Line Patterns in Free Groups
PRELIMINARY VERSION

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Abstract. We study quasi-isometric equivalence of line patterns in free groups by considering the finite cut set structure of an associated quotient of the boundary of a tree.

Contents

1. Introduction

Given a finitely generated free group F of rank greater than one and a word $w \in F$, the w-line at $g \in F$ is the set of points of the form:

$$\{gw^n \mid n \in \mathbb{Z}\}$$

We can assume that w is cyclically reduced and not a power of another element without changing the coarse equivalence class of the set. The w-line at g is the same as the w line at h if and only if $\bar{h}g$ is a power of w; the w-lines are the cosets of $\langle w \rangle$ in F.

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The *line pattern generated by* w is the collection of distinct w-lines. Similarly, if we take finitely many words w, as above, the line pattern generated by the collection is the union of the patterns generated by the individual words. We will denote the line pattern $\mathcal{L}$ when we do not wish to specify generators.

The main question is:

Question 1. *Take free groups F and F', possibly of different rank, and words $\{w_1, \ldots, w_m\} \subset F$, $\{w'_1, \ldots, w'_n\} \subset F'$. Let $\mathcal{L}$ be the line pattern in F generated by $\{w_1, \ldots, w_m\}$, and let $\mathcal{L}'$ be defined similarly for F'.*

Is there a quasi-isometry $\phi\colon F \to F'$ that preserves the patterns, in the sense that there is some constant C so that for every line $l \in \mathcal{L}$ there is an $l' \in \mathcal{L}'$ such that the Hausdorff distance between $\phi(l)$ and l' is at most C, and vice versa?

A related question is:

Question 2. *Let F be a free group and $\mathcal{L}$ a line pattern in F. What is the group $\mathcal{QI}(F, \mathcal{L})$ of quasi-isometries of F that preserve the line pattern $\mathcal{L}$?*

If $F \cong F'$ then Whitehead's Algorithm [9, 4] will tell you whether or not there is an automorphism of F that takes $\{w_1, \ldots, w_m\}$ to $\{w'_1, \ldots, w'_m\}$. Similarly, one could make F' a finite index subgroup of F and let $\{w'_1, \ldots, w'_n\}$ be the collection of words necessary to generate the induced pattern. However, not all equivalences come from algebraic constructions.

Since no two lines from a pattern are within bounded Hausdorff distance of each other, such a quasi-isometry gives a bijection between lines of the two patterns. Conversely, any reasonable bijection of lines in the patterns comes from a quasi-isometry. If l and l' are lines in the tree for F, let $\delta(l, l')$ be the minimum distance between the lines. Let ϕ_* be a bijection between the collections of lines in the two patterns. ϕ_* is a *uniformly proper pairing* if there exists some map $\rho\colon \mathbb{N} \to \mathbb{N}$ such that $\delta(\phi_*(l), \phi_*(l')) \leq \rho(\delta(l, l'))$ for all lines l, l', and conversely for ϕ_*^{-1}. If ϕ_* is the bijection induced by a quasi-isometry one could take ρ to be linear, but, because the space is hyperbolic, any ρ at all will do:

Theorem 1.1. *Every uniformly proper pairing of line patterns comes from a quasi-isometry.*

This is a special case of a theorem of Mahan Mj, [6], and is a generalization of the corresponding theorem of Schwartz for line patterns in $\mathbb{H}^n$ [8].

Thus, our goal is to construct uniformly proper pairings between line patterns. While this seems like a low hurdle, Schwartz finds that for $n \geq 3$ the only quasi-isometries preserving the line pattern are isometries of $\mathbb{H}^n$.

We have a result similar to that of Schwartz, at least for some line patterns.

We call a line pattern $\mathcal{L}$ in F *rigid* if $\mathcal{QI}(F, \mathcal{L})$ is conjugate to an isometry group in the following sense: There is a tree T with $\phi\colon F \to T$ a quasi-isometry so that

$$\phi\, \mathcal{QI}(F, \mathcal{L})\phi^{-1} = \mathrm{Isom}(T, \phi(\mathcal{L})) \subset \mathcal{QI}(T)$$

It is easy to find some examples of line patterns that are not rigid. Define the *decomposition space* associated to a line pattern to be the quotient of the boundary of the tree obtained by identifying the two endpoints of each line in the pattern. We show that a line pattern is not rigid when the decomposition space is disconnected, has cut points, or has certain bad cut pairs. However, we conjecture (Conjecture 1) that these are the only situations in which line patterns fail to be rigid.

Cut sets in the decomposition space are related to cut sets in the *Whitehead graph* of the line patterns. Theorem 4.8 shows that a line pattern is rigid if the Whitehead graph satisfies some connectivity hypotheses. We prove this by constructing the tree T in terms of the topology of the decomposition space. In particular we use the finite cut set structure of the decomposition space. We find a canonical collection of cut sets in the decomposition space and let the tree T be the dual tree, which is quasi-isometric to F. Pattern preserving quasi-isometries of F give homeomorphisms of the decomposition space, so these give isometries of the tree T.

For rigid patterns we can compute $\mathrm{Isom}(T, \phi(\mathcal{L}))$ as a graph of finite groups, so $\mathcal{QI}(F, \mathcal{L})$ is virtually a free group. In some cases it turns out that the free group F is a finite index subgroup of $\mathcal{QI}(F, \mathcal{L})$, but this is not true in general for rigid patterns.

2. Preliminaries

[*This section still under construction*]

2.1. **Cut Sets in Topological Spaces.** If X is a connected topological space, a *cut set* is a subset $S \subset X$ such that $X \setminus S$ is disconnected. A single point that is a cut set is a *cut point*; a pair of points that is a cut set is a *cut pair*, etc.

A cut set S is *minimal* if no proper subset of S is a cut set of X.

If S and S' are cut sets of X we say S' crosses S if $S' \setminus S$ has points in multiple components of $X \setminus S$. This is not a symmetric relation.

We will say S and S' are *mutually crossing* if S crosses S' and S' crosses S, and are *mutually non-crossing* if neither S nor S' crosses the other.

If S and S' are minimal cut sets then they are either mutually crossing or mutually non-crossing. To see this, suppose S' crosses S but S does not cross S'. Let S disconnect X into sets A and B so that S' intersects both A and B. Let S' disconnect X into sets C and D so that S does not intersect D. Assume $A \cap D$ is nonempty (either $A \cap D$ or $B \cap D$ is nonempty). Let $S'' = (S' \cap A) \cup (S' \cap S)$. Then the closure of $A \cap D$ is contained in $(A \cap D) \cup S''$, so S'' is a proper subset of S' that is a cut set of X, contradicting minimality of S'.

Let $\{S_i\}_{i \in I}$ be a collection of cut sets of X so that for each i, $X \setminus S_i$ has two components, A_i and B_i. Let

$$\Sigma = \{A_i\}_{i \in I} \cup \{B_i\}_{i \in I}$$

Σ is partially ordered by inclusion, and one can encode this structure in a cube complex, see Sageev [7]. If the collection $\{S_i\}_{i \in I}$ is pairwise mutually non-crossing then this cube complex is a tree, as in Dunwoody's use of tracks in [3].

Define a vertex of this dual complex to be a subset V of Σ such that:

(1) For all $i \in I$ exactly one of A_i or B_i is in V.
(2) If $C \in V$ and $C' \in \Sigma$ with $C \subset C'$ then $C' \in V$.

Two vertices are connected by an edge if they differ by only one set in Σ.

One can think of the vertices as a subset of 2^I. The i-th "coordinate" is either 0 or 1 depending on whether one is on the A_i side or the B_i side of S_i. Edges join vertices that differ in exactly one coordinate.

2.2. **Graphs.** A *graph* is a collection of vertices and edges. Each edge has two distinct endpoints, which are vertices. There may be multiple edges with the same pair of endpoints.

2.2.1. *Connectivity in Graphs.* We will say a graph $\mathcal{G}$ is (a,b)*-connected* if for any choice of a vertices $v_1,\dots,v_a$, and b edges $e_1,\dots,e_b$ one can delete the vertices v_i, all edges incident to any v_i, and the interiors of the edges e_j, and still have a non-empty, connected graph.

The choices of vertices and edges are not required to be distinct, so (a,b) connected implies (a',b')-connected for any $a'\leq a$ and $b'\leq b$.

When $a>0$ and the graph has at least $a+2$ vertices, (a,b)-connected implies $(a-1,b+1)$-connected, but the converse is not true.

A $(1,0)$-connected graph is sometimes called *connected without cut points*. A $(0,b)$-connected graph is sometimes called *b-edge-connected*.

2.2.2. *Splicing.* Given two graphs $\mathcal{G}$ and $\mathcal{G}'$ which contain vertices v and v', respectively, of equal valence, it is possible to create a new graph by *splicing $\mathcal{G}$ and $\mathcal{G}'$ at* (v,v'). Take a bijection of the edges incident to v to the edges incident to v', delete vertices v and v' from their corresponding graphs, and then combine the two graphs by identifying the loose edges according to the bijection.

Notice that the resulting graph is connected if at least one of v or v' is not a cut point.

It is not hard to see that for $a>0$, splicing two $(a,0)$-connected graphs yields an $(a,0)$-connected graph.

2.3. **The Boundary at Infinity of a Tree.** [*Establish notation and basic results about the topology of the boundary of the tree.*] [1]

∂T is a Cantor set. Since T is hyperbolic, any quasi-isometry $\phi\colon\ T\to T'$ extends to a homeomorphism $\partial\phi\colon\ \partial T\to\partial T'$.

3. Whitehead Graphs and the Topology of the Decomposition Space

Let F be the free group of rank n, with free generating set $\mathcal{B}=\{a_1,\dots,a_n\}$. For $g\in F$, let $\bar{g}$ denote g^{-1}.

Let $T=\mathcal{C}_{\mathcal{B}}(F)$ be the Cayley graph of F with respect to $\mathcal{B}$. T is a tree.

3.1. **The Decomposition Space.** Suppose $l=\{gw^n\mid n\in\mathbb{Z}\}$ is a line in the pattern. l has distinct endpoints at infinity:

$$l^+=gw^\infty=\lim_{n\to\infty}gw^n$$

and

$$l^-=gw^{-\infty}=\lim_{n\to-\infty}gw^n$$

Define the decomposition space $D_{\mathcal{L}}$ associated to a line pattern $\mathcal{L}$ to be the topological quotient of ∂T obtained by identifying the points l^+ and l^- for every line $l\in\mathcal{L}$. This is a compact, Hausdorff topological space.

Let $q\colon\ \partial T\to D$ be the quotient map to the decomposition space.

If ϕ is a quasi-isometry that takes each line l in a pattern to within bounded Hausdorff distance of a line l' in another pattern, then $\partial\phi(\{l^+,l^-\})=\{l'^+,l'^-\}$. Thus, the homeomorphism $\partial\phi\colon\ \partial T\to\partial T'$ descends to a homeomorphism of the decomposition spaces.

3.2. **Whitehead Graphs.** Let $w \in F$ be a cyclically reduced word. The *Whitehead Graph of w with respect to $\mathcal{B}$*, $Wh_{\mathcal{B}}(w)$ is the graph with $2n$ vertices labeled $a_1, \ldots, a_n, \bar{a}_1, \ldots \bar{a}_n$, and an edge between vertices v and v' for each occurrence of $\bar{v}v'$ in w (as a cyclic word). The graph depend on the choice of $\mathcal{B}$, but we will write $Wh(w)$ when $\mathcal{B}$ is clear.

For example, if $F = \langle a, b \rangle$ here are some Whitehead Graphs:

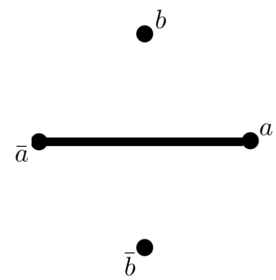

FIGURE 1. $Wh(a)$

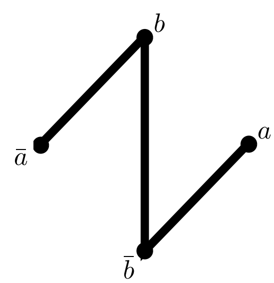

FIGURE 2. $Wh(ab^2)$

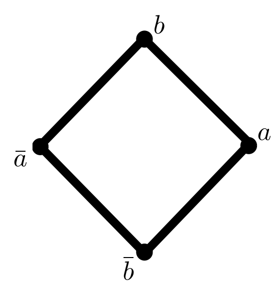

FIGURE 3. $Wh(ab\bar{a}\bar{b})$

Notice that $Wh(a)$ is disconnected; the vertices b and $\bar{b}$ are isolated. $Wh(ab^2)$ is connected but has cut vertices. $Wh(ab\bar{a}\bar{b})$ is connected and has no cut vertices.

Proposition 3.1 ([5])**.** *If $Wh(w)$ has a cut vertex there is an automorphism of ϕ of F so that $Wh(\phi(w))$ has no cut vertices.*

Consider the word ab^2. Let ϕ be the automorphism defined by

$$\phi\colon \begin{cases} a \mapsto a\bar{b} \\ b \mapsto b \end{cases}$$

Then $\phi(ab^2) = ab$, and $Wh(ab)$ is disconnected.

The following Theorem is attributed to M. Bestvina in [5].

Theorem 3.2. *For any $w \in F - \{e\}$, the following are equivalent:*

1. *w is contained in a proper free factor of F.*
2. *The width of w is strictly less than the rank of F.*
3. *The decomposition space associated to the pattern generated by w is disconnected.*
4. *For any free basis of F, the generalized Whitehead graph is disconnected for some $n \geq 1$.*
5. *Any Whitehead graph for w with no cut vertices is disconnected.*
6. *There exists a disconnected Whitehead graph of w.*

Everything that we have said so far generalizes to line patterns generated by finitely many words, so we will use $Wh(\mathcal{L})$ to denote the Whitehead graph corresponding to the pattern $\mathcal{L}$.

3.2.1. *Geometric Interpretation.* Consider the subset of the free group consisting of all words whose first letter is a. This gives a rooted tree in the Cayley graph of the free group. We can think of the vertices of the Whitehead graph as the roots of the four rooted trees obtained by taking the complement of the open ball of radius 1 about the identity vertex. An edge in the Whitehead Graph corresponds to a line from the pattern that travels between the corresponding components.

From this point of view it is easy to extend the definition of the Whitehead Graph to larger regions. Let C be a connected subset of the tree with finitely many vertices. Define the Whitehead graph $Wh(C, \mathcal{L})$ as follows: Let the vertices be the roots of the various components of the complement of the set $\bigcup_{v \in \mathcal{V}C} star(v)$. (That is, take the complement of the union of the open stars around each vertex of C.) Add an edge between two vertices if there is a line of $\mathcal{L}$ that goes between the associated components.

The standard Whitehead Graph is just $Wh(\mathcal{L}) = Wh(\{e\}, \mathcal{L})$; the set C is just the identity vertex and the star around the identity vertex is just the open ball of radius 1.

We will continue to use the notation $Wh(\mathcal{L})$ when C is a single vertex. Since $\mathcal{L}$ is equivariant we get the same graph for any vertex.

3.2.2. *Splicing.* We can build up Whitehead graphs $Wh(C, \mathcal{L})$ by splicing together copies of $Wh(\mathcal{L})$ for each of the vertices of C. We can make the splicing easier to visualize if we draw the Whitehead graphs with "loose ends" at the vertices. Consider again the Whitehead graph in F_2 for the word $ab\bar{a}\bar{b}$, but with "loose ends":

Now take a copies of this graph at e and at a and splice them together. We get the splicing map by considering the word w. At e, the edge from b to a corresponds to the pair $\bar{b}a$ in $ab\bar{a}\bar{b}$. The next letter would be b, so the next pair is ab. So the edge from b to a at the vertex e gets spliced to the edge from $\bar{a}$ to b at the vertex a. Similarly, the edge from $\bar{b}$ to a at e gets spliced to the edge from $\bar{a}$ to $\bar{b}$ at the vertex a.

Note. Unless noted otherwise, figures are drawn so that the splicing map is achieved by an orientation preserving isometry of the page.

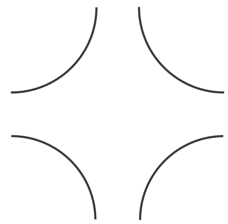

Figure 4. $Wh(\{e\}, ab\bar{a}\bar{b})$

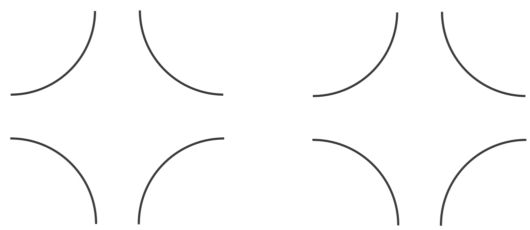

Figure 5. $Wh(\{e\}, ab\bar{a}\bar{b})$ and $Wh(a, ab\bar{a}\bar{b})$

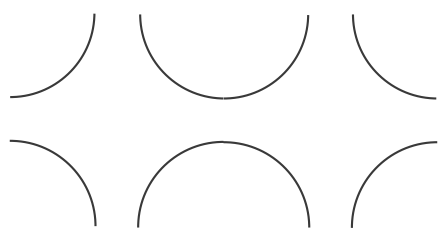

Figure 6. $Wh([e, a], ab\bar{a}\bar{b})$

It will be useful to define Whitehead graphs for unbounded C. For any connected subset of the tree we could build a Whitehead graph $Wh(C, \mathcal{L})$ using the geometric interpretation, or as a limit of splicing copies of $Wh(\mathcal{L})$. These methods produce the same graph when C has finitely many vertices. If $C \subset \overline{T}$ contains any endpoints of lines of $\mathcal{L}$, then splicing does not actually produce a graph.

If both endpoints of a line l are in $\overline{C}$ then after finitely many splices there is an edge corresponding to l, but in the limit the edge grows to be a bi-infinite line. The vertices escape to infinity. This line does not occur if we follow the geometric definition, because it is not joining two different components of the complement of C in T.

Similarly, if only one endpoint of l is in $\overline{C}$ then splicing produces a graph $\mathcal{G}$ with a half line attached.

We will define $Wh(C, \mathcal{L})$ by the geometric definition. This is equivalent to using the splicing definition and deleting any "edges" that do not have two endpoints.

3.3. **Topology of the Decomposition Space.** [*Some basic results about the topology of the Decomposition space. I don't think we'll actually use this information.*]

3.4. **Cuts Sets in the Decomposition Space and Whitehead Graph.** We can distinguish line patterns by the topology of the decomposition space. We already mentioned that the decomposition space is disconnected if and only if any Whitehead graph with no cut vertices is disconnected.

Assume we have chosen the free basis of F so that $Wh(\mathcal{L})$ is connected without cut vertices.

There are two kinds of points in the decomposition space: those whose preimage in ∂T is a single point, and those whose preimage in ∂T is the pair of endpoints of a line from the pattern. We will call points whose preimage is a pair of point *special*, and points whose preimage is a single point *ordinary*.

A *minimal finite cut set* (MF cut set) is a finite set of points in the decomposition space such that the complement is disconnected, and such that the complement of any proper subset is connected. So an MF cut set of size 1 is just a cut point. An MF cut set of size 2 is a cut pair of points, neither of which is a cut point, etc.

An *MFS cut set* is a minimal finite cut set consisting exclusively of special points.

Pattern preserving quasi-isometries extend to homeomorphisms of the decomposition space that preserve special points. Such a homeomorphism preserves the property of being an MFS cut set and the size of such a set.

Every special point ξ belongs to some MFS cut set. To see this consider the line in the pattern corresponding to ξ. Pick any edge of the tree that this line crosses. There are finitely many other lines from the pattern that also cross this edge. The set of special points corresponding to these lines is a special cut set, which can then be reduced to an MFS cut set containing ξ.

In fact, every special point ξ is either a cut point or belongs to infinitely many MFS cut sets. Our patterns are generated by finitely many words of finite length, so there is some maximum number of edges that can be shared by any two lines. For any special point, take the corresponding line in the tree, and select a sequence of edges that are spaced farther apart than the maximum overlap. The argument in the previous paragraph implies that there is an MFS cut set containing ξ associated to each of these edges. The intersection of any two of these is ξ, so if ξ is not a cut point these sets are all distinct.

Define $q_*\colon \mathcal{L} \to D$ by $q_*(l) = q(l^+) = q(l^-)$.

We have an easy sufficient condition to see that a set $\{l_1, \dots, l_k\} \subset \mathcal{L}$ gives a cut set of D.

Proposition 3.3. *Let $\{l_1, \dots, l_k\}$ be a collection of lines in $\mathcal{L}$. Let C be any convex set in the tree. In $Wh(C, \mathcal{L})$, delete the interior of any edge corresponding to one of the lines l_i. If the result is disconnected then $q_*(\{l_1, \dots, l_k\})$ is a cut set.*

This proposition is a special case of the next.

The converse is not quite true. The problem is that, while there are no cut points in the Whitehead graph, there may, in fact, be cut points in the decomposition space.

Let $\{S_i\}$ be a collection of convex subsets of $\overline{T}$. Let H be the convex hull of $\{S_i\}$. Define the *core* C of $\{S_i\}$, to be the smallest convex set such that $H \setminus C$ is a collection of disjoint infinite (open) geodesic rays $r_j\colon (0, \infty) \to T$. We use $r_j(0)$ to denote the vertex of the core which is $\lim_{t\to 0} r_j(t)$, but we do not consider this point to be in the image of the ray. The vertex set of $r_j((0, \infty))$ is the same as the vertex set of $r_j([1, \infty))$.

For a finite collection of lines $\{l_1, \ldots, l_k\} \subset \mathcal{L}$, the core is a finite tree. The convex hull minus the core is a collection of $2k$ disjoint open rays:

$$\{\gamma_i^*\colon (0,\infty) \to T \mid \lim_{t\to\infty} \gamma_i^*(t) = l_i^*,\, * \in \{+,-\},\, i = 1 \ldots k\}$$

Proposition 3.4. *Let $l_1, \ldots, l_k$ be a collection of lines in $\mathcal{L}$. Let H be the convex hull of these lines, and let C be the core. Assume that for all i, $q_*(l_i)$ is not a cut point in D. Then there is a bijection between connected components of $D \setminus q_*(\{l_1, \ldots, l_k\})$ and connected components of $Wh(C, \mathcal{L}) \setminus \{\mathring{e}_1, \ldots, \mathring{e}_k\}$, where $\mathring{e}_i$ denotes the interior of the edge of $Wh(C, \mathcal{L})$ corresponding to l_i.*

Before proving Proposition 3.4 we prove a few preliminary results.

Lemma 3.5. *Pick an edge e in T. Let $A \subset \partial T$ be all the boundary points on one side of the edge. $q(A)$ is connected in D.*

The proof is essentially the same as the proof of some of the cases of Theorem 3.2.

Proof. Suppose there are open sets X and Y of D such that $q(A) \subset X \cup Y$ and $q(A) \cap X \cap Y = \emptyset$. A is open in ∂T, so $A' = A \cap q^{-1}(X)$ and $A'' = A \cap q^{-1}(Y)$ are open. Assuming that A' is nonempty, we will show that A'' must be empty, which implies $q(A)$ is connected.

A is closed in ∂T, so A' and A'' are closed. ∂T is compact, so A' and A'' are compact clopens. Let $*$ be the vertex of e on the A side. For any $\xi \in \partial T$, the sets $N_R(\xi, *) = \{x \in \bar{T} \mid x = [*, \xi](t),\, t > R\}$ give a neighborhood basis. Since A' is compact and open, there are finitely many vertices $x_1, \ldots, x_a$ so that A' is the union of the boundaries of the cones at x_i with respect to $*$. In other words, A' consists of the points $\xi \in \partial T$ such that for some i, x_i is on the geodesic ray $[*, \xi]$. There is a similar finite collection $y_1, \ldots, y_b$ that determines A''.

Let C be the core of $\{x_i\}_{i=1}^a \cup \{d_j\}_{j=1}^b \cup *$. $Wh(C, \mathcal{L})$ is connected without cut points, since it can be constructed by splicing together finitely many copies of $Wh(\mathcal{L})$, which is connected without cut points. In particular, $*$ is not a cut point.

Assume A' is nonempty. If $x_1 = *$ then $A' = A$, so $A'' = \emptyset$, and we are done. Otherwise, x_1 is a vertex of $Wh(\mathcal{L}) \setminus *$. An edge of $Wh(\mathcal{L}) \setminus *$ incident to x_1 corresponds to a line $l \in \mathcal{L}$ with one endpoint in the cone of x_1 and the other endpoint in the cone of a vertex $v \in \{x_2, \ldots, x_a\} \cup \{y_1, \ldots, y_b\}$, corresponding to some other vertex of $Wh(\mathcal{L}) \setminus *$. In the decomposition space these two endpoints are identified, and we already know that the image of the first endpoint is in X. This means that v must be in $\{x_2, \ldots, x_a\}$. Since $Wh(\mathcal{L}) \setminus *$ is connected we conclude that all the vertices of $Wh(\mathcal{L}) \setminus *$ belong to $\{x_1, \ldots, x_n\}$, so $A'' = \emptyset$. Thus, $q(A)$ is connected in D. □

Lemma 3.6 (Hull Determines Connectivity)**.** *Let S be a nonempty, finite subset of D that is not just a single ordinary point. Let H be the convex hull of $q^{-1}(S)$. There is a bijection between connected components of $Wh(H, \mathcal{L})$ and connected components of $D \setminus S$.*

Proof. Components $\{A_i\}$ of $T \setminus H$ are the kind of sets in Lemma 3.5. Therefore, $q(\partial A_i)$ is connected in D. ∂A_i is open in ∂T. For a subcollection $\{A_{i_j}\}$ corresponding to a connected component of $Wh(H, \mathcal{L})$, $q(\bigcup \partial A_{i_j})$ is an open connected set in $D \setminus S$, as in the proof of Lemma 3.5. The complement of this set in $D \setminus S$

is either empty or is a union of sets of a similar form, corresponding to other connected components of H. Thus, $q(\bigcup \partial A_{i_j})$ closed, and is therefore a connected component of $D \setminus S$. □

Pick any vertex $* \in T$. If $\xi \in D$ is an ordinary point, the previous argument applies if we take H to be the ray $[*, \xi)$. If $Wh(\mathcal{L})$ is connected without cut points then $Wh(H, \mathcal{L})$ is connected. Therefore:

Corollary 3.7. *No ordinary point of D is a cut point.*

Now Lemma 3.6 tells us that for $l \in \mathcal{L}$, l gives a cut point in the decomposition space if $Wh(l, \mathcal{L})$ is disconnected. Pick a vertex $* \in l$. For any vertex $v \in Wh(l, \mathcal{L})$ there is a path in $Wh(l, \mathcal{L})$ from v to $Wh(*, \mathcal{L}) \setminus l$. This follows from the assumption that $Wh(\mathcal{L})$ is connected without cut points. Therefore, $Wh(l, \mathcal{L})$ has at most as many components as $Wh(*, \mathcal{L}) \setminus l$. Similarly, for any edge $e \subset l$, $Wh(l, \mathcal{L})$ has at most as many components as the number of lines of $\mathcal{L} \setminus \{l\}$ crossing e. The same reasoning applies in the infinite direction as well, which implies that l^+ and l^- are both limit points of every component of $Wh(l, \mathcal{L})$.

We can determine whether l gives a cut point by considering a finite graph:

Proposition 3.8. *There is a constant R depending on $\mathcal{L}$ so that for any line $l \in \mathcal{L}$ and for any segment s of l of length at least R, $q_*(l)$ is a cut point in D if and only if, $Wh(s, \mathcal{L}) \setminus \overline{e}$ is connected, where we delete not just the interior of the edge e corresponding to l, but the entire edge, including the two endpoints.*

Proof. l is a line corresponding to a coset of some word w. The result follows from the fact that $\langle w \rangle$ acts cocompactly on $Wh(l, \mathcal{L})$.

Let $\sigma\colon \mathbb{R} \to T$ be an isometric embedding with image l. The number of components of $Wh(\sigma([0, t], \mathcal{L}) \setminus \sigma(\{-1, t+1\})$ is less than or equal to the number E of lines of $\mathcal{L} \setminus \{l\}$ that cross the edge $\sigma([-1, 0])$. Label these lines $l_1, \dots, l_E$. The action of w takes an l_i to a line l'_i of $\mathcal{L} \setminus \{l\}$ that cross the edge $\sigma([-1 + |w|, |w|])$. For each i choose some $\rho(i)$ such that l_i and $l'_{\rho(j)}$ are in the same component of $Wh(\sigma([0, |w|], \mathcal{L}) \setminus \sigma(\{-1, |w| + 1\})$. Every l_i connects to some l'_j, and vice versa, so we can choose ρ to be a permutation of $\{1, \dots, E\}$.

Now the action of $w^{|\rho|}$ on $Wh(l, \mathcal{L})$ preserves components, so $Wh(\sigma([0, |\rho||w|], \mathcal{L}) \setminus \sigma(\{-1, |\rho||w|+1\})$ is connected if and only if $Wh(l, \mathcal{L})$ is connected, so we can determine connectivity of $Wh(l, \mathcal{L})$ by considering any segment of l of length $R = E! \cdot |w|$.

To get a uniform R replace E and $|w|$ by their respective maximums over all equivalence classes edges of the tree and generators of $\mathcal{L}$. □

Proposition 3.9. *If $Wh(\mathcal{L})$ is $(2, 0)$-connected then there are no cut points in the decomposition space and there are no ordinary points in any minimal cut set.*

Proof. Let l be a line in the pattern and let σ be as in the proof of the previous proposition. If $Wh(\mathcal{L})$ is $(2, 0)$-connected then $Wh(\sigma([0, t]), \mathcal{L})$ is $(2, 0)$-connected for any t, so $Wh(\sigma([0, |\rho||w|]), \mathcal{L}) \setminus \sigma(\{-1, |\rho||w| + 1\})$ is connected, which implies that $q_*(l)$ is not a cut point.

Suppose S is a cut set that includes an ordinary point ξ. Let H be the convex hull of $q^{-1}(S)$ and let C be the core. There is a ray γ in $H \setminus C$ tending to ξ. Since $Wh(\mathcal{L})$ is $(2, 0)$-connected, $Wh(\gamma([1, \infty)), \mathcal{L}) \setminus \gamma(0)$ is connected. Therefore, components of $Wh(H, \mathcal{L})$ are in bijection with components of $Wh(H \setminus \gamma, \mathcal{L})$, which means that $S \setminus \xi$ is still a cut set, so S was not a minimal cut set. □

Here is an example that illustrates these cut set results. Consider the pattern $\mathcal{L}$ generated by the pair of words a and $ab\bar{a}\bar{b}$. The Whitehead graph for this pattern is shown in Figure 7.

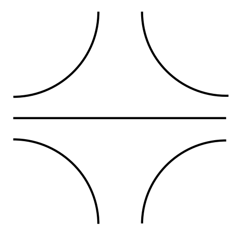

FIGURE 7. $Wh(\mathcal{L})$

For any compact, connected S, $Wh(S, \mathcal{L})$ looks like a circle with a number of disjoint chords (see Figure 8).

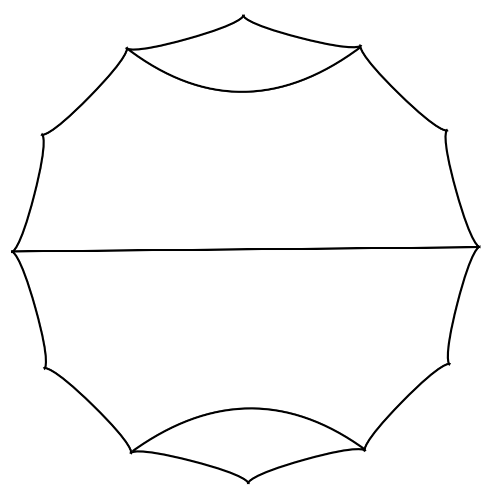

FIGURE 8. $Wh(B_2e, \mathcal{L})$

The edges of the circle correspond to $ab\bar{a}\bar{b}$-lines, and the chords correspond to a-lines. This graph has no cut points, so deleting the interior of an edge does not disconnect it, but deleting any one of the chords, including the endpoints, does disconnect the graph. For any finite segment of an a-line, the Whitehead graph always looks like a circle with one chord. We conclude that the a-lines give cut points in the decomposition space, so this pattern can not be equivalent to any pattern that does not have cut points in its decomposition space.

Now compare Figure 7 to the Whitehead graph for the pattern generated by the word $a^2b\bar{a}\bar{b}$, shown in Figure 9. The Whitehead graphs are the same, but with different splicing.

If we splice together the Whitehead graphs for this pattern at vertices e and a we get Figure 10.

Now if we close up the vertices and delete the edge that we thought corresponded to the a-line in the pattern, along with its endpoints and any other incident edges,

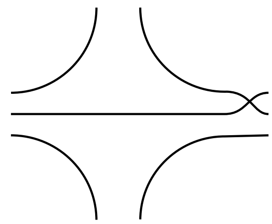

FIGURE 9. $Wh(e, a^2b\bar{a}\bar{b})$

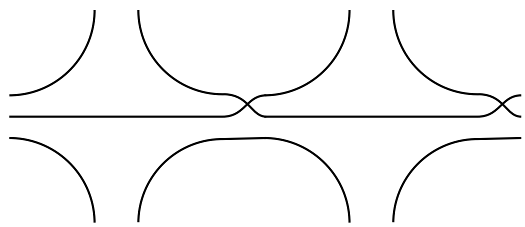

FIGURE 10. $Wh([e,a], a^2b\bar{a}\bar{b})$

we find that the graph is still connected. The line does not give us a special cut point in the decomposition space. In fact, none of the lines do; there are no special cut points in the decomposition space for this pattern.

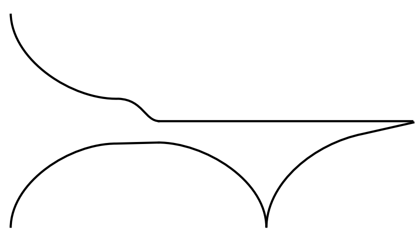

FIGURE 11. $Wh([e,a], a^2b\bar{a}\bar{b}) \setminus a$-line

Proof of Proposition 3.4. Recall that $H \backslash C$ consists of $2k$ disjoint open rays $\gamma_i^* \colon (0, \infty) \to T$.

The graph $Wh(H, \mathcal{L})$ is obtained from the graph $Wh(C, \mathcal{L}) \setminus \{\mathring{e}_i\}_{i=1}^k$ by splicing on $Wh(\gamma_i^*(0,\infty), \mathcal{L})$ at $\gamma_i^*(0)$, for each $1 \leq i \leq k$ and $* \in \{+, -\}$.

As none of the l_i give cut points in the decomposition space, each $Wh(\gamma_i^*(0,\infty), \mathcal{L}) \setminus \gamma_i^*(0)$ is connected. Therefore, connected components of $Wh(H, \mathcal{L})$ are in bijection with connected components of $Wh(C, \mathcal{L}) \setminus \{\mathring{e}_i\}_{i=1}^k$. By Lemma 3.6, these are in bijection with components of $D \setminus q_*(\{l_i\})$. □

Proposition 3.10. *Let S be an MFS cut set of size greater than one. $D \setminus S$ has exactly two components.*

Proof. Let $S = q_*(\{l_1, \ldots, l_k\})$. Let C be the core of $\{l_1, \ldots, l_k\}$. By Proposition 3.4, components of $D \backslash S$ are in bijection with components of $Wh(C, \mathcal{L}) \backslash \{e_i\}_{i=1}^k$. This is a finite graph, so $D \setminus S$ has only finitely many components.

Let $A_1, \ldots, A_m$ be a list of the components of $D \setminus S$.

If $q_*(l_i)$ is not a limit point of A_j in D then A_j is still a connected component in $D \setminus (S \setminus q_*(l_i))$. This contradicts minimality of S, so each point of S is a limit point in D of every A_j. This implies that for each i and j, at least one of the points l_i^+ and l_i^- is a limit point of $q^{-1}(A_j)$.

Now $H \setminus C$ is a collection of disjoint rays γ_i^*. We saw in the proof of Proposition 3.4 that $Wh(\gamma_i^*, \mathcal{L}) \setminus \gamma_i^*(0)$ is connected, so no l_i^+ or l_i^- is a limit point of more than one $q^{-1}(A_j)$.

Thus, there are exactly two components A_1 and A_2 of $D \setminus S$, and each line l_i has one endpoint in $q^{-1}(A_1)$ and the other in $q^{-1}(A_2)$. □

Combining Proposition 3.10 with Proposition 3.4 we have:

Corollary 3.11. *If there are no cut points in the decomposition space, then take any edge e of the tree, and let $\{l_1, \ldots, l_k\}$ be the set of lines of $\mathcal{L}$ that cross e. $q_*(\{l_i\})$ is an MFS cut set.*

Proposition 3.12. *Suppose $Wh(\mathcal{L})$ is $(2,0)$-connected and $(1,b)$-connected. There are no cut sets of size less than $b+2$, and any minimal cut set S with $|S| \leq 2b+1$ is equivalent to one that is visible in $Wh(\mathcal{L})$.*

Proof. Let S be a minimal cut set. By Proposition 3.9, $|S| > 1$ and S contains no ordinary points. Let C be the core of $q^{-1}(S)$. Let $q_*^{-1}(S) = \{l_1, \ldots, l_k\}$.

C is a finite tree. Suppose C is not a single vertex, then it has a leaf v, a valence 1 vertex of C. Let v' be the unique vertex of C adjacent to v. The portion of $Wh(C, \mathcal{L})$ at the vertex v is $Wh(v, \mathcal{L}) \setminus v'$.

Some of the lines $l \in q_*^{-1}(S)$ pass through v. Notice that if some l passes through both v and v' then the edge corresponding to l in $Wh(v, \mathcal{L})$ is incident to v' in $Wh(v, \mathcal{L})$, so this edge is deleted in $Wh(v, \mathcal{L}) \setminus v'$. Now $Wh(v, \mathcal{L})$ is $(1,b)$-connected, so $Wh(v, \mathcal{L}) \setminus v'$ is $(0,b)$-connected. Thus, $Wh(v, \mathcal{L}) \setminus v'$ is connected unless there are at least $b+1$ lines in $q_*^{-1}(S)$ that pass through v but not through v'.

If $Wh(v, \mathcal{L}) \setminus v'$ is connected then v does not contribute to the number of components of $D \setminus S$: components of $Wh(C, \mathcal{L}) \setminus \{\mathring{e}_i\}_{i=1}^k$ are in bijection with components of $Wh(C \setminus v, \mathcal{L}) \setminus \{\mathring{e}_i\}_{i=1}^k$

In this way we reduce the number of vertices of C until we are left with either a single vertex or a tree in which every leaf has at least $b+1$ lines $q_*^{-1}(S)$ that do not go deeper into the core.

If we reduce and are left with a reduced core with two or more leaves, then $|S| \geq 2b+2$, since for each leaf there are at least $b+1$ lines that do not go deeper into the core.

If we reduce to a single vertex then for S to be a cut set we need to disconnect $Wh(\mathcal{L})$. $Wh(\mathcal{L})$ is $(1,b)$-connected, so it is $(0,b+1)$-connected. Therefore, in this case $|S| \geq b+2$.

□

Remark. We can replace the hypothesis that $Wh(\mathcal{L})$ is $(2,0)$-connected in the previous proposition with the weaker hypothesis that $D_{\mathcal{L}}$ has no cut points and no ordinary points in any minimal cut set.

4. Classification of Patterns

4.1. When the Whitehead Graph is a Circle. We will show in this section that when the Whitehead graph is a circle we get a quasi-isometrically flexible line pattern.

Theorem 4.1. *The following are equivalent:*

(1) *Every Whitehead graph $Wh_{\mathcal{B}}(\mathcal{L})$ that has no cut vertex is a circle.*
(2) *Some Whitehead graph $Wh_{\mathcal{B}}(\mathcal{L})$ is a circle.*
(3) *D is a circle.*

Proof. Clearly $1 \implies 2$, because Whitehead automorphisms will eliminate cut vertices.

If some Whitehead graph $Wh_{\mathcal{B}}(\mathcal{L})$ is a circle then we can realize the free group F_n as the fundamental group of a surface with boundary, and the generators of the line pattern $\mathcal{L}$ as the boundary labels. We can give this surface a hyperbolic metric so that the universal cover is just T fattened, and the boundary components are horocycles that are in bijection with the lines of $\mathcal{L}$. This gives us a homeomorphism between the decomposition space and $S^1 = \partial\mathbb{H}^2$. Thus $2 \implies 3$.

Assume that D is a circle. Then there are no cut points, and every pair of points is a minimal cut set. Choose a free basis for F_n so that $Wh(\mathcal{L})$ is connected without cut points. The edges incident to a vertex of $Wh(\mathcal{L})$ correspond to an MFS cut set by Corollary 3.11. However, every minimal cut set of D has size two. Therefore, $Wh(\mathcal{L})$ is a finite, connected graph with all valences equal to two, hence, a circle. Thus, $3 \implies 1$. □

Theorem 4.2. *Let $\mathcal{L}$ be a line pattern in F_n and $\mathcal{L}'$ a line pattern in F_m. Suppose $Wh(\mathcal{L})$ is a circle. There is a quasi-isometry $F_n \to F_m$ that takes $\mathcal{L}$ to $\mathcal{L}'$ if and only if any $Wh(\mathcal{L}')$ with no cut vertex is a circle.*

Proof. By Theorem 4.1, if $Wh(\mathcal{L})$ is a circle then $D_{\mathcal{L}}$ is a circle. If the patterns are quasi-isometric then $D_{\mathcal{L}'}$ is also a circle, so any $Wh(\mathcal{L}')$ with no cut vertex is a circle.

If the Whitehead graphs of the two patterns are circles then, as in the proof of Theorem 4.1 we can associate each pattern with the boundary curves of the universal cover of a surface with boundary. There are many boundary preserving quasi-isometries of such surfaces. □

[*Not only are circle patterns quasi-isometric, there is a lot of flexibility. We can rescale lines. We can put a coloring on the set of lines and insist that the quasi-isometry is color-preserving.*]

4.2. When the Whitehead Graph is Complete. In this section we will show that if a line pattern gives a Whitehead graph which is the complete graph (without multiple edges) then the line pattern is quasi-isometrically rigid. The arguments in this section will be a model for more general rigid patterns in the next section.

Let F_n be the free group of rank $n > 1$ with a fixed free generating set. Let K_{2n} be the complete graph on $2n$ vertices, the graph consisting of $2n$ vertices with exactly one edge joining each pair of vertices.

Suppose $\mathcal{L}$ is a line pattern in F_n so that $Wh(\mathcal{L}) = K_{2n}$. K_{2n} is $(2n-1, 0)$-connected, so it is $(2, 0)$ and $(1, 2n-3)$-connected.

By Proposition 3.12 any cut set of the decomposition space has size at least $2n-1$, and any cut set of size less than $4n-4 > 2n-1$ is visible in $Wh(\mathcal{L})$.

By Corollary 3.11, D has MFS cut sets of size $2n-1$ corresponding to the set of edges incident at a vertex of the Whitehead graph. These are the only choice of $2n-1$ edges that will disconnect K_{2n}, so the only minimal cut sets of D of size $2n-1$ are the MFS cut sets corresponding to edges of the tree.

Consider line patterns $\mathcal{L}$ in F_n and $\mathcal{L}'$ in F_m that give Whitehead graphs K_{2n} and K_{2m}, respectively. There is no quasi-isometry from F_n to F_m that respects these line patterns if $m < n$, because then $D_{\mathcal{L}'}$ has cuts sets of size $2m-1$, but $D_{\mathcal{L}}$ does not.

However, much more is true. We have shown that the tree, which depends on the choice of generating set, can be reconstructed from the topology of the decomposition space.

Consider the collection of cut sets of size $2n-1$ in $D_{\mathcal{L}}$. The collection is equivariant. For any two of these cut sets, say S and S', we have $S' \setminus S$ is contained in one component of $D \setminus S$, and vice versa. We call such cut sets *mutually non-crossing.* This collection of cut sets gives a dual tree, with the cut sets corresponding to edges of the tree. For this pattern the tree is a regular $2n$-valent tree, it is just the tree T. We can also construct a line pattern in this tree by considering intersections of the cut sets. In this case we get back exactly the line pattern $\mathcal{L}$. In particular, in the star of each vertex the line pattern looks like the graph K_{2n}.

Since we have defined this tree and line pattern in terms of topological data of the decomposition space, the only quasi-isometries are isometries of the tree. We have not specified the splicing data needed to match the line pattern in the star of one vertex to the star of an adjacent vertex, but in this case that doesn't matter because of the symmetry of the pattern.

$\mathcal{L}$ and $\mathcal{L}'$ could be any two line patterns in a tree of valence $2n$ such that the pattern restricted to the star of any vertex is K_{2n}, there is an isometry of the tree that takes one to the other. In fact, you can choose a vertex to send to a vertex, an incident edge to send to an incident edge, and an identification of the $2n-1$ lines in the cut set corresponding to the edge, and this determines the isometry on the rest of the tree.

This discussion proves the following two theorems:

Theorem 4.3. *Suppose $\mathcal{L}$ is a line pattern in F_n such that $Wh(\mathcal{L}) = K_{2n}$, and $\mathcal{L}'$ is a line pattern in F_m. There is a pattern-respecting quasi-isometry $F_n \to F_m$ if and only if $D_{\mathcal{L}'}$ has the following properties:*

(1) *There are no cut sets of size less than $2n-1$.*
(2) *The collection of cut sets of size $2n-1$ is pairwise mutually non-crossing, and the dual tree is valence $2n$.*
(3) *The line pattern given by intersection of the cut sets restricts to the complete graph K_{2n} in the star of any vertex.*

If $\mathcal{L}$ is a line pattern in F let $\mathcal{QI}(F, \mathcal{L})$ denote the group of quasi-isometries that preserve the pattern $\mathcal{L}$. Similarly, let $\mathrm{Isom}(T, \mathcal{L}$ denote the group of isometries of the tree T that preserve the line pattern $\mathcal{L}$.

Theorem 4.4. *Let $\mathcal{B}$ be a free generating set for $F = F_n$ and let $\mathcal{L}$ be a line pattern in F such that $Wh_{\mathcal{B}}(\mathcal{L}) = K_{2n}$. Let $T = \mathcal{C}_{\mathcal{B}}(F)$ be the Cayley graph. Then $\mathcal{QI}(F, \mathcal{L}) = \mathrm{Isom}(T, \mathcal{L}) \subset \mathcal{QI}(T)$. Furthermore, F is a finite index subgroup of*

$\mathcal{QI}(F, \mathcal{L})$, and we can present this group explicitly as a graph of groups:

$$\mathrm{Isom}(T, \mathcal{L}) \cong \mathrm{Sym}_{2n} *_{\mathrm{Sym}_{2n-1}} \mathrm{Sym}_{2n-1} \times \mathrm{Sym}_2$$

Where Sym_k denotes the symmetric group on k elements.

4.3. **Rigid Patterns.** Motivated by the examples where the Whitehead graph is the complete graph, we make the following definition:

Definition 4.5. A line pattern $\mathcal{L}$ in F *rigid* if $\mathcal{QI}(F, \mathcal{L})$ is conjugate into an isometry group in the following sense: There is a tree T with $\phi\colon F \to T$ a quasi-isometry so that

$$\phi\, \mathcal{QI}(F, \mathcal{L})\phi^{-1} \subset \mathrm{Isom}(T, \phi(\mathcal{L})) \subset \mathcal{QI}(T)$$

Proposition 4.6. *If $\mathcal{L}$ is a rigid pattern we can present $\mathcal{QI}(F, \mathcal{L}) \cong \mathrm{Isom}(T, \phi(\mathcal{L}))$ as a graph of groups by considering the isometric pattern preserving action on T. Moreover, since there are finitely many lines through any edge or vertex, $\mathcal{QI}(F, \mathcal{L})$ is virtually free.*

Remark. We saw in the case when a Whitehead graph is complete that the free group is finite index in $\mathrm{Isom}(T\mathcal{L})$, but this is not true in general for rigid patterns.

Proposition 4.7. *A line pattern $\mathcal{L}$ in F is not rigid if $D_{\mathcal{L}}$ has cut points or cut pairs of ordinary points.*

Proof. Let S be either a cut point or a cut pair of ordinary points in $D_{\mathcal{L}}$. Let H be the convex hull of $q^{-1}(S)$. H is a line in the Cayley graph T of F. (It is a line of $\mathcal{L}$ if and only if S was a cut point.)

By assumption $Wh(H, \mathcal{L})$ is not connected. There are finitely many components $C_1, \dots, C_k$. We can define a pattern preserving quasi-isometry from $T \to T$ that is the identity map on each C_i but shears the C_i relative to each other along H by any amount we want. □

Conjecture 1. *A line pattern $\mathcal{L}$ in F is rigid if and only if $D_{\mathcal{L}}$ is connected with no cut points and no cut pairs of ordinary points.*

We can prove the conjecture if we add some hypothesis:

Theorem 4.8. *A line pattern $\mathcal{L}$ in F is rigid if $Wh(\mathcal{L})$ is $(2, 0)$ and $(1, b)$-connected and if all the vertices of $Wh(\mathcal{L})$ have valence less than or equal to $2b + 1$.*

Proof. By Proposition 3.12, $D_{\mathcal{L}}$ has no cut sets of size less than $b + 2$, and any cut sets of size less than or equal to $2b + 1$ are equivalent under the group action to a cut set that is visible in $Wh(\mathcal{L})$.

Consider the largest collection of minimal cut sets of $D_{\mathcal{L}}$ of size between $b + 2$ and $2b + 1$ that is equivariant with respect to the action of F and also pairwise mutually non-crossing.

The cut sets corresponding to the edges of $\mathcal{C}_{\mathcal{B}}(F)$ belong to this collection. Take the tree T dual tree to this collection of cut sets. If the edge cut sets are all the sets of the collection then T is just the Cayley graph of F, but there may be more. In general there is an equivariant quasi-isometry from $\mathcal{C}_{\mathcal{B}}(F)$ to T that blows up each vertex of $\mathcal{C}_{\mathcal{B}}(F)$ to a finite subtree.

One can see this subtree as the dual tree to the collection of pairwise mutually non-crossing cut sets of size between $b + 2$ and $2b + 1$ in $Wh(\mathcal{L})$. □

5. Applications

[*This section still to be written.*]

5.1. **Graphs of Free Groups with Cyclic Edge Groups.**

5.2. **Graphs of Free $\times\ \mathbb{Z}$ Groups with Cyclic Edge Groups.** See [2].

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