pattern symmetric, and linear map that takes the line pattern to itself is an isometry followed by a homothety, so all distances are multiplied by a common factor.

3. The Procedure

In this section we give the procedure for computing the maximum slope, using the group $BB(p,r)$ to illustrate.

Step 1: Compute the Height Changes:
Start with a graph of groups decomposition. Call the vertices $v_1, \dots v_j$. Symmetrize vertex groups and compute height changes across each edge.

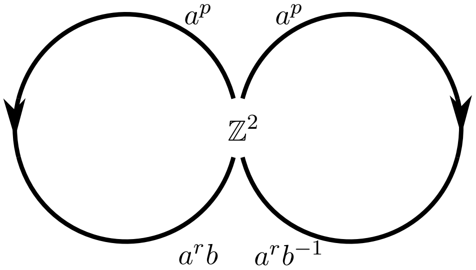

Figure 1. graph of groups for $BB(p,r)$

Let x and y be stable letters corresponding to the loop on the left and right, respectively. The labels at the ends of the edges indicate that these elements of the vertex group are conjugate by the stable letter associated to the edge. Then the fundamental group of the graph of groups is:

$$BB_{p,r} = \left\langle a, b, x, y \mid [a,b] = 1,\, x^{-1}a^p x = a^r b,\, y^{-1}a^p y = a^r b^{-1} \right\rangle$$

The matrix A that determines the symmetric metric is:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & r\sqrt{3} \end{pmatrix}$$

The height change across x is

$$H = -\log_2 \frac{|A\binom{r}{1}|}{|A\binom{p}{0}|} = -\log_2 \frac{2r}{p}$$

The height change across y is the same. Note that H is negative since $r > p$.

Step 2: Identify Parallel Edges:
At each vertex, join edges with an arc if their groups inject into a common maximal cyclic subgroup. We call such edges *parallel* at the vertex.

If there is a loop in the graph such that for every edge-vertex-edge sequence in the loop the two edge groups are parallel and such that the net height change around the loop is not zero, stop. The maximum slope is infinite.

In $BB(p,r)$ the two groups map into the maximal cyclic subgroup $\langle a\rangle$ at the top of Figure 2. There are no loops where each edge-to-edge transition is between parallel edges.

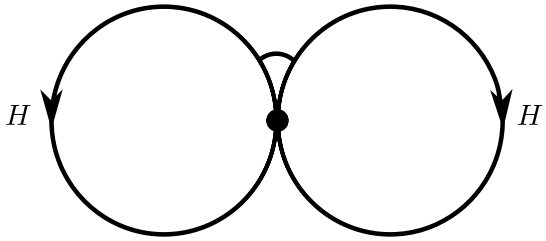

FIGURE 2. height change and parallel edges for $BB(p,r)$

Step 3: Fold Parallel Edges:
Subdivide each edge by adding a vertex at the midpoint. Call these new vertices $m_1, \dots m_k$.

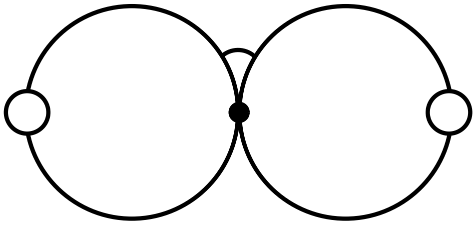

FIGURE 3. subdivided edges for $BB(p,r)$

At each v_i, for each collection of edges incident to the vertex and joined by arcs, fold them together by identifying them up to their midpoints.

In the resulting *graph of P–sets*, label each edge with a height change in such a way that the height changes between the original vertices is preserved. This is always possible. One way to accomplish this is to look at each remaining m_i. It is adjacent to some of the original vertices $v_{i_1}, \dots, v_{i_l}$. Among these there is some α so that for all β, the height change from v_{i_α} to v_{i_β} is non-negative. Give the edge from v_{i_α} to m_i height change 0, and give the edge from m_i to v_{i_β} the height change equal to the height change from v_{i_α} to v_{i_β}.

We will leave edges with height change 0 unlabeled and undirected.

Step 4: Find the Embedded Loop of Maximum Slope:
Now *slopes* are computed by taking a loop in the graph, adding the height change around the loop, and dividing by the length of the loop. (Here we consider edges to have length one half, because we subdivided edges in Step 3.)

If the graph is a tree then the maximum slope is 0. Otherwise, the maximum slope can be realized by an embedded loop. To see this, consider a loop that is not embedded and has non-zero slope. It contains some embedded sub-loops, and the slope of the entire loop is less than or equal to the maximum of the slopes of the embedded sub-loops.

For $BB(p,r)$ the maximum slope is $|H| = \log_2 \frac{2r}{p}$. Recall that the isoperimetric exponent for $BB(p,r)$ is $2\log_2 \frac{2r}{p}$, exactly twice the maximum slope.

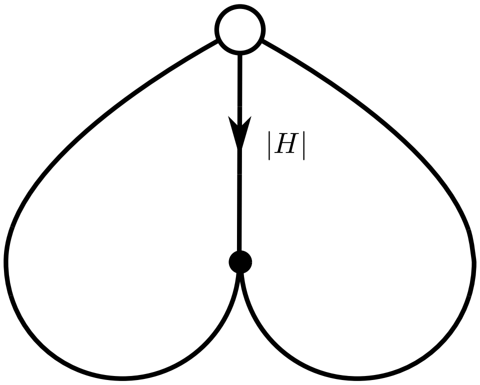

FIGURE 4. graph of P–sets for $BB(p, r)$

4. AN EXAMPLE

Consider the snowflake group, $G = G_{r,P}$ of [2], with $P = (4)$, $p = 4$, $q = 1$. Let $r = 2^p = 16$. This snowflake group has Dehn function x^4.

G is the fundamental group of a graph of groups consisting of one vertex and four edges.

All four edge groups are infinite cyclic.

The vertex group V has generators $v_1, \ldots, v_4$, and $c = v_1v_2v_3v_4$.

V is the fundamental group of a tree of groups consisting of three $\mathbb{Z}^2$ vertices and two $\mathbb{Z}$ edges.

$$V = \langle a_i, b_i \mid [a_i, b_i], b_1 = a_2b_2, b_2 = a_3b_3\rangle$$

Define $a_4 = b_3$, so that $b_2 = a_3b_3 = a_3a_4$ and $b_1 = a_2b_2 = a_2a_3a_4$.

Let $c = a_1b_1 = a_1a_2a_3a_4$.

Then

$$V = \left\langle a_i \mid [a_i, \prod_{j>i} a_j] \right\rangle$$

$$G = \left\langle a_i, s_i \mid i = 1 \ldots 4,\ [a_i, \prod_{j>i} a_j],\ s_i^{-1}a_i^{16}s_i = c \right\rangle$$

Each vertex has three lines, a_i, b_i, and a_ib_i. Conveniently, symmetrization in this case does not stretch the lines, so stretch factors are just ratios of indices. The edge that goes from a_i^{16} to c therefore has stretch factor $\frac{1}{16}$, and height change $-\log_2 \frac{1}{16} = 4$.

Figure 5 gives a graph of groups diagram for this group.

Now if we fold this graph down to a graph of P–set, Figure 6, we quickly see that the maximum slope in the tree of P–sets for this group is 4 (the smallest loop in the figure realizes the maximum slope). The isoperimetric exponent was also 4.

However, recall that $BB(p, r)$ has isoperimetric exponent equal to twice the maximum slope. $BB(1, 2)$ has Dehn function x^4 but a maximum slope of 2. It is not quasi-isometric to the snowflake group.

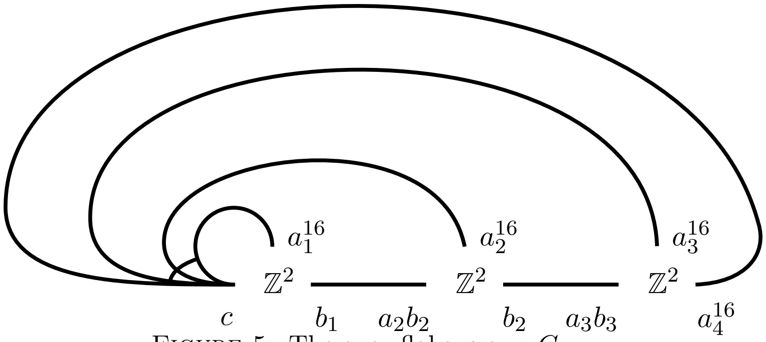

FIGURE 5. The snowflake group $G_{16,(4)}$

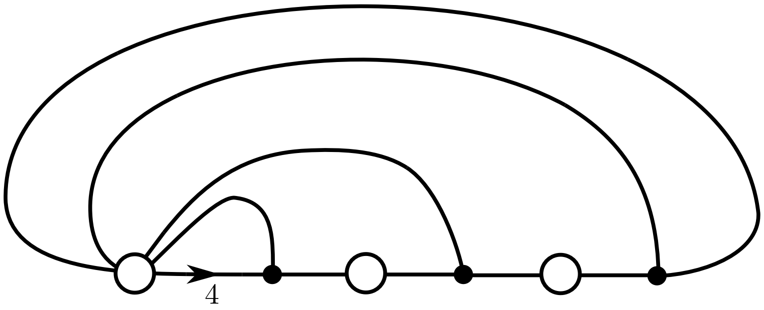

FIGURE 6. The P–set graph for $G_{16,(4)}$

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