Quasi-isometries Among Tubular Groups

BY

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THESIS

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To Jodie, Connor, and baby girl.

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SUMMARY

A *tubular group* is a graph of groups consisting of vertex groups quasi-isometric to $\mathbb{Z}^2$ or $\mathbb{Z}$, amalgamated along edge groups quasi-isometric to $\mathbb{Z}$. The geometric models for these groups contain certain subspaces, called *P-set spaces*, which are invariant under quasi-isometries. P-set spaces are internally flexible, but their interconnections can be rigid. A *tree of P-sets* is built to model the decomposition of the geometric model space into P-set spaces. Two tubular groups are quasi-isometric if and only if there is an allowable isomorphism between their trees of P-sets. There is an algorithm that determines whether or not there is such an isomorphism.

CHAPTER 1

INTRODUCTION

1.1 Background

One line of attack in Gromov's program to classify finitely generated groups up to quasi-isometry has been to study splittings of groups. Given a splitting of a group G into a graph of groups, one would like to know whether there is a similar splitting for a finitely generated group H quasi-isometric to G, and what constraints such splittings impose upon quasi-isometries between the groups.

A result of this type applies when G and H are accessible groups. Paposoglu and Whyte showed that G and H are quasi-isometric if and only if they have the same number of ends and if, in terminal splittings over finite subgroups, they have the same sets of quasi-isometry classes of one ended vertex groups [9].

Mosher, Sageev, and Whyte prove splitting rigidity results for graphs of coarse Poincaré duality groups under some additional hypotheses [7], [8]. These results apply, in particular, to graphs of groups with $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups. Under appropriate hypothesis, such splittings are quasi-isometrically rigid. Furthermore, the patterns of attachment of edge groups to vertex groups give quasi-isometry invariants of the groups. In contrast to the Paposoglu-Whyte results, these quasi-isometry invariants do not fully determine the quasi-isometry class of a group. In fact, among groups with the same sets of edge patterns there may be infinitely many different quasi-isometry classes.

In this paper we show that there is an algorithm to determine whether or not two groups which split as graphs of $\mathbb{Z}^2$ vertex groups with $\mathbb{Z}$ edge groups are quasi-isometric.

Such a group can be thought of as the fundamental group of a finite 2-complex consisting of a disjoint union of tori glued together by annuli. Martin Bridson has described such groups as “tubular”.

We will define a class of *tubular groups* which incorporates these key examples, is closed under quasi-isometry, and rules out some degenerate cases. Examples of tubular groups have occurred in different contexts.

Right-Angled Artin groups whose defining graphs are trees of diameter at least three are tubular groups. Such groups are also graph-manifold groups, and were shown to be quasi-isometric to each other by Behrstock and Neumann as a special case of their quasi-isometry classification of graph-manifold groups [1].

Examples of tubular groups were used in work of Brady and Bridson [2], as well as Brady, Bridson, Forester, and Shankar [3], to show that the isoperimetric spectrum is dense in $[2, \infty)$ and that $\mathbb{Q} \cap [2, \infty) \subset IP$, respectively. In the latter work these examples were termed “snowflake groups”.

The fundamental groups of the Torus Complexes of Croke and Kleiner [6] are also examples of tubular groups. These Torus Complexes provided examples of finite, non-positively curved, homeomorphic 2-complexes whose universal covers have non-homeomorphic ideal boundaries.

1.2 Quasi-isometries Among Tubular Groups

Let Γ be a graph of groups whose fundamental group, $G = \pi_1(\Gamma)$, is a tubular group. The geometric model for G is a tree of spaces X with a G-equivariant projection to the Bass-Serre tree, $D\mathit{\Gamma}$, of Γ.

The preimage of a vertex of the tree is called a *vertex space*, and the preimage of the midpoint of an edge is called an *edge space*. The vertex spaces are Euclidean planes or lines, and the edge spaces are lines. The preimage of an entire edge is topologically an infinite strip $\mathbb{R} \times [0, 1]$ whose two boundary components glue to vertex spaces along lines. Metrically these edge strips are horostrips in non-positively curved planes. In a given vertex space, the lines to which incident edges attach generate an affine pattern.

When these patterns are complicated enough, the results of Mosher, Sageev, and Whyte apply and show that a quasi-isometry of G must preserve edge patterns, so must restrict to an affine homothety on the vertex spaces [8]. However, the Bass-Serre tree is not quasi-isometrically rigid; a quasi-isometry between tubular groups need not induce an isomorphism between their Bass-Serre trees. This fact provides flexibility, but there is still a geometric relation that creates some rigidity in the Bass-Serre trees. We say that vertices of $D\mathit{\Gamma}$ *satisfy Relation P* if their corresponding vertex spaces are close on unbounded sets. Section 3.4 explores this relation, and defines *P-sets*, which are maximal subsets of $D\mathit{\Gamma}$ such that any two members satisfy Relation P. P-sets are invariant under quasi-isometry, and they intersect in at most a single vertex. This implies that a quasi-isometry of a tubular group must respect the decomposition of its Bass-Serre tree into P-sets.

We define the *tree of P-sets*, to model the decomposition of the Bass-Serre tree of a tubular group into P-sets. A quasi-isometry between tubular groups induces an isomorphism between their trees of P-sets.

We also define a notion of *height change* between vertices of $D\Gamma$, and extend this to the tree of P-sets. Height change has been used before in studying quasi-isometries. Whyte showed, in Lemma 4.1 of [13], that a quasi-isometry between Baumslag-Solitar groups induces a coarsely height preserving quasi-isometry between their Bass-Serre trees. Lemma 1, which is an adaptation of Whyte's argument, shows that height change must be coarsely preserved on *rigid components* of the tree of P-sets. A tree isomorphism which respects the affine patterns of edge inclusions in vertex spaces, and which is coarsely height preserving on rigid components is called *allowable*.

The main results of this paper are as follows:

For $i = 1, 2$, let $G_i = \pi_1(\Gamma_i)$ be a tubular group and T_i its tree of P-sets.

Theorem 1. *There is an algorithm which in finite time decides whether or not there is an allowable isomorphism* $T_1 \to T_2$.

Theorem 2. *There is an allowable isomorphism* $\psi : T_1 \to T_2$ *if and only if there is a quasi-isometry* $\Phi : G_1 \to G_2$.

We will construct allowable tree isomorphisms and quasi-isometries of tubular groups by starting from some chosen basepoint and extending the map inductively to larger neighborhoods. The invariants provided by Mosher, Sageev, and Whyte provide local obstacles to this process of building out. There is also the large scale restriction that the maps be coarsely height preserving.

Informally, the proof of Theorem 1 is based on the idea that a P-set vertex in the tree of P-sets knows nothing of its past except for the height change back to some chosen basepoint. Thus, if we are building a tree isomorphism, ψ, and wish to extend ψ to a neighborhood of a P-set vertex, R, we need two strategies for R, one for when the height of R is higher than expected, and one for when it is lower than expected. A strategy for R will contain information for extending ψ to a neighborhood of R in a way that looks locally allowable.

The algorithm attempts to build a *set of strategies* in such a way that a locally allowable isomorphism can be built according to the strategies in the set. From such a set of strategies a system of inequalities is derived. This system will have solutions, and the set of strategies will be called *consistent*, if and only if the set of strategies can be used to build an isomorphism which is coarsely height preserving on rigid components of the tree of P-sets.

An isomorphism which is coarsely height preserving on rigid components and locally allowable is allowable, so if the set of strategies is consistent, the algorithm is done. If the set is inconsistent then the algorithm backtracks and tries to find a different set of strategies. The number of possible sets of strategies is bounded, so the algorithm will either find a consistent set or will exhaust the possibilities.

It is possible to derive a consistent set of strategies from an allowable isomorphism, so if an allowable isomorphism exists, the algorithm will be successful.

Theorem 2 proves the converse of Lemma 1, G_1 and G_2 are quasi-isometric if there exists an allowable isomorphism $T_{G_1} \to T_{G_2}$.

P-sets in $D\Gamma$ are infinite, but they get collapsed in to a point in T_G. Thus, there is no guarantee that a given allowable isomorphism of trees of P-sets can be extended to a quasi-isometry between model spaces.

Instead, we use Theorem 1 to produce a set of strategies for building a new tree isomorphism, possibly different from the one we started with. Success of the algorithm is guaranteed, since we know there exists an allowable tree isomorphism.

We use the set of strategies to build a quasi-isometry between the groups P-set by P-set. This is accomplished via Lemmas 2 and 3, which essentially say that the preimages of two P-sets in the geometric models are quasi-isometric if and only if the P-sets are either both of bounded height change or both of unbounded height change, and that the quasi-isometry constants can be bounded in terms of height. Since we are using a consistent set of strategies, the height errors, and hence the quasi-isometry constants, are uniformly bounded. We then piece together these quasi-isometry of P-set spaces to get a quasi-isometry of the groups.

In Section 5.1 we define *coarse slope* of a geodesic ray in a rigid component of T_G and show that there is a maximum such slope in each rigid component with no P-sets of unbounded height change. These maximum slopes are quasi-isometry invariants of G. However, Example 5 gives an example of a family of infinitely many tubular groups with the same sets of affine edge patterns, no P-sets of unbounded height change, one rigid component each, and all having the same maximum slope, such that no two are quasi-isometric.

CHAPTER 2

PRELIMINARIES

This chapter contains standard definitions and constructions, which could also be found in [8] or [5]. For the most part, notation will follow that of [8].

2.1 Coarse Geometry

Let (X, d_X) and (Y, d_Y) be metric spaces. For $A \subset X$, $N_r(A)$ is the closed r-neighborhood of A in X. If B is also a subset of X, A is *coarsely contained in* B, $A \overset{c}{\subset} B$, if $\exists r \geq 0$ such that $A \subset N_r(B)$. Subsets A and B are coarsely equivalent, $A \overset{c}{=} B$, if $A \overset{c}{\subset} B$ and $B \overset{c}{\subset} A$.

A subspace A is *coarsely dense* in X if X is coarsely contained in A.

A subspace C of X is a *coarse intersection* of A and B, $C = A \overset{c}{\cap} B$, if, for sufficiently large r, $C \overset{c}{=} N_r(A) \cap N_r(B)$.

A map $f : X \to Y$ is K-*bilipschitz*, for $K \geq 1$, if

$$\frac{1}{K} d_X(x, y) \leq d_Y(f(x), f(y)) \leq K d_X(x, y)$$

for all $x, y \in X$. The map f is a (K, C)-*quasi-isometric embedding*, for $K \geq 1$, $C \geq 0$, if

$$\frac{1}{K} d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq K d_x(x, y) + C$$

for all $x, y \in X$. Furthermore, f is a (K, C)-*quasi-isometry* if it is a (K, C)-quasi-isometric embedding and the image is C-coarsely dense in Y.

Two maps $f, g : X \to Y$ are *bounded distance* from each other, $f \overset{c}{=} g$, if there is a $C \geq 0$ such that, $\forall x \in X$, $d_Y(f(x), g(x)) \leq C$. Two maps, $f : X \to Y$ and $\bar{f} : Y \to X$, are *coarse inverses* if $f \circ \bar{f} \overset{c}{=} Id_Y$ and $\bar{f} \circ f \overset{c}{=} Id_X$. If f is a quasi-isometry, there is a coarse inverse, $\bar{f} : Y \to X$, of f which is also a quasi-isometry, with constants depending on those of f.

2.2 Bass-Serre Theory

If Γ is a graph, let $\mathcal{V}\mathit{\Gamma}$ denote the vertex set, and $\mathcal{E}\mathit{\Gamma}$ the edge set. Let $\mathcal{V}\mathcal{E}\mathit{\Gamma} = \mathcal{V}\mathit{\Gamma} \cup \mathcal{E}\mathit{\Gamma}$. The set of ends of edges of Γ is $\mathcal{E}\mathit{\Gamma} \times \{0, 1\}$.

Each edge has two ends, of the form $\eta = (e, i)$, and each is identified with some vertex $v(\eta)$ such that $e = e(\eta)$ is incident to $v(\eta)$. The second coordinate of an end can be taken mod 2, so that (e, i) and $(e, i + 1)$ are the two ends of e, regardless of the value of i.

A *graph of groups*, $(\Gamma, \{G_\gamma\}, \{\phi_\eta\})$, is a graph, Γ, equipped with a *local group* G_γ for each $\gamma \in \mathcal{V}\mathcal{E}\mathit{\Gamma}$, and *edge injections* $\phi_\eta \in Hom(G_{e(\eta)}, G_{v(\eta)})$ for each end η. We will generally use Γ to denote the graph of groups, and refer to the *underlying graph of* Γ if we wish to consider only the graph itself.

Note. A graph of groups is *of finite type* if the underlying graph is finite, the vertex groups are finitely presented, and the edge groups are finitely generated. All the graphs of groups of interest in this paper are of finite type, so, from this point forward, finite type can be taken as an implicit hypothesis for any statement about graphs of groups.

Associated to a graph of groups there is a finitely presented group, $G = \pi_1((\Gamma, \{G_\gamma\}, \{\phi_\eta\}))$, the *fundamental group of the graph of groups.*[11]

A graph of groups is *reducible* if there is an edge e such that the vertices $v(e,0)$ and $v(e,1)$ are distinct, and such that one of the edge homomorphisms $\phi_{(e,i)}$ is surjective. In this case it is possible to simplify the graph of groups without changing the fundamental group. Remove e and $v(e,i)$ from Γ, and for any other end with $v(e',j) = v(e,i)$, replace $\phi_{(e',j)}$ with

$$\phi_{(e,i+1)} \circ \phi^{-1}_{(e,i)} \circ \phi_{(e',j)}$$

Scott and Wall gave a topological realization of G [10]. Build a *graph of spaces*, $\mathcal{K}$, for $(\Gamma, \{G_\gamma\}, \{\phi_\eta\})$ by choosing *local spaces*, $\mathcal{K}_\gamma$, for each $\gamma \in \mathcal{VE}\Gamma$. For each γ, choose $\mathcal{K}_\gamma$ to be a pointed, connected, compact CW-complex, with a map $\pi_1(\mathcal{K}_\gamma) \to G_\gamma$. This map should be an isomorphism if $\gamma \in \mathcal{V}\Gamma$, and an epimorphism if $\gamma \in \mathcal{E}\Gamma$. For each end η of Γ, choose an *edge map*, a pointed CW-map $f_\eta : \mathcal{K}_{e(\eta)} \to \mathcal{K}_{v(\eta)}$, such that the induced map on fundamental groups agrees with the edge injection.

Now, let $\mathcal{K}$ be the finite CW-complex obtained from the disjoint union

$$\coprod_{v \in \mathcal{V}\Gamma} \mathcal{K}_v \amalg \coprod_{e \in \mathcal{E}\Gamma} \mathcal{K}_e \times [0,1]$$

by using $f_{(e,i)}$ to glue $\mathcal{K}_{e \times \{i\}}$ to $\mathcal{K}_{v(e,i)}$, for each end (e,i). The fundamental group $\pi_1(\mathcal{K})$ is well defined, up to isomorphism, and, by van Kampen's Theorem, is isomorphic to G.

Consider the universal cover $X = \tilde{\mathcal{K}}$, with covering map p and metric lifted from $\mathcal{K}$. The group G acts properly discontinuously and cocompactly by isometries by deck transformations on X, so they are quasi-isometric by the Švarc-Milnor Lemma [5]. Thus, X serves as a *geometric model* for G. For questions of the coarse geometry of G, it is sufficient to study X.

The space X can be decomposed into path connected lifts of the local spaces $\mathcal{K}_\gamma$, and the action of G on X respects this decomposition. Let $D\mathit{\Gamma} = q(X)$ be the quotient space of the decomposition. The quotient is a tree on which G acts without edge inversion, and $D\mathit{\Gamma}$ is G-equivariantly isomorphic to the Bass-Serre Tree of Γ, which is also known as the development of Γ. Call X the *Bass-Serre Complex*, and $X \xrightarrow{q} D\mathit{\Gamma}$ the *Bass-Serre tree of spaces* for Γ.

For $v \in \mathcal{V}D\mathit{\Gamma}$, $X_v = q^{-1}(v)$ is called a *vertex space*, and $\mathcal{V}X = \bigcup_{v \in \mathcal{V}D\mathit{\Gamma}} X_v$ is the set of vertex spaces. The set of vertex spaces is $\frac{1}{2}$-coarsely dense in X. For $e \in \mathcal{E}D\mathit{\Gamma}$, $X_e = q^{-1}(\text{midpoint of } e)$ is called an *edge space*.

For $t \in D\mathit{\Gamma}$, $Stab_G(t) = Stab_G(X_t)$. The group $Stab_G(t)$ is conjugate in G to $G_{p(t)}$, and $Stab_G(t)$ acts on X_t as the deck transformation group of the covering map $X_t \to \mathcal{K}_{p(t)}$.

CHAPTER 3

TUBULAR GROUPS

3.1 Definitions and Rigidity Results

Consider the class, $\mathcal{T}$, of finite graphs of groups with edge groups $\mathbb{Z}$ and vertex groups $\mathbb{Z}^2$. It is not true that a finitely generated group quasi-isometric to a group in $\mathcal{T}$ must also be in $\mathcal{T}$. We give a different definition for a class of *tubular groups* which includes much of $\mathcal{T}$ and is also closed under quasi-isometry.

Definition 1. A *tubular group* is the fundamental group of a finite, connected graph of groups satisfying the following conditions:

1. Every edge group is finitely generated and quasi-isometric to $\mathbb{Z}$.
2. Every vertex group is finitely generated and quasi-isometric to either $\mathbb{Z}$ or $\mathbb{Z}^2$.
3. There is at least one vertex group quasi-isometric to $\mathbb{Z}^2$.
4. For every vertex quasi-isometric to $\mathbb{Z}^2$, the "crossing graph condition" is satisfied.

Remarks.

1. Graphs of groups with all local groups quasi-isometric to $\mathbb{Z}$ were classified by Whyte [13]. Condition (3) excludes these from consideration.

2. Condition (4) is satisfied if every vertex of the Bass-Serre tree whose stabilizer is quasi-isometric to $\mathbb{Z}^2$ has incident edges whose edge spaces are not coarsely contained in one another. This will be discussed further in Section 3.1.1.

3. Suppose $v \in \mathcal{V}\Gamma$ with G_v quasi-isometric to $\mathbb{Z}$. Suppose $e \in \mathcal{E}\Gamma$ such that $v(e,i) = v$ and $v(e, i+1) \neq v$. We can assume that $\phi_{(e,i)}(G_e)$ is a subgroup of G_v of index at least two. Otherwise the graph of groups is reducible, and we could simplify it without changing the fundamental group.

3.1.1 Rigidity Results of Mosher-Sageev-Whyte

Mosher, Sageev, and Whyte provided tools for studying quasi-isometries of graphs of groups [8]. In this section we recall some of their rigidity results and apply them to tubular groups.

3.1.1.1 Quasi-isometric Rigidity

In a graph of groups, a *depth zero* vertex group is one which is not strictly coarsely contained in any other vertex group. Let $\mathcal{V}_0 D\Gamma$ be the set of depth zero vertices of $D\Gamma$. Let $\mathcal{V}_0 X = \bigcup_{v \in \mathcal{V}_0 D\Gamma} X_v$.

In a tubular group, those are precisely the vertex groups quasi-isometric to $\mathbb{Z}^2$.

Given a graph of groups, G, with Bass-Serre tree of spaces $X \to T$, the following hypothesis will be needed:

1. G is finite type, irreducible, and finite depth.

2. No depth zero raft of the Bass-Serre tree T is a line.

3. Each depth zero vertex group is coarse PD.

4. The crossing graph condition holds for each depth zero vertex of T which is a raft.

5. Each vertex and edge group of G is coarse finite type.

Proposition 1. *Tubular groups satisfy these hypothesis.*

Proof. The depth zero vertices are all rafts, and these are the only depth zero rafts. Condition (4) from the definition of tubular group ensures that the crossing graph condition is satisfied. Virtually abelian groups are coarse PD and coarse finite type. □

Theorem (Quasi-isometric Rigidity Theorem, Theorem 1.5 of [8])**.** *Let G be a graph of groups satisfying (1)-(5) above. If H is a finitely generated group quasi-isometric to $\pi_1 G$ then H is the fundamental group of a graph of groups satisfying (1)-(5).*

Theorem (Quasi-isometric Classification Theorem, Theorem 1.6 of [8])**.** *Let G, G' be graphs of groups satisfying (1)-(5) above. Let $X \to T$, $X' \to T'$ be Bass-Serre tree of spaces for G, G', respectively. If $f : X \to X'$ is a quasi-isometry then f coarsely respects vertex and edge spaces. To be precise, for any $K \geq 1$, $C \geq 0$ there exists a K', C' quasi-isometry $f_\# : \mathcal{VE}(T) \to \mathcal{VE}(T')$ such that the following hold:*

- *If $a \in \mathcal{VE}(T)$ then $d_{\mathcal{H}}(f(X_a), X'_{f_\#(a)}) \leq C'$*

- *If $a' \in \mathcal{VE}(T')$ then there exists $a \in \mathcal{VE}(T)$ such that $d_{\mathcal{H}}(f(X_a), X'_{a'}) \leq C'$*

Corollary 1. *The class of tubular groups is closed under quasi-isometry. That is, any finitely generated group quasi-isometric to a tubular group is itself a tubular group. Furthermore, any quasi-isometry between tubular groups coarsely respects vertex and edge spaces.*

Note that in a tubular group, a vertex quasi-isometric to $\mathbb{Z}^2$ can not be bounded Hausdorff distance from any other vertex space. Thus, we can change a quasi-isometry by a bounded amount so that it actually respects such vertex spaces.

3.1.1.2 Affine Patterns

If $V \subset \mathbb{R}^n$ is a linear subspace, let P_V be the set of affine subspaces of $\mathbb{R}^n$ parallel to V. For a finite collection, F, of linear subspaces, the *affine pattern* induced by F, P_F, is the union of the P_V for $V \in F$. An affine pattern is *rigid* if for every K, C, R there is an R' such that if $f : \mathbb{R}^2 \to \mathbb{R}^n$ is a K, C quasi-isometry which R-coarsely respects each P_V, then f is within R' of an affine homothety.

Lemma (Lemma 7.12 of [8])**.** *If F is a finite collection of linear subspaces of $\mathbb{R}^n$ which contains $n+1$ hyperplanes in general position, then F is rigid.*

In particular, a collection of at least three distinct lines in the plane is rigid.

Corollary (Corollary 7.11 of [8])**.** *Let Γ be a graph of groups with all vertex and edge groups finitely generated abelian groups. Assume that for each depth zero, one vertex raft v in the Bass-Serre tree, the collection of edge spaces at the vertex space of v is a rigid affine pattern. Assume also that there are no line-like rafts of depth zero. If H is any finitely generated group quasi-isometric to $G = \pi_1(\Gamma)$, then H splits as a graph of virtually abelian groups and the quasi-isometry $G \to H$ is affine along each depth zero, one vertex raft. Moreover, the set of affine equivalence classes of edge patterns is the same for H as it is for G.*

3.2 Geometric Models for Tubular Groups

3.2.1 Affine Patterns of Lines in the Plane

Let $F = \{l_1, l_2, .., l_n\}$, $n \geq 3$, be a finite collection of distinct lines through the origin in $\mathbb{R}^2$, with the usual Euclidean metric. Two such collections, F and F', are *linearly equivalent* if there exists $A \in \mathrm{GL}_2\,\mathbb{R}$ such that $AF = \{Al_i\} = F'$. Scalar matrices in $\mathrm{GL}_2\,\mathbb{R}$ will preserve any such F, so projectivise and consider the slopes of the lines instead.

Let $PF = \{m_1, m_2, .., m_n\}$ be a collection of slopes in $\mathbb{RP}$. There is a finite subgroup, $L_{PF} \subset \mathrm{PGL}_2\,\mathbb{R}$, which fixes PF set-wise. Fixing an ordering on the elements of PF gives an isomorphism between this subgroup of $\mathrm{PGL}_2\,\mathbb{R}$ and a subgroup of S_n, the symmetric group on n elements.

Let $L_F = L_{PF} Z(\mathrm{GL}_2\,\mathbb{R}) \cap \{A \in \mathrm{GL}_2\,\mathbb{R} \mid \det(A) = \pm 1\}$. This is a group of linear self equivalences of F which double covers L_{PF}. The two elements in the preimage of an element of L_{PF} differ by $\pm Id$, which is a trivial difference as far as actions on lines are concerned. We call L_{PF} the *group of symmetries* of F, and call F *symmetric* if L_F acts by isometries.

Proposition 2. *For any collection F of $n \geq 3$ distinct lines through the origin, there is a symmetric representative of the linear equivalence class of F. Choosing a symmetric representative is equivalent to choosing a Euclidean metric on $\mathbb{R}^2$ for which F is symmetric.*

Proof. Define a new metric on the plane by

$$\langle x, y\rangle_{L_F} = \frac{1}{|L_F|} \sum_{A \in L_F} \langle Ax, Ay\rangle$$

The group L_F acts isometrically on the plane with this metric. There is an $A_F \in \mathrm{GL}_2 \mathbb{R}$ such that $\langle x, y \rangle_{L_F} = \langle A_F x, A_F y \rangle$, so $A_F F$ is a symmetric representative of the linear equivalence class of F.

Conversely, suppose F' is a symmetric representative for the linear equivalence class of F. Suppose A_1 and A_2 are matrices such that $|\det(A_i)| = \pm 1$ and $A_i F = F'$. Then $A_1 A_2^{-1} \in L_{F'}$. Since F' is symmetric, this means the A_i differ by an isometry, so

$$\langle A_1 x, A_1 y \rangle = \langle A_2 x, A_2 y \rangle \qquad \square$$

For $n = 3$ there is a single linear equivalence class. There is, up to isometry, a unique symmetric representative, consisting of three lines meeting at angles $\frac{\pi}{3}$. The group of symmetries is isomorphic to S_3.

For $n = 4$ there are infinitely many linear equivalence classes, indexed by the cross ratios $[-1, 0)$. Each has, up to isometry, a unique symmetric representative, and a transitive group of symmetries. The symmetric representative consists of two pairs of orthogonal lines, offset by an angle depending on the cross ratio. When the cross ratio is -1 this angle is $\frac{\pi}{4}$ and the group of symmetries is isomorphic to the dihedral group of order 8. Otherwise, the group of symmetries is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For $n = 5$ there is no longer a unique symmetric representative for every linear equivalence class. Indeed, there are five-line patterns with trivial group of symmetries. In such a class, every member is symmetric, so there is no canonical choice of metric.

For $n = 2$ we will consider two orthogonal lines to be a symmetric pattern. The group of symmetries in this case is not a finite group; it is isomorphic to $(\mathbb{R}^+, \times) \rtimes \mathbb{Z}/2\mathbb{Z}$.

3.2.2 Coarse Bass-Serre Complex

In Section 2.2, we saw that the Bass-Serre complex provides a geometric model for a graph of groups. In fact, it is possible to relax some of the group theoretic restrictions and still get a geometric model quasi-isometric to $G = \pi_1\Gamma$. Following the proof of Lemma 2.9 of [8], we construct a *coarse Bass-Serre complex* $Y \to D\Gamma$ to serve as a geometric model for G.

Let Γ be a graph of groups for a tubular group $G = \pi_1\Gamma$.

Let v be a depth zero vertex of Γ, and e an edge of Γ incident to v at end η.

Suppose A is a group quasi-isometric to $\mathbb{Z}^n$. The group A has a finite index normal subgroup $B \cong \mathbb{Z}^n$ [4]. Since B has finite index, it is coarsely dense in A, so there is a quasi-isometry, $f_B : A \to B$, at bounded distance from Id_A, with $f_B|_B = Id_B$.

Suppose $\langle z \rangle < A$ is an infinite cyclic subgroup. Since A/B is a finite group, some power of z is in B. Therefore, $f_B(\langle z \rangle)$ is bounded distance from a cyclic subgroup of B.

Applying this reasoning to G_v and G_e,

$$\begin{array}{ccc} G_e & \xrightarrow[qi]{f_{\mathbb{Z}}} & \mathbb{Z} \\ {\scriptstyle \phi_\eta}\downarrow & & \downarrow \\ G_v & \xrightarrow[qi]{f_{\mathbb{Z}^2}} & \mathbb{Z}^2 \end{array}$$

the image of the map $f_{\mathbb{Z}^2} \circ \phi_\eta \circ \bar{f}_{\mathbb{Z}}$ is bounded distance from a cyclic subgroup

$$\left\langle x^a y^b \right\rangle < \langle x, y \rangle \cong \mathbb{Z}^2$$

The usual inclusion of $\mathbb{Z}^2$ into $\mathbb{R}^2$ is a quasi-isometry, and the subgroup $\langle x^a y^b \rangle$ includes into the line through the origin with rational slope $\frac{b}{a}$. In this way, each edge incident to v is associated to a line in $\mathbb{R}^2$. Let F_v be the set of distinct lines. The affine pattern induced by this set is called the *affine pattern of edge inclusions*, and is well defined up to linear equivalence. If F_v contains n distinct lines, we will say that v *has n lines* or *is an n-line vertex.* Furthermore, we can choose a new Euclidean metric on $\mathbb{R}^2$ to make this pattern symmetric.

Let $X \xrightarrow{q} D\mathit{\Gamma}$ be a Bass-Serre tree of spaces for Γ. For each $\gamma \in \mathcal{VE}D\mathit{\Gamma}$, let $h_\gamma : X_\gamma \to Y_\gamma$, where Y_γ is either $\mathbb{R}$ or $\mathbb{R}^2$ and h_γ is the quasi-isometry given by restricting to a finite index normal abelian subgroup and then including into Y_γ.

For each vertex $v \in \mathcal{V}D\mathit{\Gamma}$, and each incident edge $e \in \mathcal{E}D\mathit{\Gamma}$, let $F_{ev} : X_e \to X_v$ be the attaching map. Define a new attaching map by $F'_{ev} = f_v \circ F_{ev} \circ \bar{f}_e : Y_e \to Y_v$.

Define Y by taking the disjoint union

$$\coprod_{v \in \mathcal{V}D\mathit{\Gamma}} Y_v \amalg \coprod_{e \in \mathcal{E}D\mathit{\Gamma}} Y_e \times [0,1]$$

and gluing according to the attaching maps.

The h_γ have uniform quasi-isometry constants, since the underlying graph was finite, so they piece together to give a quasi-isometry $h : X \to Y$. Since G was quasi-isometric to X, we now have that G is quasi-isometric to Y. The action of G on X is quasi-conjugated by h to give a proper, cobounded quasi-action of G on Y. The space Y is still a tree of spaces over $D\Gamma$, and $Y \to D\Gamma$ is called a coarse Bass-Serre complex.

When v is a depth zero vertex, the edge spaces of the incident edges attach to Y_v along lines of the affine pattern in Y_v, within bounded distance.

3.2.3 Contraction Factors and Height Change

Let e be an edge of $D\Gamma$, and let v_i be the vertex at end (e, i) of e, for $i = 0, 1$. For each i, F'_{ev_i} maps $Y_e = \mathbb{R}$ within bounded distance of a line in Y_{v_i}, and there is a factor l_i such that $d_{Y_{v_i}}(F'_{ev_i}(x), F'_{ev_i}(y)) = l_i d_{Y_e}(x, y)$ to within bounded additive error. Define the *contraction factor across* e to be $\frac{l_1}{l_0}$ and the *height change across* e to be $h(e) = -\ln(\frac{l_1}{l_0})$.

Metrize the strip $Y_e \times [0, 1]$ as the strip $0 \leq y \leq 1$ in the plane with metric $(\frac{l_1}{l_0})^{2y} dx^2 + dy^2$, so the edge strips are horostrips in a plane of constant curvature $-(\ln \frac{1_1}{l_0})^2$. Without changing the quasi-isometry type, we can change Y by a bounded amount so that F'_{ev_i} glues $Y_e \times \{i\}$ isometrically along a line in Y_{v_i}.

Let p be vertical projection from the bottom $(i = 0)$ of the strip to the top $(i = 1)$ of the strip. The map p is closest point projection. If a and b are points in the bottom edge of the strip, and d_i is the distance in $Y_e \times \{i\}$, then

$$\frac{d_1(p(a), p(b))}{d_0(a, b)} = \frac{l_2}{l_1} = \exp(-h(e))$$

For vertices $v, w \in D\mathit{\Gamma}$, the *height change from* v *to* w, $h(v,w)$, is the sum of the height changes across the edges of the geodesic between v and w. This quantity will sometimes be called the *height of* w *relative to* v.

If a base vertex v_0 has been specified, then the *height*, $h(v)$, of a vertex v is $h(v_0, v)$. The height function can be extended to all of $D\mathit{\Gamma}$ by extending linearly across edges.

3.2.4 Geometric Models for Two Tubular Groups

The coarse Bass-Serre complex, Y, will serve as the geometric model for its tubular group. We will have no further use for the original Bass-Serre complex.

From this point forward, given two tubular groups, $G_i = \pi_1(\Gamma_i)$, for $i = 1, 2$, X will always refer to the geometric model for G_1, and Y will always refer to the geometric model for G_2.

There were choices involved in metrizing the geometric models. For each depth zero vertex in the underlying graph we chose a symmetric representative for the equivalence class of affine pattern induced by the edge inclusions. It will be convenient to assume that these choices were consistent. That is, if vertex spaces of X and Y have equivalent affine patterns of edge inclusions, assume that we have chosen the same symmetric representative for each of them.

3.3 Examples of Tubular Groups

Let F be the set of lines through the origin of distinct slopes 0, $\frac{s}{r}$, and $\frac{u}{t}$. Let

$$A' = \begin{pmatrix} 1 & -\frac{1}{2}\frac{ru+st}{su} \\ 0 & \frac{\sqrt{3}}{2}\frac{ru-st}{su} \end{pmatrix}$$

Let $\alpha = \frac{1}{\sqrt{|\det(A')|}}$. The matrix $A = \alpha A'$ makes the three-line pattern generated by F symmetric.

Remark. In the following examples there is an extra decoration included on each diagram of a graph of groups. A small arc is drawn between incident edges at a vertex whose edge groups are coarsely contained in one another. This extra decoration does not give new information, it is simply a visual cue that highlights an important relation between the edges.

If no edge group is specified it is assumed to be $\mathbb{Z}$. Arrows pointing to elements of the vertex groups indicate the image of the generator of the edge group.

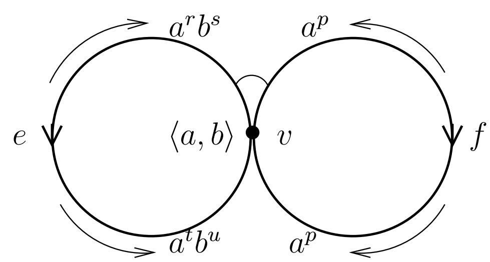

Figure 1. One torus group with contraction on one edge.

Example 1 (Contraction on One Loop)**.** Let Γ be the graph of groups in Figure 1.

$$G = \pi_1\Gamma = \left\langle a, b, x, y \mid [a, b] = 1, xa^p x^{-1} = a^p, ya^r b^s y^{-1} = a^t b^u \right\rangle$$

This is the fundamental group of one torus with two annuli glued on. One annulus is flat; the contraction factor is 1. The contraction factor across the other edge is

$$\lambda = \frac{|A\binom{r}{s}|}{|A\binom{t}{u}|} = \left|\frac{s}{u}\right|$$

Given any positive rational λ and indices m, n, and p, we can find a group realizing λ as the contraction factor with the specified indices for the edge inclusions.

If $\beta \in \mathbb{Q}$, let $N(\beta)$ be the numerator of β in reduced form, and let $D(\beta)$ be the denominator. Choose some c such that, if n does not divide $D(\lambda)$, then $n|c$ and, if m does not divide $N(\lambda)$, then $m|c$. Let $s = N(\lambda)c$ and let $u = D(\lambda)c$.

Choose any nonzero r' and t' such that $\gcd(r', \frac{s}{m}) = 1$ and $\gcd(t', \frac{u}{n}) = 1$ and $r' \neq \frac{n\lambda}{m}t'$. Let $r = mr'$, and let $t = nt'$.

Then $\gcd(r, s) = m$, $\gcd(t, u) = n$, and $\frac{s}{u} = \lambda$, so the presentation above gives an example of a group with the desired properties. ♦

Figure 2. A one torus group with contraction on both edges.

Example 2 (Contraction on two loops)**.** Let Γ be the graph of groups in Figure 2.

$$G = \pi_1\Gamma = \left\langle a, b, x, y \mid [a, b] = 1,\ xa^p x^{-1} = a^r b^s,\ ya^q y^{-1} = a^t b^u \right\rangle$$

Let $\nu = \left|\frac{ru-st}{su}\right|$. The contraction factor across e is

$$\lambda = \frac{|A\binom{r}{s}|}{|A\binom{p}{0}|} = \left|\frac{s}{p}\right| \nu$$

The contraction factor across f is

$$\mu = \frac{\left|A\binom{t}{u}\right|}{|A\binom{q}{0}|} = \left|\frac{u}{q}\right| \nu$$

A special case is a group $G_{p,r}$ where $p = q > r = t > 0$, $s = 1$, and $u = -1$. For such a group the contraction factors across the two edges are

$$\lambda = \mu = \frac{2r}{p}$$

and $0 < \lambda < 2$.

Brady and Bridson have shown [2] that the group $G_{p,r}$ has Dehn function

$$n^{4-2log_2\lambda}$$

This proved that there are no gaps in the isoperimetric spectrum beyond 2.

Suppose we are given positive contraction factors λ and μ in $\mathbb{Q}$ and the four indices of edge inclusions, p, q, $m = \gcd(r, s)$, and $n = \gcd(t, u)$. There are conditions under which these can be realized in a one torus group.

If these did arise from a tubular group, the matrix A which was used to define the metric on the torus allows us to realize the group G_v as a lattice in $\mathbb{R}^2$. In this lattice, $(1, 0)$, $\frac{\lambda p}{m}(1/2, \sqrt{3}/2)$, and $\frac{\mu q}{n}(-1/2, \sqrt{3}/2)$ must be primitive elements, since these are the images of generators of maximal cyclic subgroups in G. These three elements generate a lattice in which each of them is primitive only if, as reduced rationals, $\frac{\lambda p}{m}$ and $\frac{\mu q}{n}$ have the same numerators and relatively prime denominators.

Conversely, suppose $\frac{\mu q}{n} = \frac{i}{j}$ and $\frac{\lambda p}{m} = \frac{i}{k}$ where i, j, and k are pairwise coprime. Choose any $g \in \mathbb{Z}$ such that

$$g \equiv -\frac{1}{j} \bmod k$$

and such that g and j are coprime.

Choose any h coprime to g, $\frac{1+gj}{k}$, and i.

Let $s = hj$, $u = hk$, $t = gi$, and $r = \frac{i}{k}(1 + gj)$.

Then r and s are coprime, and t and u are coprime.

Consider the group

$$G = \left\langle a, b, x, y \mid [a, b] = 1,\ xa^p x^{-1} = a^{mr} b^{ms},\ ya^q y^{-1} = a^{nt} b^{nu} \right\rangle$$

This group has the desired indices, and the contraction factors are

$$\left|\frac{ms}{p}\frac{mrnu - msnt}{msnu}\right| = \left|\frac{m}{p}\left(r - s\frac{t}{u}\right)\right| = \left|\frac{m}{p}\frac{i}{k}\right| = \lambda$$

and

$$\left|\frac{nu}{q}\frac{mrnu - msnt}{msnu}\right| = \left|\frac{n}{q}\left(\frac{r}{s}u - t\right)\right| = \left|\frac{n}{q}\frac{i}{j}\right| = \mu$$

Thus, it is possible to construct a one torus tubular group with any two positive rationals as the contraction factors, but the choice of contraction factors puts some constraints on the possible indices of the edge inclusions. ♦

Example 3 (Right Angled Artin Groups)**.** Consider a Right Angled Artin Group whose defining graph is a tree of diameter at least three. Construct a tubular graph of groups as follows. Retain each non-leaf vertex of the graph and associate to it a local group $\mathbb{Z}$ generated by the corresponding generator of the Artin Group. Insert a vertex with local group $\mathbb{Z}^2$ at the midpoint of each interior edge, generated by the Artin Group generators associated to the two adjacent vertices.

For a leaf edge, delete the edge and replace it by a loop. Let the generator for the leaf vertex be the stable letter associated to the loop; let the local group of the loop be $\mathbb{Z}$; let both edge to vertex injections map the generator of the local group to the generator of the vertex, thus making the stable letter commute with the vertex generator.

This graph of groups will not be irreducible; reduce it.

Since the diameter of the original graph was at least three, there was at least one interior edge, hence the new graph has at least one vertex with a $\mathbb{Z}^2$ local group and crossing incident edges. Therefore, the result is a tubular graph of groups in which every depth zero vertex has exactly two lines. ♦

3.4 Relation P

No two depth zero vertex spaces are bounded Hausdorff distance from one another, but we can give another characterization of closeness of vertices.

Definition 2. Two elements $x, y \in \mathcal{VED}\varGamma$ *satisfy Relation P* if $\exists C > 0$ such that

$$\{p \in X_x \mid d(p, X_y) \leq C\}$$

is unbounded.

Proposition 3. *Relation P is invariant under quasi-isometry.*

This is clear, since quasi-isometries preserve unboundedness and finiteness of distance.

Relation P is symmetric, but not transitive.

The 'P' in 'Relation P' is for 'parallel'. The next proposition shows that this describes how vertex spaces satisfying Relation P are connected.

Proposition 4. *Suppose there exist elements $\gamma_1, \gamma_2, \gamma_3$ of $\mathcal{VED}\varGamma$ with γ_2 adjacent to both γ_1 and γ_3. That is, either γ_2 is a vertex with γ_1 and γ_3 incident edges, or γ_1 and γ_3 are adjacent vertices joined by the edge γ_2.*

Let $L_i = X_{\gamma_2} \cap N_1(X_{\gamma_i})$. *Then* v_1 *and* v_3 *satisfy Relation* $P \iff L_1$ *and* L_3 *have unbounded coarse intersection.*

Proof. If γ_2 is an edge or a positive depth vertex then $L_i = X_{\gamma_2}$, which is unbounded, so the result is clear.

Suppose γ_2 is a depth zero vertex.

The edge strip $X_{\gamma_i} \times [0,1]$ attaches to X_{γ_2} along some line, and L_i is coarsely equivalent to this line. The intersection $L_1 \overset{c}{\cap} L_3$ is unbounded if and only if these two lines are parallel.

If these lines are parallel lines in X_{v_2}, let c be the distance between them plus one. The edge spaces X_{γ_1} and X_{γ_3} are unbounded and have Hausdorff distance at most c.

If the lines are not parallel then every intersection of bounded neighborhoods of L_1 and L_2 is a bounded neighborhood of the point of intersection of the lines. Every path from X_{γ_1} to X_{γ_3} passes through both L_1 and L_2. This means that any geodesic segment of bounded length, c, between X_{γ_1} and X_{γ_3} passes through a bounded set. Thus, the set of points in X_{γ_1} which is c-close to X_{γ_3} is also c-close to a bounded set, hence is itself bounded. □

Induction then gives the following Corollary:

Corollary 2. *If* $\gamma_1, \ldots, \gamma_k$ *are the consecutive edges and vertices of a geodesic segment in* $D\Gamma$, *then* v_1 *and* v_k *satisfy Relation* $P \iff$ *for any (equivalently, for every)* $1 < i < k$, *the following all hold:*

1. γ_i *and* γ_1 *satisfy Relation* P.

2. γ_i *and* γ_k *satisfy Relation* P.

3. If γ_i is a depth zero vertex, $\left(N_1(X_{\gamma_{i-1}}) \cap X_{\gamma_i}\right) \overset{c}{\cap} \left(N_1(X_{\gamma_{i+1}}) \cap X_{\gamma_i}\right)$ is unbounded.

Proposition 5. *Let x, y, z_1, and z_2 be distinct elements of $\mathcal{VED}\Gamma$ such that, for each i, z_i satisfies Relation P with both x and y. Then any of the four elements satisfies Relation P with any of the others.*

Proof. This follows by applying Corollary 2. Let p be the geodesic in $D\Gamma$ joining x and y. Let w_i be the element of $\mathcal{VED}\Gamma$ on p closest to z_i.

Suppose $z_i = w_i$. Assume the points are ordered x, z_1, z_2, y on p. The points x and z_2 satisfy Relation P, so by Corollary 2, z_1 and z_2 satisfy Relation P. The points y and z_1 also satisfy Relation P, so if z_1 is an edge or positive depth vertex we are done.

If z_1 is a depth zero vertex, let e_1 be the edge of p incident to z_1 between z_1 and x. Let e_2 be the edge of p incident to z_1 between z_1 and z_2. The points z_2 and x satisfy Relation P, so $(N_1(X_{e_1}) \cap X_{z_1}) \overset{c}{\cap} (N_1(X_{e_2}) \cap X_{z_1})$ is unbounded. The edge e_2 is also the edge leading from z_1 to y, and z_1 satisfies Relation P with both x and y, so x and y satisfy Relation P.

Now suppose $z_1 \neq w_1$.

In this case, w_1 is a vertex which lies on the geodesic segment joining z_1 to x and also on the geodesic segment joining z_1 to y. Let e_x be the edge of the geodesic from w_1 to x incident to w_1. Similarly define e_y and e_{z_1}. Now either $w_1 = x$ or $w_1 = y$ or

$$X_{w_1} \cap N_1(X_{e_x}) \overset{c}{=} X_{w_1} \cap N_1(X_{e_{z_1}}) \overset{c}{=} X_{w_1} \cap N_1(X_{e_y})$$

In any of these cases, any points in the convex hull of z_1, x, and y, satisfy Relation P.

If z_2 lies in the convex hull of x, y, and z_1 then we are done. If not, then any two vertices in the convex hull of x, y, and z_2 satisfy Relation P. Some edge in the geodesic joining z_1 to z_2 lies in one of these two convex hulls, so z_1 and z_2 satisfy Relation P as well.

□

3.5 P-sets

Definition 3. A *P-set* of $D\Gamma$ is a maximal subset of $\mathcal{V}\mathcal{E}D\Gamma$ such that any two elements satisfy Relation P.

Proposition 6 (Properties of P-sets)**.**

1. *A P-set is a subtree of $D\Gamma$.*
2. *Every depth zero vertex of a P-set is adjacent to infinitely many other vertices in that P-set.*
3. *An n-line, depth zero vertex belongs to n distinct P-sets.*
4. *Edges and positive depth vertices belong to exactly one P-set.*
5. *Any two P-sets are either disjoint or intersect in exactly one vertex, which is necessarily a depth zero vertex.*

Proof.

(1) Connectivity follows from Corollary 2.

(2 and 3) Every depth zero vertex has a number of families of parallel lines to which incident edges attach. Edges which attach to parallel lines satisfy Relation P, by Proposition 4. For each of these families, the edges which attach to lines in the family belong to

a common P-set. Infinitely many edges attach to each family, and each such edge leads to another vertex in the same P-set.

(4) By Corollary 2, if two elements of $\mathcal{V}\mathcal{E}D\mathit{\Gamma}$ each satisfy Relation P with an edge or positive depth vertex, they satisfy Relation P with each other.

(5) This follows from Proposition 5. □

The fundamental group $G = \pi_1(\Gamma)$ quasi-acts on X. Quasi-isometries preserve Relation P, so the action of G on $D\mathit{\Gamma}$ induces an action of G on the set of P-sets of $D\mathit{\Gamma}$.

Within a P-set S, two depth zero vertices v and w are *of the same type* if $\exists g \in Stab_G(S)$ such that $gv = w$. Type is an equivalence relation among depth zero vertices of a fixed P-set, and will be denoted $[\cdot]_S$, where S is the P-set. For an n-line depth zero vertex, the equivalence class of the vertex under the action of the whole group splits into at most n vertex types in S.

If there is a height change between vertices of the same type in a P-set, then there is unbounded height change between vertices of the P-set. Furthermore, for any vertex v in such a P-set, and for all sufficiently large r and any M, there exist infinitely many vertices of each vertex type of height between M and $M + r$, relative to v.

If there is zero height change between every pair of vertices of the same type in a P-set then, since there are only finitely many vertex types, the vertices of the P-set occur at only finitely many heights. Thus, there is bounded height change between any two vertices of the P-set.

For each orbit of P-sets, pick a representative S_i and fix an ordered list of representative $x_{i,j}$ for the vertex types in S_i. Suppose S and R are P-sets, x and y are vertices in R, $g \in G$ such that $gR = S$, and $k \in Stab_G(R)$ such that $kx = y$. Then $gkg^{-1} \in Stab_G S$ and $gkg^{-1}gx = gkx = gy$, so the action of the group takes vertices of the same type in one P-set to vertices of the same type in another. Thus, we can fix an ordering of vertex types for each equivalence class of P-set.

We will say that $x \in \mathcal{V}_0 S$ is of type $\{[S_i], j\}$ with respect to S if there is some $g \in G$ such that $gS = S_i$ and $gx = x_{i,j}$.

Proposition 7. *Suppose $v \in \mathcal{V}_0 D\Gamma$, and R and S are P-sets containing v. There is an element $g \in Stab_G(v)$ such that $gR = S$ if and only if v is of the same type with respect to both R and S.*

Proof. Suppose $\exists g \in Stab_G(v)$ such that $gR = S$. The P-sets R and S then belong to the same equivalence class; call it $[S_i]$. Suppose $v \in \{[S_i], j\}$ with respect to R and $v \in \{[S_i], k\}$ with respect to S. Then there are elements $f, h \in G$ with $fR = S_i$, $fv = x_{i,j}$, $hS = S_i$, and $hv = x_{i,k}$. However, this would mean that $fgh^{-1} \in Stab_G(S_i)$ and $fgh^{-1}x_{i,k} = x_{i,j}$, but this is only possible if $j = k$.

Conversely, suppose $v \in \{[S_i], j\}$ with respect to both R and S. Then there are elements $f, h \in G$ with $fR = S_i$, $fv = x_{i,j}$, $hS = S_i$, and $hv = x_{i,j}$, so $g = h^{-1}f$ fixes v and takes R to S.

□

3.6 The Tree of P-sets

Definition 4. The *tree of P-sets, T_Γ*, of a tubular group, $G = \pi_1(\Gamma)$, is given by:

- $\mathcal{V}T_\Gamma = \{\text{depth zero vertices of } D\Gamma\} \amalg \{\text{P-sets}\}$
- Edges are determined by inclusion of vertices in P-sets. Each edge is assigned length $\frac{1}{2}$.

Proposition 8. *The tree of P-sets is a tree.*

Proof. Depth zero vertices are connected to an adjacent P-set by a unique edge, and every P-set contains depth zero vertices, so to prove that T_Γ is a tree, it is sufficient to consider the depth zero vertices.

Consider two depth zero vertices, v and w, and the geodesic, p, joining them in $D\Gamma$. Let $v = v_0, v_1, \ldots, v_n = w$ be the sequence of depth zero vertices along p. P-sets intersect only in depth zero vertices, so v_i and v_{i+1} belong to a common P-set and are therefore connected in T_Γ. This shows that T_Γ is connected.

Now consider a path, p, without backtracking, connecting v to w in T_Γ. Let $v = v_0, v_1, \ldots, v_n = w$ be the sequence of depth zero vertices along p. The vertices v_i and v_{i+1} are connected in T_Γ by a P-set vertex. That P-set is a subtree in $D\Gamma$, so there is a unique path connecting v_i to v_{i+1} in $D\Gamma$. Since p has no backtracking, the P-set connecting v_i to v_{i+1} is distinct from the P-set connecting v_{i+1} to v_{i+2}. Therefore, the segments connecting v_i to v_{i+1} in $D\Gamma$ piece together without backtracking, giving the unique geodesic between v and w in $D\Gamma$.

Another path without backtracking must lift to this same geodesic, so must consist of the same sequence of depth zero vertices. Thus, T_Γ is a tree. □

From the properties of P-sets we see that T_Γ consists of infinite valence vertices, the P-set vertices, and finite valence vertices, the depth zero vertices. The kind of vertex alternates, P-set vertices are adjacent only to depth zero vertices, and vice versa.

The P-set vertices have adjacent vertices of only finitely many types, but there are infinitely many vertices of each of these types.

It will be convenient to extend the notion of height change across an edge to some of the edges of T_Γ. If R is a P-set with bounded height change, there is some adjacent depth zero vertex, v, of maximal height. For any vertex w adjacent to R, define $h(w, R) = h(w, v)$.

We could make a similar definition for edges connecting to a P-set vertex of unbounded height change by choosing some adjacent depth zero vertex as a reference.

There is an action of G on $\mathcal{V}T_\Gamma$ induced by the action of G on $D\Gamma$. This action can be extended to the edges as well. Let $g \in G$ and $e \in \mathcal{E}T_\Gamma$. The edge e joins a P-set vertex R to a depth zero vertex $v \in \mathcal{V}D\Gamma$ such that $v \in R$. The action of G on $D\Gamma$ preserves P-sets, so gv is a vertex of the P-set gR; hence, there is an edge, e', in T_Γ joining gv to gR. Define $ge = e'$.

The quotient of T_Γ by the action of G is a finite graph, called the *graph of P-sets*. The graph of P-sets has vertices corresponding to equivalence classes of depth zero vertices of $D\Gamma$ and vertices corresponding to equivalence classes of P-sets.

Suppose for some equivalence classes $[R]$ and $[v]$ there are representatives $R \in [R]$ and $v \in [v]$ such that $v \in R$. Then $[R]$ and $[v]$ will be connected by edges in the graph of P-sets, and number of such edges will be equal to the number of vertex types into which $[v] \cap R$ splits. Thus, P-set vertices in the graph of P-sets have valence equal to the number of distinct vertex types of their adjacent depth zero vertices.

The equivalence class $[v]$ of depth zero vertex, on the other hand, will have valence equal to the number of distinct vertex types a representative v belongs with respect to its adjacent P-sets. Proposition 7 implies that this will be strictly less than the number of lines in X_v if and only if $Stab_G(v)$ permutes the affine pattern of edge inclusions.

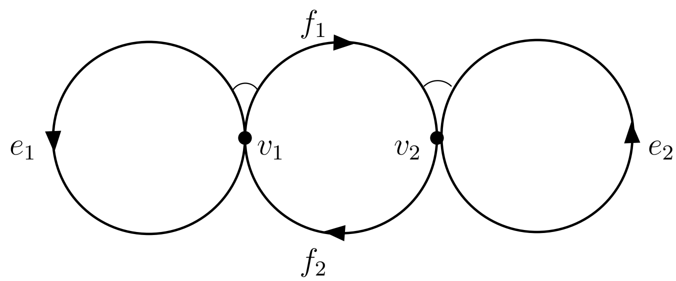

Figure 3. A graph for a two torus group.

Example 4 (A Two Torus Tubular Group)**.** Consider the graph, Γ, in Figure 3. Suppose the vertex groups are both $\mathbb{Z}^2$ and the edge groups are all $\mathbb{Z}$.

The Bass-Serre tree, $D\mathit{\Gamma}$, has two equivalence classes of P-sets. One class, $[A]$, consists of P-sets which project to $e_1 \cup f_1 \cup e_2$ in Γ. The other, $[B]$, consists of P-sets which project to f_2 in Γ.

P-sets in $[A]$ have four types of depth zero vertices. P-sets in $[B]$ have only two types of depth zero vertex.

Suppose the height change across the f edges is λ, the height change across the e_1 is μ, and the height change across e_2 is $\lambda - \mu$, with $\lambda > \mu > 0$.

In the following figures, solid black vertices are depth zero vertices. Open circles are P-set vertices. P-set vertices are infinite valence, so only representative adjacent vertices are shown. The type labels for the depth zero vertices are with respect to the P-set with the heavier black circle in the center of the figure. The vertical axis is height relative to this central P-set.

Figure 4 defines an ordering on types of vertices adjacent to a P-set in $[A]$.

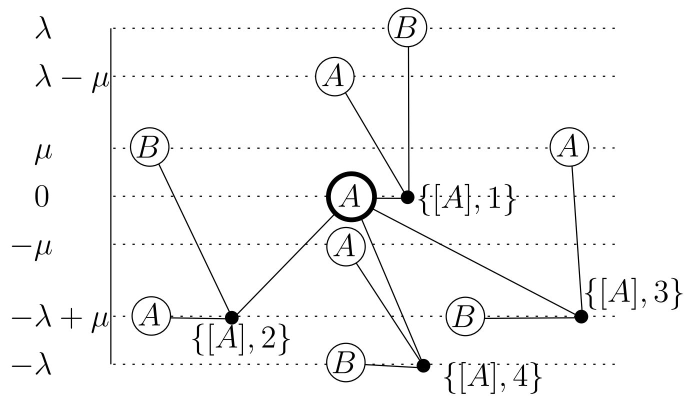

Figure 4. Vertex types and relative heights for $[A]$.

Figure 5 defines an ordering on types of vertices adjacent to a P-set in $[B]$.

The graph of P-sets is given in Figure 6.

♦

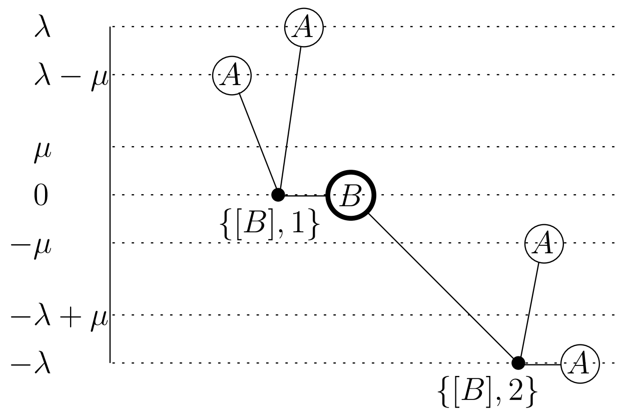

Figure 5. Vertex types and relative heights for $[B]$.

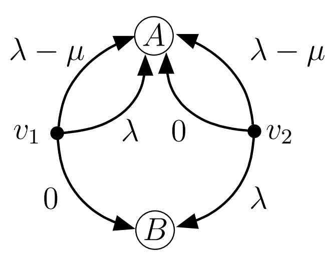

Figure 6. The graph of P-sets.

CHAPTER 4

QUASI-ISOMETRIES AMONG TUBULAR GROUPS

4.1 Allowable Isomorphisms of Trees of P-Sets

Suppose $G_1 = \pi_1(\Gamma_1)$ and $G_2 = \pi_1(\Gamma_2)$ are tubular groups, with trees of P-sets T_1 and T_2, respectively.

A quasi-isometry, $\psi : G_1 \to G_2$, respects depth zero vertices and Relation P, so it preserves adjacency in the tree of P-sets. Thus, ψ induces a tree isomorphism $\psi_\# : T_1 \to T_2$.

Recall that the P-sets $R_1, \ldots, R_n$ adjacent to a depth zero vertex v are in one-to-one correspondence with the families of parallel lines forming the affine pattern of edge inclusion in the vertex space of v. Applying the rigidity results of Mosher-Sageev-Whyte [8] we know that, up to homothety, a quasi-isometry of tubular groups acts on a depth zero vertex space by an element of the group of symmetries of the affine pattern. The vertex space of $\psi_\#(v)$ must have an equivalent affine pattern of edge inclusions, and the permutation of the pattern induced by $\{R_1, \ldots R_n\} \to \{\psi_\#(R_1), \ldots, \psi_\#(R_n)\}$ must be in the group of symmetries of the pattern.

Define a *rigid component* of T_i to be a connected component of

$$T_i \setminus \{\text{2-line, depth zero vertices in } T_i\}$$

An induced isomorphism $\psi_\#$ must take 2-line, depth zero vertices to 2-line, depth zero vertices, so $\psi_\#$ takes rigid components to rigid components.

Lemma 1 (Adaptation of Lemma 4.1 of [13]). *A quasi-isometry between tubular groups induces an isomorphism between their trees of P-sets which is coarsely height preserving on rigid components.*

Proof. Let R be a P-set in a tubular group $G = \pi_1(\Gamma)$. In (X_R, d_{X_R}), for any vertex, v, there is a well defined closest point projection $p_{(R,v)} : X_R \to X_v$.

Suppose v_0 and v_1 are two vertices in R. In the geodesic of $D\Gamma$ joining v_0 and v_1, let e_i be the edge incident to v_i. Let x and y be an arbitrary pair of reference points in $X_{v_0} \overset{c}{\cap} X_{e_0}$. Let $p = p_{(R,v_1)}$. Define the fiber distortion from v_0 to v_1 to be

$$\frac{d_{X_{v_1}}(p(x), p(y))}{d_{X_{v_0}}(x, y)}$$

$d_{X_{v_1}}(p(x), p(y)) = e^{-h(v_0,v_1)} d_{X_{v_0}}(x, y)$, so the distortion reflects height change between vertices.

A quasi-isometry preserves closest point projection up to a multiplicative factor depending on the quasi-isometry constant, and an additive factor depending on the quasi-isometry constants and the distance between v_i and v_{i+1}. A quasi-isometry restricted to a vertex space with at least 3 lines is just a homothety, at least up to isometry and to within uniformly bounded distance. In particular, on the large scale the expansion is the same in all directions.

Let v_0, R_1, v_1, R_2,…, R_n, v_n be vertices of a geodesic in T_G, where the v_i are depth zero vertices with at least 3 lines and the R_i are P-set vertices. Define the fiber distortion from v_0 to v_n to be the product of the fiber distortions from v_i to v_{i+1}. In taking the product of the distortions, additional uniform additive errors are added in each factor. However, for $1 \leq i \leq n-1$ the multiplicative errors cancel, since the edge space coming in to v_i and the edge space leaving v_i are scaled by the same amount.

Taking the limit as each pair of reference points becomes infinitely far apart, the additive errors drop out, leaving the limit of the distortion bounded above and below by multiples of $e^{h(v_0,v_n)}$. So the height change $h(\psi(v_0), \psi(v_n))$ differs from $h(v_0, v_n)$ by a uniform additive error. □

Definition 5. A tree isomorphism $\phi : T_1 \to T_2$ is *allowable* if

1. For every depth zero vertex v of T_1, the vertex spaces of v and $\phi(v)$ have equivalent affine patterns of edge inclusions. If the P-set vertices adjacent to v are $R_1, \ldots, R_n$ then the permutation of the affine edge patterns of v and $\phi(v)$ induced by $\{R_1, \ldots R_n\} \to \{\phi(R_1), \ldots, \phi(R_n)\}$ is in the group of symmetries of the pattern.

2. ϕ is coarsely height preserving on the rigid components of T_1.

The Lemma 1 and the preceding material show that quasi-isometries of tubular groups induce allowable isomorphisms on their trees of P-sets.

A tree isomorphism will be called *locally allowable* if it satisfies condition (1) of Definition 5.

4.2 Finding Allowable Isomorphisms

Theorem 1. *There is an algorithm which in finite time decides whether or not there is an allowable isomorphism* $T_{\Gamma_1} \to T_{\Gamma_2}$.

The idea behind the algorithm is that it should be possible to build an allowable isomorphism according to a finite set of strategies. Strategies will be defined to give instructions for extending an isomorphism in a locally allowable way.

An arbitrary allowable isomorphism need not be built in such a simple way, but from an given allowable isomorphism it is possible to extract enough information to see how another allowable isomorphism could have been built according to one of the candidate instruction sets.

We define what a set of strategies should be in such a way that there are only finitely many candidates. Given a candidate set of strategies, it is possible to check it to see whether it can be used to build an isomorphism which is coarsely height preserving on rigid components.

The number of sets of strategies is finite, but large. Rather than blindly enumerating every possibility, the algorithm tries to build a candidate which is not an obvious failure. If this is successful, then the check is performed. If a candidate passes this check, the algorithm stops and returns the set of strategies. If a candidate fails, the algorithm backtracks and attempts to build a different candidate. If the algorithm exhausts the finitely many candidates without finding one that works, then no allowable isomorphism exists.

The remainder of this chapter is devoted to proving Theorem 1.

4.2.1 Definitions

4.2.1.1 Extensions

Let $[R_1], \ldots, [R_m]$ be the equivalence classes of P-sets in Γ_1 under the action by G_1.

Let $[S_1], \ldots, [S_n]$ be the equivalence classes of P-sets in Γ_2 under the action by G_2.

Let $\rho(a)$ be the number of vertex types in $[R_a]$ and let $\sigma(b)$ be the number of vertex types in $[S_b]$.

A *match* is a pair $([R], [S])$ consisting of an equivalence class of P-set from Γ_1 and one from Γ_2.

An *extension* for $([R_a], [S_b])$ is a $\rho(a) \times \sigma(b)$ matrix, (m_{ij}), with entries described below. An extension must have at least one non-zero entry in each row and in each column.

Recall that we had chosen representatives $x_{a,i} \in \{[R_a], i\}$ with respect to R_a and $y_{b,j} \in \{[S_b], j\}$ with respect to S_b.

If $X_{x_{a,i}}$ and $Y_{y_{b,j}}$ have inequivalent patterns of edge inclusions then $m_{ij} = 0$. If they do have equivalent patterns, fix a linear equivalence, τ, between them. Let $R^1, \ldots, R^p$ be the P-sets adjacent to x_{a_i}, and let $S^1, \ldots, S^p$ be the P-sets adjacent to $y_{b,j}$. A P-set adjacent to a depth zero vertex corresponds to a family of parallel lines in the pattern of edge inclusions in the vertex space. According to this correspondence, τ induces a bijection $\{R^l\}_{l=1}^p \to \{S^l\}_{l=1}^p$, and we will write $S^{\tau(l)}$ for $\tau(R^l)$. We may also pre- or post-compose τ with an element of the group of symmetries of the pattern and get such a bijection, and these are the only allowable bijections between P-sets adjacent to $x_{a,i}$ and $y_{b,j}$.

Consider the set $\{\{[R^l], k_l\}\}_{l=1}^p$, where $x_{a,i}$ is of type $\{[R^l], k_l\}$ with respect to R^l. Similarly, consider the set $\{\{[S^l], k_l'\}\}_{l=1}^p$, where $y_{b,j}$ is of type $\{[S^l], k_l'\}$ with respect to S^l. A bijection between P-sets adjacent to x_{a_i} and $y_{b,j}$ gives a bijection of these sets of types. We will let m_{ij} be any bijection of these sets which can be induced like this and which includes $\{[R_a], i\} \to \{[S_b], j\}$. If there are no such bijections then $m_{ij} = 0$.

Notice that the number of possible extensions for $([R_a], [S_b])$ is bounded above by

$$\rho(a)\sigma(b)(\max p)!$$

Proposition 9. *Let $R \in [R_a]$ and $S \in [S_b]$ be P-sets. Let $v \in \mathcal{V}_0 R$ be of type $\{[R_a], i\}$ with respect to R, and let $w \in \mathcal{V}_0 S$ be of type $\{[S_b], j\}$ with respect to S. A bijection m_{ij} as discussed above gives instructions for mapping P-sets adjacent to v to P-sets adjacent to w in an allowable way, with R mapping to S. Furthermore, the set of matches with height errors that this mapping induces is independent of the choices made.*

Proof. If v and w are both two-line vertices then there are no choices to make, since each vertex is adjacent to only two P-sets. Assume v and w each have more than two lines.

There is an isometry, θ, between $X_{x_{a,i}}$ and $Y_{y_{b,j}}$ which acts as a symmetry of edge patterns and induces the desired bijection on the associated P-sets and vertex types. There are also elements $f \in G_1$ and $h \in G_2$ such that $gR = R_a$, $gv = x_{a,i}$, $hS = S_a$, and $hw = y_{b,j}$. The element f takes X_v to $X_{x_{a,i}}$, is bounded distance from an isometry, and gives an equivalence of the edge patterns. A similar statement is true for h, so $h^{-1}\theta g : X_v \to Y_w$ gives an edge pattern symmetry between the edge patterns of X_v and Y_w. Furthermore, the action of the groups preserves vertex type, so this map induces the bijection required by m_{ij}, and this would still be true if we had chosen different f, θ, or h. In particular, we can choose f and h so that the map takes R to S.

This is true since, by Proposition 7, a vertex is of the same type with respect to two different adjacent P-sets if and only if there is an element of the vertex stabilizer which takes one P-set to the other. Thus, if $h^{-1}\theta f$ does not take R to S we could change it by pre- and post- composition by elements of $Stab_{G_1}(v)$ and $Stab_{G_2}(w)$, respectively, to ensure $R \mapsto S$.

The set of matches that this map induces is determined by the bijection m_{ij}. The height change between a depth zero vertex and an adjacent P-set depends only on the type of the vertex with respect to the P-set, so the height errors are also independent of the choices. □

Proposition 10. *Let R be a P-set of T_1 and S be a P-set of T_2. An extension for $([R], [S])$ gives instructions for mapping the ball of radius 1 around R in T_1 to the ball of radius 1 around S in T_2 in a locally allowable way.*

Proof. Suppose $R \in T_1$ and $S \in T_2$ are P-set vertices, with $R \in [R_a]$ and $S \in [S_b]$. Let $\mathcal{E} = (m_{ij})$ be an extension for $([R], [S])$. Choose a bijection between vertices adjacent to R and vertices adjacent to S such that vertices in $\{[R], i\}$ map to vertices in $\{[S], j\}$ if and only if $m_{ij} \neq 0$. This is always possible, since there are infinitely many vertices of each type adjacent to any P-set.

Suppose this bijection identifies a vertex $v \in \{[R], i\}$ to a vertex $w \in \{[S], j\}$. Use entry m_{ij} and Proposition 9 to extend to the P-sets adjacent to v and w.

If this process identifies a P-set R' adjacent to v to a P-set S' adjacent to w, we get an induced match, $([R'],[S'])$. In addition, if R, S, R', and S' are all of bounded height change, and if v and w have at least three lines, then there is a well defined height error, $E = h(R,R') - h(S,S')$. The extension may induce the same match in many different ways, with differing height errors, but the set $\{(\mathcal{M}, E)\}$ of induced matches with (possibly undefined) height errors is independent of the choices of R, S, and of the bijection $N_{\frac{1}{2}}(R) \to N_{\frac{1}{2}}(S)$. □

Conversely, given an allowable isomorphism $\phi : N_1(R) \to N_1(S)$, it is possible to "read off" an extension for $([R],[S])$. Set $m_{ij} = 0$ if $\phi(\{[R],i\}) \cap \{[S],j\} = \emptyset$. For every i and j such that $\phi(\{[R],i\}) \cap \{[S],j\} \neq \emptyset$, choose some representative vertex $v_{i,j}$ with $v_{i,j} \in \{[R],i\}$ and $\phi(v_{i,j}) \in \{[S],j\}$. The identification by ϕ of P-sets adjacent to $v_{i,j}$ to those adjacent to $\phi(v_{i,j})$ corresponds to an element of the group of symmetries of the edge patterns in the vertex spaces, since ϕ was assumed allowable. Set m_{ij} equal to the bijection of vertex types induced by this element. Then the matrix (m_{ij}) is an extension for $([R],[S])$.

It may be possible to simplify this extension. Non-zero entries may be set to zero as long as there remains at least one non-zero entry in every row and every column.

4.2.1.2 Strategies

A *strategy,* $\mathcal{S}$, for $([R],[S])$ consists of:

1. a root vertex, $ROOT(\mathcal{S})$, which is labeled by $([R],[S])$,
2. an extension $\mathcal{E} = (m_{ij})$ for $([R],[S])$,

3. a collection of terminal vertices corresponding to the set $\{(\mathcal{M}, E)\}$ of induced matches with height errors coming from $\mathcal{E}$.

The *label* of a terminal vertex is the match associated to it. A strategy can be simplified by considering at most three terminal vertices for each label: one with maximum height error, one with minimum height error, and one with undefined height error. Thus, we can assume that the number of terminal vertices of any strategy is at most $3mn$.

A strategy records how the boundary of a neighborhood of a P-set R in T_1 maps to the boundary of a neighborhood of S in T_2 when mapped according to an extension $\mathcal{E}$ for $([R], [S])$. In this definition the neighborhood of R is just $N_1(R)$, but generalizations are straightforward.

If $\mathcal{S}$ is a strategy, define

$$TERM(\mathcal{S}) = \{\text{terminal vertices of } \mathcal{S}\}$$

The strategy $\mathcal{S}$ is *non-increasing* if for all vertices $v \in TERM(\mathcal{S})$ with

$$label(v) = label(ROOT(\mathcal{S}))$$

the height error of v is non-positive.

The strategy $\mathcal{S}$ is *non-decreasing* if for all vertices $v \in TERM(\mathcal{S})$ with

$$label(v) = label(ROOT(\mathcal{S}))$$

the height error of v is non-negative.

The sign of a strategy is $+$ if it is a non-decreasing strategy, and $-$ if it is a non-increasing strategy.

Of course, it is possible for a single strategy to be both non-increasing and non-decreasing if there are no vertices in $TERM(\mathcal{S})$ with the same label as the root, or if all such vertices have height error 0 or undefined.

A *set of strategies* will consist of a graph, each of whose vertices is labeled by a match and two (not necessarily distinct) strategies for the match, one non-increasing, and one non-decreasing. There will be at most one vertex labeled by a given match, so at most mn vertices.

For each strategy we will add edges going to the vertices labeled by terminal vertices of that strategy, and these edges are completely determined by the strategies. The strategies, in turn are determined by the extensions, and there is some finite number of possible extensions for any match.

The total number of candidates is therefore at most $2mn$ times the maximum number of possible extensions for any match. This is a lot of candidates, and many of these are obviously unsuitable for building quasi-isometries between tubular groups.

Rather than enumerate all these possibilities, the algorithm attempts to build one with some potential to work.

A match is considered good until it has been declared bad. A strategy is bad if any of its vertices are labeled by bad matches.

A match $([R],[S])$ is bad if one of the classes of P-set is a class of bounded height change and the other is a class of unbounded height change. A match $([R],[S])$ is also bad unless there are both non-increasing and non-decreasing strategies for $([R],[S])$.

For such a match, no P-set vertex in $[R]$ can be mapped by an allowable isomorphism to a P-set vertex in $[S]$.

Suppose, for example, that $\mathcal{M} = ([R],[S])$ has no non-increasing strategy, and suppose $\phi(R) = S$ for some tree isomorphism ϕ. Every strategy for $\mathcal{M}$ has a terminal vertex labeled $\mathcal{M}$ with a strictly positive height error. There are only finitely many strategies for $\mathcal{M}$, so there is some $\epsilon > 0$ such that every strategy for $\mathcal{M}$ has a terminal vertex labeled $\mathcal{M}$ with height error at least ϵ.

Thus, there must be some $R' \in [R]$ and some $S' \in [S]$ at distance 1 from R and S, respectively, such that $\phi(R') = S'$ and such that $h(R,R') - h(S,S') \geq \epsilon$. $([R'],[S']) = \mathcal{M}$, so this argument can be repeated to find R'' and S'' with

$$h(R'',R) - h(S'',S) \geq h(R',R) - h(S',S) + \epsilon \geq 2\epsilon$$

Continuing in this way, we can find a path in the tree with unbounded height error, so ϕ can not be coarsely height preserving.

4.2.2 <u>The Algorithm</u>

If there are no good matches then halt the algorithm, there is no coarsely height preserving isomorphism $T_1 \to T_2$.

Constructing Sets of Strategies

Let $\mathcal{G}$ be the empty graph.

Pick some good match $\mathcal{M} = ([R], [S])$. Let $\mathcal{S}^+$ be a good, non-decreasing strategy for M, and let $\mathcal{S}^-$ be a good, non-increasing strategy for M. If such strategies do not exist, then M is a bad match; pick a new one. Add a vertex $(\mathcal{M}, \mathcal{S}^+, \mathcal{S}^-)$ to $\mathcal{G}$. This vertex is at level 1.

Extending the Graph

Build $\mathcal{G}$ inductively as follows: Suppose $(\mathcal{M}, \mathcal{S}^+, \mathcal{S}^-)$ is a vertex of $\mathcal{G}$ at level k.

Let $\mathcal{M}_1^+$, .., $\mathcal{M}_a^+$ be a list of matches which occur as labels of terminal vertices of $\mathcal{S}^+$. Starting with $i = 1$, perform the following steps:

Check if $\mathcal{M}_i^+$ already appears as a vertex label in $\mathcal{G}$. If so, add a directed edge from $(\mathcal{M}, \mathcal{S}^+, \mathcal{S}^-)$ to that vertex, labeling the edge with $+$. Check the graph.

If $\mathcal{M}_i^+$ is not already in the graph, find a good, non-decreasing strategy, $\mathcal{S}_i^+$, and a good, non-increasing strategy, $\mathcal{S}_i^-$. If there fails to be such an $\mathcal{S}_i^+$ or $\mathcal{S}_i^-$, declare $\mathcal{M}_i^+$ a bad match and scour the graph for $\mathcal{M}_i^+$.

If $\mathcal{S}_i^+$ and $\mathcal{S}_i^-$ exist then add a new vertex $(\mathcal{M}_i^+, \mathcal{S}_i^+, \mathcal{S}_i^-)$ to $\mathcal{G}$ at level $k+1$. Add a directed edge from $(\mathcal{M}, \mathcal{S}^+, \mathcal{S}^-)$ to $(\mathcal{M}_i^+, \mathcal{S}_i^+, \mathcal{S}_i^-)$ labeled by $+$. Now check the graph.

Repeat this process for $\mathcal{S}^-$, labeling any new edges with $-$.

Scouring the Graph

If, in the course of extending the graph, a match, $\mathcal{M}$ is declared bad, it is necessary to go back through the graph to see that $\mathcal{M}$ does not occur. If any vertex v of $\mathcal{G}$ contains a strategy with a vertex labeled by $\mathcal{M}$, declare that strategy bad, and remove edges originating at v with label matching the sign of that strategy. If there is another choice of good strategy for v to replace the one declared bad, then do so. Since v has been altered it will be necessary to go back and extend from the strategy of v that was replaced. If there is no suitable replacement strategy, then the match labeling v is bad, and $\mathcal{G}$ should be scoured for that match as well.

If this process divides $\mathcal{G}$ into different connected components, delete all but the component containing the level 1 vertex.

If this process deletes the level 1 vertex, then start over. At least one match that was previously considered good has now been declared bad, so the algorithm is making progress.

Checking the Graph

We need to know that a match has non-increasing and non-decreasing strategies to ensure that there are no loops of length one in the graph which will lead to uncontrolled height error. As the graph gets larger, we also need to check that there are not longer implication loops which will lead to uncontrolled height error. This is determined by whether or not there is a solution to a certain set of inequalities.

For P-sets of bounded height error, the type of a vertex determines it's relative height. Choosing an extension therefore determines height errors in a neighborhood of the P-set.

For P-sets of unbounded height error this is not true, there is more flexibility. For any vertex type, and large enough K, there are infinitely many vertices of the specified type in any height interval of size K. This means we do not need to worry about height change when it comes to P-sets of unbounded height change; they have enough flexibility to correct any height error which may occur.

Let $\mathcal{M}_1, \ldots$ be a list of the matches of P-sets of bounded height change which label vertices of $\mathcal{G}$. For each i, add to the set of inequalities:

$$L_i \leq E_i \leq U_i$$

Let $\mathcal{S}_i^+$ and $\mathcal{S}_i^-$ be the non-decreasing and non-increasing strategies chosen for $\mathcal{M}_i$.

Suppose $v \in TERM(\mathcal{S}_i^+)$ has a defined height error, and the label of v is $\mathcal{M}_j$. Let E be the height error of v in $\mathcal{S}_i^+$. Add the following inequalities to the set:

$$M_i + E \geq L_j$$

$$U_i + E \leq U_j$$

Suppose $v \in TERM(\mathcal{S}_i^-)$ has a defined height error, and the label of v is $\mathcal{M}_j$. Let E be the height error of v in $\mathcal{S}_I^-$. Add the following inequalities to the set:

$$M_i + E \leq U_j$$

$$L_i + E \geq L_j$$

If the system of inequalities has a solution then height error is well controlled. The L_i and U_i provide bounds. Continue building the graph.

If such a system of inequalities has a solution, then it has a solution such that all the U_i are positive, all the L_i are negative, and all the M_i are zero. Furthermore, given any fixed B, there is a solution such that for all i, $U_i > B$ and $L_i < -B$.

If the system has no solutions, then backtrack through $\mathcal{G}$ and make a different choice. That is, first go to the highest level of vertices, and see if any have alternate choices of strategies. If so, try replacing a strategy, deleting all edges originating at that vertex whose labels match the sign of the strategy, and extending from there. If there are no alternate choices, check the next highest level of vertices, and so on. If the backtracking goes all the way back to the level 1 vertex, and no choices have yielded a consistent graph, then declare the match labeling the level 1 vertex a bad match, and start over.

Finishing the Graph

$\mathcal{G}$ is complete when the appropriate edges have been added to every vertex. Vertices in $\mathcal{G}$ are never labeled by the same match, and have only finitely many edges coming out of them, so $\mathcal{G}$ is a finite graph.

4.2.3 Using a Set of Strategies

Suppose the algorithm halts with a set of strategies, $\mathcal{G}$. It is possible to use $\mathcal{G}$ to build an allowable isomorphism between trees of P-sets. The isomorphism is built inductively by building out from a P-set vertex according to an extension. As described previously, this requires choosing a bijection between the infinite sets of depth zero vertices adjacent to the P-sets. For the purpose of building tree isometries the particular choice of bijection does not matter. However, with a little more care, the bijections can be chosen so that the isomorphism that is constructed is actually induced by a quasi-isometry between the groups. For this reason, the construction of an allowable isomorphism from a set of strategies is postponed until the proof of Theorem 2.

4.2.4 Deriving a Set of Strategies

Not every allowable isomorphism arises from a set of strategies as described. However, if such an isomorphism exists, it can be used to produce a set of strategies. This set of strategies could then be used to build a different allowable isomorphism, possibly with larger bounds on the height errors.

Suppose ϕ is an allowable isomorphism $T_1 \to T_2$ of trees of P-sets.

For each match $([R], [\phi(R)])$ with $[R]$ and $[\phi(R)]$ of unbounded height change, pick some representative R and read off a strategy. Strategies for P-sets of unbounded height change are vacuously non-decreasing and non-increasing, so this is the only strategy we need for the match $([R], [\phi(R)])$.

Suppose there is a class of P-set vertices of T_1 such that every depth zero vertex in a member of the class has 2 lines. Pick some representative, R, for such a class, and read off a strategy. This strategy is vacuously non-decreasing and non-increasing, so this is the only strategy we need for the match $([R], [\phi(R)])$.

The only matches left are pairs of P-sets of bounded height change which contain at least one depth zero vertex with at least 3 lines. Pick such a P-set, R, and depth zero vertex, v_0.

Every rigid component, $\mathcal{C}$, of T_1 contains a vertex, $v_{\mathcal{C}}$, closest to v_0. The vertex $v_{\mathcal{C}}$ is necessarily a 2-line, depth zero vertex.

For any depth zero vertex, w, in the rigid component containing v_0, define $h(w) = h(v_0, w)$. For any depth zero vertex, w', in the rigid component $\mathcal{C}$, define $h(w') = h(v_{\mathcal{C}}, w')$.

Make the corresponding definitions for heights of depth zero vertices of T_2.

Let $\mathcal{M}_1, \ldots, \mathcal{M}_a$ be a list of matches which occur as $([R], [\phi(R)])$, with R and $\phi(R)$ of bounded height change and having a depth zero vertex with at least 3 lines. Let

$$U_i = \sup\{h(R) - h(\phi(R)) \mid ([R], [\phi(R)]) = \mathcal{M}_i\}$$

$$L_i = \inf\{h(R) - h(\phi(R)) \mid ([R], [\phi(R)]) = \mathcal{M}_i\}$$

These quantities exist since ϕ is uniformly coarsely height preserving on rigid components.

For any match there are only finitely many possible strategies. For each $1 \leq i \leq a$ if U_i is achieved, pick an R with $U_i = height(R) - height(\phi(R))$ and $([R], [\phi(R)]) = \mathcal{M}_i$ and read off a strategy $\mathcal{S}_i^-$ for $([R], [\phi(R)])$. If U_i is not achieved, let $\mathcal{S}_i^-$ be any strategy which occurs for R with height error arbitrarily close to U_i These strategies are necessarily non-increasing.

Pick non-decreasing strategies $\mathcal{S}_i^+$ analogously using the L_i.

Pick an $\mathcal{M}_i$ to be the level 1 vertex, and build a set of strategies using these chosen strategies. Since these strategies came from an allowable isomorphism, all inequalities of the following forms will hold:

$$L_i \leq U_i$$

$$U_i + E \leq U_j$$

$$L_i + E \leq U_j$$

$$U_i + E \geq L_j$$

$$L_i + E \geq L_j$$

The system of inequalities has solutions given by:

$$M_i = \frac{L_i + U_i}{2}$$
$$U_i' = U_i + \frac{1}{2}\max\{U_i - L_i\}$$
$$L_i' = L_i - \frac{1}{2}\max\{U_i - L_i\}$$

4.3 Quasi-isometries

In this section we construct isometries of P-set spaces according to an extension. These then serve as the building blocks for the quasi-isometry of tubular groups constructed in Theorem 2.

4.3.1 Quasi-Isometries of P-set Spaces

Let R be a P-set in $D\Gamma$. X_R, the P-set space of R, is the union of edge spaces and vertex spaces for the edges and vertices of R, with the induced path metric. Let v_0 be some depth zero vertex of R. Assume that the height of v_0 has been defined. Height for other points of R can then be defined relative to v_0.

X_{v_0} is topologically $\mathbb{R} \times \mathbb{R}$. Put orthogonal coordinates (t, r) on X_{v_0} in such a way that the edges of R glue onto X_{v_0} along lines of the form $t \times \mathbb{R}$. Furthermore, choose the coordinates so that $d_{X_{v_0}}(t, r) = \exp(-h(v_0))d_{\mathbb{R}\times\mathbb{R}}(t, r)$.

The second coordinate can be projected over all of X_R, giving coordinates on X_R of the form $tree \times \mathbb{R}$, with the $\mathbb{R}$ factor scaled according $\exp(-\text{height})$. With these coordinates, an edge space or a positive depth vertex space is just $(t, \mathbb{R})$, for some t, and the length of $(t, [a, b])$ is $\exp(-h(t))|b - a|$.

If v is a depth zero vertex of R, edges of R incident to v have edge spaces that attach to X_v along a family of parallel lines. In the chosen coordinates, $X_v = (l, \mathbb{R})$, where l is a line, and $(l, 0)$ is a line in X_v that runs perpendicular to the lines of edge attachment. Let the point of l closest to X_{v_0} be 0, and parameterize l so that $d_{X_v}((a, 0), (b, 0)) = \exp(-h(v))d_{\mathbb{R}\times\mathbb{R}}((a, 0), (b, 0))$.

Define a map $B : R \to tree$ by lifting to X_R and projecting to zero in the second coordinate. In $B(R)$ the depth zero vertices get “blown up” into lines.

$B(R)$ is a simplicial tree. Edges of $B(R)$ contained in a depth zero vertex space will be called *horizontal*, since height is constant on a vertex space. Conversely, edges of $B(R)$ which cross an edge space will be called *vertical*, since there is (potentially) a height change across an edge.

A *horizontal component* of $B(R)$ is the blow up of some depth zero vertex. A *vertical component* is a connected component of the complement of the interiors of the horizontal edges.

The next lemma gives a quasi-isometry of a P-set space for a P-set of bounded height change. These will be used to build a quasi-isometry of a tubular group, P-set by P-set. The various parameters just say that this is a quasi-isometry, built according to an extension $\mathcal{E}$, starting from a previously defined map on one of the depth zero vertex spaces.

Recall that $\mathcal{V}_0X$ is the union of the depth zero vertex spaces of X. Similarly, $\mathcal{V}_0X_R = \bigcup_{v\in\mathcal{V}_0R} X_v$.

Lemma 2. *Let $G_1 = \pi_1(\Gamma_1)$ and $G_2 = \pi_1(\Gamma_2)$ be two tubular groups. Let $R \subset D\varGamma_1$ and $S \subset D\varGamma_2$ be P-sets of bounded height change. Let $x_0 \in R$ and $y_0 \in S$ be depth zero vertices. Let $[x_1]_R, \ldots, [x_a]_R$ and $[y_1]_S, \ldots, [y_b]_S$ be lists of the types of vertices which occur in R and S, respectively. Let $\mathcal{E} = (m_{ij})$ be an extension for $([R], [S])$. Let α and β be arbitrary positive real numbers. Then there is a quasi-isometry $\Phi = \Phi^0_{\mathcal{E},(R,x_0,\alpha),(S,y_0,\beta,)} : X_R \to Y_S$ with the following properties:*

1. $\Phi(X_{x_0}) = Y_{y_0}$
2. *Φ takes depth zero vertex spaces to depth zero vertex spaces, and induces a bijection, $\Phi_\#$, between depth zero vertices of R and S.*

3. *Up to isometric permutation of the affine patterns,* $\Phi|_{X_v}$ *is a homothety with expansion by* $\frac{\beta}{\alpha}\exp(h(v) - h(\Phi_{\#}(v)))$.

4. *Excluding* x_0 *and* y_0*, there exist vertices of type* $[x_i]_R$ *mapping to vertices of type* $[y_j]_S$ $\iff$ m_{ij} *is non-zero.*

5. Φ *is a bilipschitz bijection* $(\mathcal{V}_0 X_R, d_{X_R}|_{\mathcal{V}_0 X_R}) \to (\mathcal{V}_0 Y_S, d_{Y_S}|_{\mathcal{V}_0 Y_S})$.

Proof. For any $x \in R$, $h(x) = h(x_0, x)$, and similarly for $y \in S$ with respect to y_0. These quantities are constants for vertex types.

Let M be the maximum and m be the minimum of the set

$$\{1\} \cup \{\exp\left(h([x_i]_R) - h([y_j]_S)\right) \mid m_{ij} \neq 0\}$$

The logarithms of these values are bounds for height error, and they depend only on the equivalence classes of R and S.

Let x be a depth zero vertex of R. Let e be an edge of R incident to x. Since $Stab_{G_1}(e)$ has infinite index in $Stab_{G_1}(x)$, the orbit of e by $Stab_{G_1}$ contains infinitely many other edges incident to x. The edge spaces for these edges glue on to X_x along parallel lines, spaced a bounded distance apart.

Pick some L such that for every depth zero vertex x in R, any open interval of length L in $B(x)$ has at least two incident vertical edges from every $Stab_{G_1}(x)$ equivalence class of edge incident to the vertex, and at least three total vertical incident edges.

Increase L if necessary so that for every depth zero vertex y in S, any open interval of length $Lm\frac{\beta}{\alpha}$ in $B(y)$ has at least two incident vertical edges from every $Stab_{G_2}(y)$ equivalence class of edge incident to the vertex, and at least three total vertical incident edges.

Choose coordinates for X_R as discussed above.

For every $x \in R$ do the following: Define the basepoint ξ_x of $B(x)$ to be the point of $B(x)$ with coordinate zero. Slide any edge incident to the open interval

$$\left(-\frac{1}{2}\exp(h(x))L, \frac{1}{2}\exp(h(x))L)\right)$$

in $B(x)$ to 0. Note that for each x this interval is of length L, so there are at least two incident vertical edges covering every edge incident to the projection of x to Γ, and at least three total vertical incident edges.

Choose coordinates for Y_S in a similar fashion, with the following provision. If we already have a map $X_{x_0} \to Y_{y_0}$, choose the coordinates of Y_{y_0} so that the origin of Y_{y_0} is the image of the origin of X_{x_0} and so that map preserves the orientation of the second coordinate.

Define the basepoint ξ_y of $B(y)$ to be the point of $B(y)$ with coordinate 0. For every $y \in S$, slide any edge incident to the open interval

$$\left(-\frac{1}{2}\frac{\beta}{\alpha}\exp(h(x))L, \frac{1}{2}\frac{\beta}{\alpha}\exp(h(x))L\right)$$

in $B(y)$ to 0. Again, this interval has length $L\frac{\beta}{\alpha}\exp(h(x) - h(y))$ in Y_y, so there are at least three incident vertical edges and at least two edges covering every edge in the underlying graph.

These sliding operations change distances between points in different horizontal components, but preserve distance within a fixed horizontal component.

For the vertices of the vertical components which are points of intersection with horizontal components, the *type* of the vertex is just the type of the vertex in $D\mathit{\Gamma}$ corresponding to that horizontal component. After sliding, what were formerly vertical components are still connected sets, but the new vertical components are composed of unions of the original vertical components. In particular, new vertical components containing ξ_{x_0} and ξ_{y_0} are bounded valence trees, call them V and W.

The image of V in Γ_1 is the same as the image of R in Γ_1. Therefore, the set vertices of V of any particular vertex type are coarsely dense in V. The diameter of the image of R in Γ_1 provides a coarseness constant.

Similar statements are true for W as well. Let δ be the greater of the diameters of R in Γ_1 and S in Γ_2.

Every vertex of V and W is distance at most δ from a vertex of valence at least three. We will say V and W are δ-*bushy*. Also, the valence of V and W is bounded above.

It would be convenient if we had a bijection between vertex types, but that is not guaranteed, since $\mathcal{E}$ may have more than one nonzero entry in a row or column. However, it is possible to refine the vertex types to get a bijection. V and W are bushy trees, so they are nonamenable. Since each vertex type is coarsely dense in each vertical component, it is possible to divide the set of vertices of a given type into finitely many subsets, each of which is still coarsely dense, though with a larger constant. For each type $[x_i]_R$, subdivide $B([x_i]_R) \cap V$ into a number of subsets equal to the number of non-zero entries in column i of $\mathcal{E}$. Similarly, subdivide $B([y_j]_S) \cap W$ into a number a subsets equal to the number of non-zero entries in row j of $\mathcal{E}$. Assume that we would like to map the subset of $[x_i]_R \cap V$ corresponding to a non-zero matrix entry m_{ij} to the subset of $[y_j]_S \cap W$ corresponding to this entry.

Bounded valence bushy trees are quasi-isometric with quasi-isometry constants depending only on the valence bound and the bushiness constant. Therefore, there is a quasi-isometry $f : V \to W$. The quasi-isometry f can be changed a bounded amount so that it is a bilipschitz bijection of depth zero vertices and respects the partitions of vertices by type; see Lemma 4 in Appendix A for a proof.

Let ϕ be the bilipschitz bijection provided by this lemma, and let K be the bilipschitz constant.

This process will be called extension along a vertical component. Note that every depth zero vertex in V or W is the basepoint of its horizontal component.

Suppose extension along a vertical component identifies ξ_x to ξ_y.

For $n = 1, 2, \ldots$, consider the half-open intervals

$$\left[\frac{2n-1}{2}\exp(h(x))L, \frac{2n+1}{2}\exp(h(x))L\right) \subset B(x)$$
$$\left(\frac{-2n-1}{2}\exp(h(x))L, \frac{-2n+1}{2}\exp(h(x))L\right] \subset B(x)$$
$$\left[\frac{2n-1}{2}\frac{\beta}{\alpha}L, \frac{2n+1}{2}\frac{\beta}{\alpha}L\right) \subset B(y)$$
$$\left(\frac{-2n-1}{2}\frac{\beta}{\alpha}L, \frac{-2n+1}{2}\frac{\beta}{\alpha}L\right] \subset B(y)$$

In each of these half-open intervals, slide all incident vertical edges to the closed endpoint. Extend ϕ by $\phi|_{B(x)} : B(x) \to B(y) : r \mapsto \pm\frac{\beta}{\alpha}r$. The choice of orientation here is determined by the element m_{ij} associated to x, y. For either choice of orientation, the map takes closed endpoints of the half-open intervals in $B(x)$ to closed endpoints of the half-open intervals of $B(y)$ and expands distance by $\frac{\beta}{\alpha}\exp(h(x) - h(y))$.

The intervals in $B(x)$ have length L. The intervals in $B(y)$ have length $\frac{\beta}{\alpha}\exp(h(x)-h(y))L$. By our choice of L, each of these half-open intervals has at least three incident vertical edges. Thus, every point which is the endpoint of one of the half-open intervals has at least three incident vertical edges. Furthermore, every other depth zero vertex in the vertical component containing such a point is the basepoint of its horizontal component. We arranged for these points to have at least three incident vertical edges in the first step, so this vertical component is a bushy tree of bounded valence. We can extend ϕ along these vertical components.

Alternate extending along collections of vertical and horizontal components to build ϕ.

We make the following observations about this construction:

1. Every vertical edge has at most one end which slides.

2. No edge slides more than once.

3. Sliding an edge only changes distances between points on opposite sides of the edge.

4. No edge in $B(R)$ slides more than a distance L.

5. No edge in $B(S)$ slides more than a distance $\frac{\beta}{\alpha}LM$.

Thus, edge sliding in $B(R)$ changes distances by at most a multiplicative factor of $1+L$. Edge sliding in $B(S)$ changes distances by at most a multiplicative factor of $1+\frac{\beta}{\alpha}LM$. There is a uniform upper bound on the number of edges accumulated at any point, so there is a uniform upper bound on the valence of any vertical component that is created. Thus, we can take a uniform constant K for the bilipschitz maps used to extend along vertical components. Extending along horizontal components changes distances by at most a multiplicative factor of $\max(M, \frac{1}{m})$. The product of these four factors gives a bilipschitz constant for ϕ.

Define $\Phi : B(R) \times \mathbb{R} \to B(S) \times \mathbb{R}$ by $(t, u) \mapsto (\phi(t), \frac{\beta}{\alpha}u)$. Restricted to a depth zero vertex space X_x, this map expands distances by $\frac{\beta}{\alpha}\exp(h(x) - h(\Phi_{\#}(x)))$.

Geodesics in X_R and Y_S can be approximated within bounded multiplicative error by paths in which only one coordinate changes at a time, so Φ is bilipschitz on $\mathcal{V}_0 X_R$.

The constants δ, M, m, K, and the valence bound on vertical components are invariants of the equivalence classes of R and S, so the multiplicative constants of Φ depend on $[R]$, $[S]$, $\mathcal{E}$,

$\frac{\beta}{\alpha}$. The union of vertex spaces is dense in a P-set space, so Φ gives a quasi-isometry of P-set spaces. □

We have a similar result for P-sets of unbounded height change.

Lemma 3. *Let $G_1 = \pi_1(\Gamma_1)$ and $G_2 = \pi_1(\Gamma_2)$ be two tubular groups. Let $R \subset D\Gamma_1$ and $S \subset D\Gamma_2$ be P-sets of unbounded height change. Let $x_0 \in R$ and $y_0 \in S$ be depth zero vertices. Let $[x_1]_R, .., [x_a]_R$ and $[y_1]_S, .., [y_b]_S$ be lists of the types of vertices which occur in R and S, respectively. Let $\mathcal{E} = (m_{ij})$ be an extension for $([R], [S])$. Let α and β be arbitrary positive real numbers. Then there is a quasi-isometry $\Phi = \Phi^{\infty}_{\mathcal{E},(R,x_0,\alpha),(S,y_0,\beta,)} : X_R \to Y_S$ with the following properties:*

1. *$\Phi(X_{x_0}) = Y_{y_0}$, and this map is, up to isometry, just a homothety with expansion by $\frac{\beta}{\alpha}$.*
2. *Φ takes depth zero vertex spaces to depth zero vertex spaces, and induces a bijection, $\Phi_{\#}$, between depth zero vertices of R and depth zero vertices of S.*
3. *Up to isometric permutation of the affine patterns, $\Phi|_{X_v}$ is a homothety with uniformly bounded expansion factor.*
4. *Excluding x_0 and y_0, there exist vertices of type $[x_i]_R$ mapping to vertices of type $[y_j]_S$ iff m_{ij} is non-zero.*
5. *Φ is a bilipschitz bijection $(\mathcal{V}_0 X_R, d_{X_R}|_{\mathcal{V}_0 X_R}) \to (\mathcal{V}_0 Y_S, d_{Y_S}|_{\mathcal{V}_0 Y_S})$.*

Proof. The proof is almost exactly the same as the proof of Lemma 2 except for the extension along vertical components. In the previous Lemma, only the vertex types mattered, and vertex type determined vertex height. In the unbounded height change case, vertices of every type occur with unbounded height.

For $i = 1, 2$, let δ_i be the diameter of the projection of the P-set to the graph Γ_i, and let μ_i be the largest height change across an edge in the projection. Let

$$K = \sum_i 4\mu_i + 3 + 2\delta_i\mu_i$$

Take the height error bounds M and m from the previous Lemma to be $M = \exp(K)$ and $m = \exp(-K)$.

After sliding edges, vertical components, $V \subset B(R)$ and $W \subset B(S)$, will be trees such that there is a K-coarsely height preserving quasi-isometry between them. This fact is provided by Lemma 5, which is proved in Appendix A.

Let ξ_{x_0} and ξ_{y_0} be the basepoints for V and W, respectively. There is some vertex ξ_y in W at bounded distance from ξ_{y_0} such that $|h(\xi_{y_0}, \xi_y) + \ln\frac{\beta}{\alpha}| \leq \mu_2$. Let ϕ be a K-coarsely height preserving quasi-isometry between V and W that maps ξ_{x_0} to ξ_y. Let ϕ' be equal to ϕ for all points other than ξ_{x_0}, and define $\phi'(\xi_{x_0}) = \xi_{y_0}$. This makes ϕ' a quasi-isometry, and if the height error of ξ_{x_0} is known to be $-\ln\frac{\beta}{\alpha}$ then the height error of all other vertices is at most $K + \mu_2$.

It is important to note that while $\frac{\beta}{\alpha}$ affects the quasi-isometry constants, the height error bounds for all vertices other than ξ_{x_0} depend only on the δ_i and μ_i, which are invariants of $[R]$ and $[S]$.

As in the previous Lemma, ϕ', can then be changed a bounded amount to give a bijection of depth zero vertices which respects the partitions by type. This map will still have bounded height error, since moving by a bounded amount produces a bounded height change, and this is the map we use to extend along vertical components.

□

4.3.2 Building Quasi-isometries of Tubular Groups

The tree of P-sets of a tubular group throws out all the metric information from within the P-sets. Vertices of $D\Gamma$ which belong to a common P-set are distance 1 in T, while in X there is no bound on the distance between vertex spaces in a common P-set. Thus, it would be too much to ask that an allowable isomorphism of trees of P-sets lift to a quasi-isometry of groups. However, the existence of such an isomorphism does gives information about how to build a quasi-isometry.

Theorem 1 gives a set of strategies for building a new allowable isomorphism. The following Theorem shows that this set of strategies can actually be used to build a quasi-isometry between the tubular groups. Lemmas 2 and 3 provide the basic building blocks, and the quasi-isometry constants are controlled by controlling height error. Two-line, depth zero vertices and P-sets of unbounded height change are flexible enough that height error will not be an issue. The danger of compounding height error comes from P-sets of bounded height change connected by depth zero vertices with at least three lines, but these height errors are controlled by the set of strategies.

If $\chi : X \to Y$ takes vertex spaces to vertex spaces and P-set spaces to P-set spaces, let $\chi_\#$ denote the induced map $T_1 \to T_2$.

Theorem 2. *For $i = 1, 2$, let $G_i = \pi_1(\Gamma_i)$ be a tubular group and T_i its tree of P-sets. There is an allowable isomorphism $\psi : T_1 \to T_2$ if and only if there is a quasi-isometry $\Phi : G_1 \to G_2$.*

Proof. In Section 4.1 we saw that quasi-isometries induce allowable isomorphisms.

Assume that $\psi : T_1 \to T_2$ is an allowable isomorphism.

In the proof of Theorem 1 we demonstrated how to derive a consistent set of strategies from a given allowable isomorphism of trees of P-sets. Let $\mathcal{G}$ be such a set of strategies derived from ψ. Let $\mathcal{M}_i$ be the matches of P-sets of bounded height change occurring in $\mathcal{G}$, and let $\{U_i\}$ and $\{L_i\}$ be solutions to the set of inequalities from the algorithm.

Choose a uniform K such that, for any P-set R of unbounded height change, there is a quasi-isometry $\Phi^\infty : X_R \to Y_\psi(R)$ as in Lemma 3 with height error bounded by K.

In $D\Gamma_1$ and $D\Gamma_2$ there are finitely many equivalence classes of P-sets of bounded height change, so there is some maximum height change, $\frac{J}{2}$, that can occur between depth zero vertices in such a P-set.

Assume the U_i are greater than $K + J$ and the L_i are less than $-J - K$.

Pick any P-set vertices R_0 of T_1 and S_0 of T_2 such that $([R_0], [S_0])$ is the match of the level 1 vertex of $\mathcal{G}$. Set $\phi(R_0) = S_0$.

Base Case 1

If R_0 is of bounded height change, pick a depth zero vertex $x_0 \in R_0$ of maximal height. Declare x_0 to be the basepoint, so x_0 has height zero and all other heights are determined relative to x_0.

Also assume that $\mathcal{M}_1 = ([R_0], [S_0])$.

The level 1 vertex of $\mathcal{G}$ has a non-decreasing strategy, $\mathcal{S}$, with an extension, $\mathcal{E}$, for $([R_0], [S_0])$. If x_0 is of type $\{[R_0], i\}$ find some j such that the i, j entry of $\mathcal{E}$ is nonzero and choose a depth zero vertex $y_0 \in S_0$ of type $\{[S_0], j\}$.

Declare y_0 to be height zero, and define heights in S_0 relative to y_0.

With these choices, $h(R_0) = 0$, and $h(S_0) \geq 0$, so the height error of R is in $[-\frac{J}{2}, 0]$.

Let $\Phi_{R_0} = \Phi^0_{\mathcal{E},(R_0,x_0,1),(S_0,y_0,1)}$ from Lemma 2.

Define $\Phi|_{X_{R_0}} = \Phi_{R_0}$.

Φ_{R_0} induces a map $(\Phi_{R_0})_\# : N_{\frac{1}{2}}(R_0) \to N_{\frac{1}{2}}(S_0)$ which is a tree isomorphism. Extend ϕ from R_0 according to $\mathcal{E}$, using the bijection of depth zero vertices furnished by $(\Phi_{R_0})_\#$.

The height errors of the depth zero vertices of R_0 is in $[-J, \frac{J}{2}]$.

Base Case 2

If R_0 is a P-set of unbounded height change pick a depth zero vertex, $x_0 \in \{[R_0], 1\}$, and declare its height to be zero. Height relative to x_0 determines heights for all the other depth zero vertices of R_0.

The level 1 vertex of $\mathcal{G}$ has a strategy, $\mathcal{S}$, with an extension, $\mathcal{E}$, for $([R], [S])$. If x_0 is of type $\{[R_0], i\}$ find some j such that the i, j entry of $\mathcal{E}$ is nonzero, and choose a depth zero vertex

$y_0 \in S_0$ of type $\{[S_0], j\}$. Declare the height of y_0 to be zero, and define height in S_0 relative to y_0.

Let $\Phi_{R_0} = \Phi^{\infty}_{\mathcal{E},(R_0,x_0,1),(S_0,y_0,1)}$ from Lemma 3.

Define $\Phi|_{X_{R_0}} = \Phi_{R_0}$.

Extend ϕ from R_0 according to $\mathcal{E}$, using the bijection of depth zero vertices furnished by $(\Phi_{R_0})_{\#}$.

By construction, the height errors of the depth zero vertices of R_0 are in $[-K, K]$.

Induction Steps

Suppose Φ has been defined for a P-set space X_R, and ϕ has been defined on $N_1(R)$. Let R' be a P-set vertex at distance 1 from R, and let v be the depth zero vertex joining R to R'.

There are a number of situations to consider, depending on whether R and R' are of bounded or unbounded height change, and whether v has 2 lines or more than 2 lines.

In each of these cases, we give a quasi-isometry $\Phi_{X_{R'}}$ from the P-set space of R' to the P-set space of $\phi(R')$ according to some extension $\mathcal{E}$.

To finish each case, define $\Phi|_{X_{R'}} = \Phi_{R'}$, and extend ϕ from R' according to $\mathcal{E}$, using the bijection of depth zero vertices furnished by $(\Phi_{R'})_{\#}$.

v has more than 2 lines

In these cases, R and R' are in the same rigid component, so height in R' is already defined. The same goes for $\phi(R)$ and $\phi(R')$

R and R' of bounded height change

Suppose $([R],[\phi(R)]) = \mathcal{M}_i$ and $([R'],[\phi(R')]) = \mathcal{M}_j$. If the height error of R was in $[L_i, U_i]$, then the height error of R' is in $[L_j, U_j]$.

If the height error of R' is positive, let $\mathcal{S}$ be the non-increasing strategy for $\mathcal{M}_j$ in $\mathcal{G}$. If the height error of R' is non-positive, let $\mathcal{S}$ be the non-decreasing strategy for $\mathcal{M}_j$ in $\mathcal{G}$.

The strategy $\mathcal{S}$ has an extension, $\mathcal{E}$, for $\mathcal{M}_j$.

Let $\alpha = \exp(-h(v))$; let $\beta = \exp(-h(\phi(v)))$.

Let $\Phi_{R'} = \Phi^0_{\mathcal{E},(R',v,\alpha),(\phi(R'),\phi(v),\beta)}$ from Lemma 2.

The height errors of depth zero vertices of R' are in $[L_j - J, U_j + J]$.

R of unbounded height change, R' of bounded height change

Suppose $([R'],[\phi(R')]) = \mathcal{M}_j$. Since R was of unbounded height change, we can assume that the height error of v is in $[-K, K]$. Thus, the height error of R' is in $[-K - J, K + J]$.

Proceed as in the previous case.

The height errors of depth zero vertices of R' are in $[-K - J, K + J]$.

R' of unbounded height change, both cases

There is only one strategy for $([R'],[\phi(R')])$ in $\mathcal{G}$. Let $\mathcal{E}$ be the extension given by this strategy.

Let $\alpha = \exp(-h(v))$; let $\beta = \exp(-h(\phi(v)))$.

Let $\Phi_{R'} = \Phi^0_{\mathcal{E},(R',v,\alpha),(\phi(R'),\phi(v),\beta)}$ from Lemma 3.

By construction, the height errors of depth zero vertices of R' (other than v) are in $[-K, K]$.

v has 2 lines

In these cases, R and R' are in different rigid components. Define height in R' relative to v and define height in $\phi(R')$ relative to $\phi(v)$.

R' of bounded height change

The P-sets R' and $\phi(R')$ have heights defined relative to v and $\phi(v)$, respectively, and the height error of R is in $[-J, J]$. If the height error of R is positive, let $\mathcal{S}$ be the non-increasing strategy of $\mathcal{G}$ for $([R'], [\phi(R')])$. If the height error of R is non-positive, choose the non-decreasing strategy. Let $\mathcal{E}$ be the extension given by $\mathcal{S}$.

Let $\Phi_{R'} = \Phi^0_{\mathcal{E},(R',v,1),(\phi(R'),\phi(v),1)}$.

The height errors of depth zero vertices of R' are in $[-J, J]$.

R' of unbounded height change

There is only one strategy for $(R', \phi(R'))$ given in $\mathcal{G}$. Let $\mathcal{E}$ be the extension associated to this strategy.

Let $\Phi_{R'} = \Phi^\infty_{\mathcal{E},(R',v,1),(\phi(R'),\phi(v),1)}$.

By construction, the height errors of the depth zero vertices of R' are in $[-K, K]$.

This completes the induction steps.

The various maps Φ_R were constructed so that they agree when they overlap on any depth zero vertex spaces of at least three lines. They may not agree on two-line vertex spaces, but this can be fixed, since two-line patterns are not rigid.

Suppose v is a two-line vertex joining P-sets R and R'. Suppose $\Phi_R|_{X_v}$ is expansion by a factor C, and $\Phi_{R'}|_{X_v}$ is expansion by a factor C'. Change these maps so that they both expand by a factor of C along the lines where the R' edges glue on, and expand by a factor of C' along the lines where the R edges glue on. This does not change the maps that we had on $B(v) \subset B(R)$ and $B(v) \subset B(R')$.

The map Φ is built from various pieces Φ_R. The Φ_R are bilipschitz bijections on $\mathcal{V}_0 X_R$, and agree on vertex spaces where their domains intersect. We have arranged that the bilipschitz constants of Φ_R depend on $[R]$ and $[\phi(R)]$ and on height error. Tubular groups come from finite graphs of groups, so there are only finitely many equivalence classes of P-set. Thus, invariants of equivalence classes of P-sets can be uniformly bounded. Height error is bounded by $J + \max\{U_i\} \cup \{|L_i|\}$.

The map Φ is thus a bilipschitz bijection $\mathcal{V}_0 X \to \mathcal{V}_0 Y$. The set $\mathcal{V}_0 X$ is dense in X, and $\mathcal{V}_0 Y$ is dense in Y, so $\Phi : X \to Y$ is a quasi-isometry, and $\phi = \Phi_\#$ is an allowable tree isomorphism. $\square$

CHAPTER 5

CONSEQUENCES AND EXAMPLES

5.1 Lines of Maximum Slope

Suppose G is a tubular group with at least 3 lines in every vertex and having bounded height change in every P-set. Then there is a notion of height change for every edge in the tree of P-sets.

If γ is a geodesic ray in T_G, γ *has coarse slope* m, $slope(\gamma) = m$, if there exists a $C > 0$ such that for all t,

$$tm - C \leq h(\gamma(0), \gamma(t)) \leq tm + C$$

Proposition 11. *There is some* $m_G = \max_{\gamma \subset T_G} slope(\gamma)$*, and the set of slopes of geodesic rays of* T_G *is dense in the interval* $[-m_G, m_G]$.

Proof. Let γ be a geodesic ray in T_G. Consider the image of γ in the graph of P-sets. The graph of P-sets has only finitely many vertices, so the image of γ can be written as some initial segment of bounded length followed by loops in the graph. The initial segment contributes a bounded amount to height, so it can be discarded. Thus, it suffices to consider products of loops in the graph of P-sets.

Suppose $e_1, \ldots, e_n$ are the edges of the graph, and let $\alpha = \prod e_i^{a_i}$ and $\beta = \prod e_i^{b_i}$ be loops.

$$slope(\alpha) = \frac{\sum a_i h(e_i)}{\sum |a_i|}$$

and similarly for β.

If, for each i, $|a_i + b_i| = |a_i| + |b_i|$, then α and β *add without cancellation.* In this case,

$$slope(\alpha\beta) = \frac{\sum a_i h(e_i) + \sum b_i h(e_i)}{\sum |a_i| + \sum |b_i|} \leq \max\{slope(\alpha),\, slope(\beta)\}$$

Suppose $\gamma = \prod e_i^{c_i}$ is a loop. If any c_j is larger than 1, rewrite γ as a product of loops α and β which add without cancellation and with $a_j = 1$ and $b_j = c_j - 1$. Then

$$slope(\gamma) \leq \max\{slope(\alpha),\, slope(\beta)\},$$

so it would be possible to reduce γ without reducing its slope. Thus, it is possible to choose a loop, γ, of maximal slope from the finitely many loops of the form $\prod e_i^{c_i}$ with $c_i \in \{0, \pm 1\}$.

Loops which are homotopically trivial have 0 height change, so we can get a geodesic ray in T_G of slope arbitrarily close to any $m \in [-m_G, m_G]$ by concatenating lifts of $\pm\gamma$ with lifts of trivial loops. □

Corollary 3. *There is a maximum coarse slope in each rigid component of T_G which contains no P-set of unbounded height change.*

Suppose G_1 and G_2 are tubular groups with at least three lines in every vertex and having bounded height change in every P-set. Also, suppose that G_1 and G_2 have the same sets of equivalence classes of affine patterns in their vertices, and that each of these patterns has a unique (up to isometry) symmetric representative. This occurs, for instance, if every vertex has 3 or 4 lines.

Suppose $m_{G_1} > m_{G_2}$, and γ is a geodesic ray of slope m_{G_1} in T_{G_1}. Under a coarsely height preserving tree isomorphism, the image of γ will still be a geodesic ray of slope m_{G_1}, contradicting maximality of m_{G_2}. This means m_G is a quasi-isometry invariant of G.

When vertices of G have affine edge patterns without a unique symmetric representative, the value of m_G depends on the choice of representatives.

5.2 Examples

Having the same maximum slope is a necessary, but not sufficient condition for determining if the groups are quasi-isometric, as illustrated by the following example.

Example 5. Suppose G_1 and G_2 are groups as in Example 2. Each G_i has only one equivalence class of P-set, $[C_i]$, and each P-set has three vertex types.

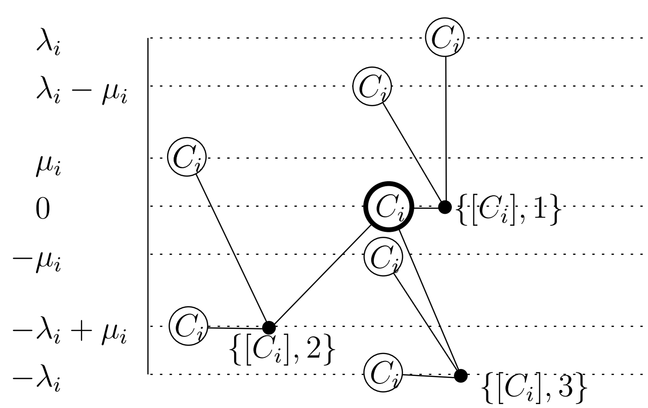

Figure 7. Vertex types and relative heights for $[C_i]$.

Assume that $\lambda_i \geq \mu_i \geq 0$, and that $\lambda_i > 0$. λ_i is the maximum slope in T_i, so assume that $\lambda_1 = \lambda_2$.

First we attempt to build an extension $\mathcal{E} = (m_{jk})$ that gives a non-decreasing strategy.

The largest negative height change that occurs is $-\lambda$ in both trees, so we must identify these to get a non-decreasing strategy. In each case, if C_i is a P-set of T_i, then any other P-set C_i' of T_i with $h(C_i, C_i') = -\lambda$ connects to C_i through a vertex of type $\{[C_i], 3\}$ with respect to C_i. Thus, m_{33} must be non-zero. Furthermore, if we choose this entry to identify the P-sets at height $-\lambda$, we are forced to identify P-sets at height $-\mu_1$ with P-sets of height $-\mu_2$. Thus, a non-decreasing strategy exists only if $\mu_1 \leq \mu_2$.

We get the reverse inequality by considering a non-increasing strategy. The largest positive height change that occurs is λ in both trees, so we must identify these to get a non-increasing strategy. This forces an identification of P-sets at height $\lambda - \mu_1$ with P-sets of height $\lambda - \mu_2$, so a non-increasing strategy exists only if $\mu_1 \geq \mu_2$.

Thus, if $\mu_1 \neq \mu_2$, the two groups can not be quasi-isometric.

When $\mu_1 = \mu_2$, define an extension $\mathcal{E}$ by

$$m_{11} = m_{22} = m_{33} = \left\{ \begin{array}{c} \{[C_1], 1\} \mapsto \{[C_2], 1\} \\ \\ \{[C_1], 2\} \mapsto \{[C_2], 2\} \\ \\ \{[C_1], 3\} \mapsto \{[C_2], 3\} \end{array} \right\}$$

Let all other $m_{ij} = 0$.

This extension gives rise to a strategy which is both non-increasing and non-decreasing. The set of strategies consists of just this one strategy, since there is only one match. The groups are then quasi-isometric. ♦

Example 6. Consider two tubular groups all of whose depth zero vertices have two lines, and all of whose P-sets are of bounded height change. There is no obstruction to building a consistent set of strategies for two such groups, so they are quasi-isometric. In particular, Right Angled Artin Groups whose defining graphs are trees of diameter at least three fall into this category. This recovers a result of Behrstock and Neumann [1]. ♦

The next example illustrates a case where building a quasi-isometry requires distinct non-increasing and non-decreasing strategies.

Example 7. Let $G_1 = \pi_1(\Gamma_1)$ be a one torus group as in Example 5 with height changes $\lambda > \mu > 0$ across the two edges.

Let $G_2 = \pi_1(\Gamma_2)$ be the group from Example 4.

In $D\mathit{\Gamma}_1$ there is only one equivalence class of P-set, labeled C. The P-sets in class $[C]$ have three types of vertices.

In $D\mathit{\Gamma}_2$ there are two equivalence classes of P-sets, labeled A and B. The P-sets in class $[B]$ have two types of vertices. The P-sets in class $[A]$ have four types of vertices.

For $([C],[A])$, consider the extension $\mathcal{E}_A = (m_{ij})$ defined as follows:

$$m_{11} = \begin{Bmatrix} \{[C],1\} \mapsto \{[A],1\} \\ \{[C],2\} \mapsto \{[A],2\} \\ \{[C],3\} \mapsto \{[B],2\} \end{Bmatrix} \quad m_{22} = \begin{Bmatrix} \{[C],1\} \mapsto \{[A],1\} \\ \{[C],2\} \mapsto \{[A],2\} \\ \{[C],3\} \mapsto \{[B],2\} \end{Bmatrix}$$

$$m_{23} = \begin{Bmatrix} \{[C],1\} \mapsto \{[B],2\} \\ \{[C],2\} \mapsto \{[A],3\} \\ \{[C],3\} \mapsto \{[A],4\} \end{Bmatrix} \quad m_{34} = \begin{Bmatrix} \{[C],1\} \mapsto \{[B],1\} \\ \{[C],2\} \mapsto \{[A],3\} \\ \{[C],3\} \mapsto \{[A],4\} \end{Bmatrix}$$

All other $m_{ij} = 0$.

This extension says that given P-sets C and A, we should split the vertices of type $\{[C],2\}$ with respect to C and map half of them to the vertices of type $\{[A],2\}$ with respect to A and the other half to vertices of type $\{[A],3\}$ with respect to A.

Figure 8 shows the result of extending by $\mathcal{E}_A$.

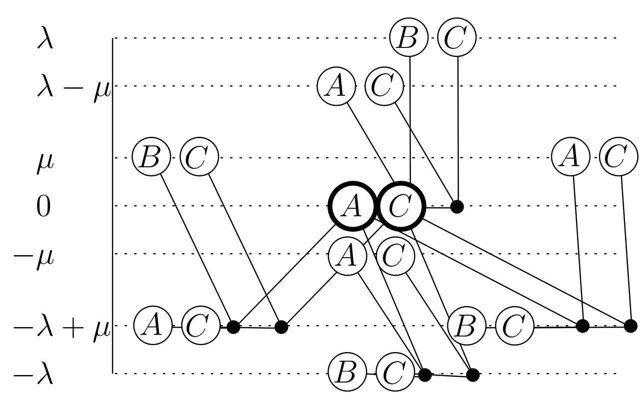

Figure 8. Extension by $\mathcal{E}_A$.

All the matches induced by this extension have height error 0, so the strategy $\mathcal{S}_A$ for $\mathcal{E}_A$, shown in Figure 9, is both non-increasing and non-decreasing.

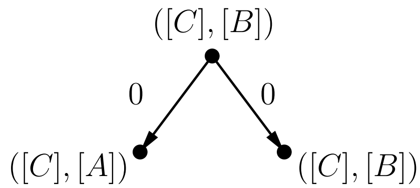

Figure 9. Strategy for $([C],[A])$

For $([C],[B])$ we define extensions $\mathcal{E}_B^+$ in Figure 10 and $\mathcal{E}_B^-$ in Figure 13. A P-set in class $[B]$ has neighboring P-sets only in class $[A]$, so strategies for $([C],[B])$ are trivially non-increasing and non-decreasing. However, we will need two different strategies here to produce a consistent set of strategies.

Figure 11 shows the result of extending by $\mathcal{E}_B^+$.

The extension $\mathcal{E}_B^+$ induces matches $([C],[A])$ with height errors 0, μ, and 2μ, so it gives rise to the non-decreasing strategy $\mathcal{S}_B^+$ in Figure 12.

Figure 14 shows the result of extending by $\mathcal{E}_B^-$.

This extension induces matches $([C],[A])$ with height errors 0, $-2(\lambda-\mu)$, $\lambda-\mu$, so it gives rise to the non-increasing strategy $\mathcal{S}_B^-$ in Figure 15.

$$\begin{pmatrix} \left\{\begin{matrix} \{[C],1\} \mapsto \{[B],1\} \\ \{[C],2\} \mapsto \{[A],3\} \\ \{[C],3\} \mapsto \{[A],4\} \end{matrix}\right\} & 0 \\ 0 & \left\{\begin{matrix} \{[C],1\} \mapsto \{[A],1\} \\ \{[C],2\} \mapsto \{[B],2\} \\ \{[C],3\} \mapsto \{[A],2\} \end{matrix}\right\} \\ 0 & \left\{\begin{matrix} \{[C],1\} \mapsto \{[A],1\} \\ \{[C],2\} \mapsto \{[A],2\} \\ \{[C],3\} \mapsto \{[B],2\} \end{matrix}\right\} \end{pmatrix}$$

Figure 10. The extension $\mathcal{E}_B^+$.

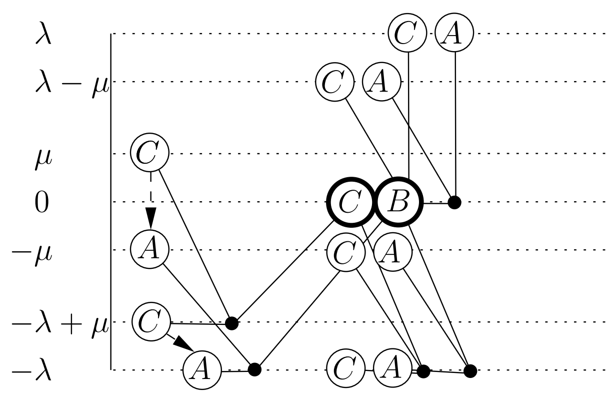

Figure 11. Extension by $\mathcal{E}_B^+$.

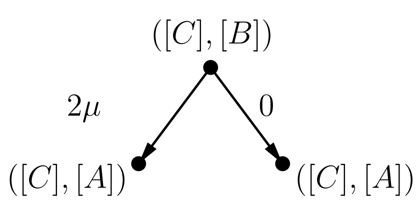

Figure 12. Non-decreasing strategy for $([C],[B])$

$$\begin{pmatrix} \left\{\begin{matrix} \{[C],1\} \mapsto \{[B],1\} \\ \{[C],2\} \mapsto \{[A],3\} \\ \{[C],3\} \mapsto \{[A],4\} \end{matrix}\right\} & 0 \\ \left\{\begin{matrix} \{[C],1\} \mapsto \{[A],1\} \\ \{[C],2\} \mapsto \{[B],2\} \\ \{[C],3\} \mapsto \{[A],2\} \end{matrix}\right\} & 0 \\ 0 & \left\{\begin{matrix} \{[C],1\} \mapsto \{[A],1\} \\ \{[C],2\} \mapsto \{[A],2\} \\ \{[C],3\} \mapsto \{[B],2\} \end{matrix}\right\} \end{pmatrix}$$

Figure 13. The extension $\mathcal{E}_B^-$.

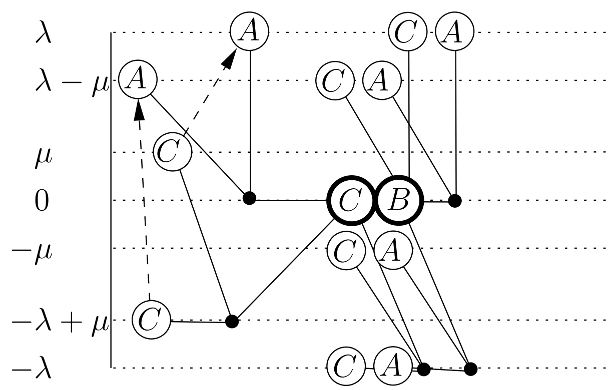

Figure 14. Extension by $\mathcal{E}_B^-$.

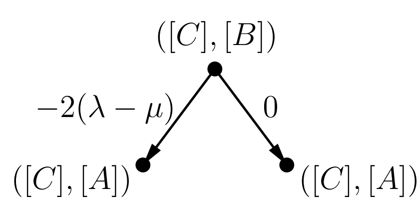

Figure 15. Non-increasing strategy for $([C],[B])$

These strategies give the set of strategies in Figure 16.

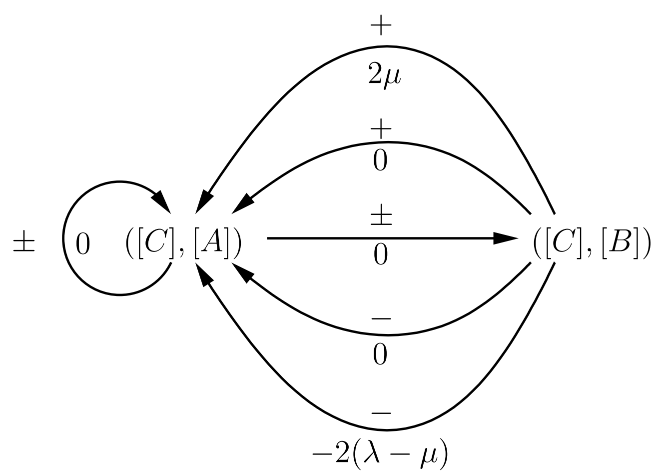

Figure 16. The set of strategies

This set of strategies gives rise to a system of inequalities:

$M_1 \leq U_1$	$U_1 + 0 \leq U_2$	$M_1 + 0 \leq U_2$	$U_2 - 0 \leq U_1$	$M_2 + 0 \leq U_2$
$L_1 \leq M_1$	$M_1 + 0 \geq L_2$	$L_1 + 0 \geq L_2$	$M_2 - 0 \geq L_1$	$L_2 + 0 \geq L_2$
$M_2 \leq U_2$	$U_1 + 0 \leq U_1$	$M_1 + 0 \leq U_1$	$U_2 - 2(\lambda - \mu) \leq U_1$	$M_2 + 2\mu \leq U_1$
$L_2 \leq M_2$	$M_1 + 0 \geq L_1$	$L_1 + 0 \geq L_1$	$M_2 - 2(\lambda - \mu) \geq L_1$	$L_2 + 2\mu \geq L_1$

This system has solutions, for instance:

$$U_1 = U_2 = 2\lambda$$

$$M_1 = M_2 = 0$$

$$L_1 = L_2 = -2\lambda$$

so groups of this form are quasi-isometric. ♦

APPENDIX

HALL'S SELECTION THEOREM AND SOME LEMMA'S ABOUT TREES

This chapter contains details on the use of non-amenability and Hall's Selection Theorem needed in Lemma 2.

Theorem (Hall's Selection Theorem)**.** *[12] Let $\alpha : X \to FIN(Y) =$ finite subsets of Y. There exists $\phi : X \to Y$ an injection with $\phi(x) \in \alpha(x)$ iff for all finite $S \subset X$, $|S| \leq | \cup_{s \in S} \alpha(s)|$*

When $n = 1$, the following Lemma is a special case of Theorem 4.1 of [12].

Lemma 4. *Let X and Y be δ-bushy trees with edges of length 1, both with variable valence at most b. Suppose $f : X \to Y$ is a quasi-isometry. Given finitely many δ-coarsely dense subsets $X_1, \dots, X_n \subset X$ and $Y_1, \dots, Y_n \subset Y$, there exists $\chi : \cup X_i \to \cup Y_i$ a bilipschitz bijection which respects the partitions, $\chi(X_i) \subset Y_i$ for all i. The map χ extends to a quasi-isometry $X \to Y$. Furthermore, χ is bounded distance from f, and the quasi-isometry constants depend only on b, δ, and the quasi-isometry constants of f.*

Proof. X and Y are both quasi-isometric to the Cayley graph of the free group of rank 2, so they are both non-amenable. Suppose f is a (λ, ϵ)-quasi-isometry.

For each i, X_i and Y_i are dense, so, for each i, there is a quasi-isometry $f_i : X_i \to Y_i$ which is distance at most δ from f. The map f_i is a $(\lambda, 2\delta(\lambda + \epsilon))$-quasi-isometry.

APPENDIX (Continued)

Let $C = 1 + b\left(\frac{(b-1)^{\delta(\lambda+\epsilon)}-1}{b-2}\right)$. This is the maximum number of vertices in a ball of diameter $2\delta(\lambda+\epsilon)$ in X. For any $S \in FIN(X_i)$, $|S| \leq C|f_i(S)|$.

The tree Y is nonamenable, so by Lemma 2.1 of [12], for any K there exists an r_K, such that for any $S \in FIN(Y)$, $K|S| \leq |N_{r_K}(S)|$. Furthermore, r_K depends only on K and b.

Here $N_r(S)$ is the closed r-neighborhood of S in Y, and $N_r^i(S) = N_r(S) \cap Y_i$.

There are at most $a = 1 + b\frac{(b-1)^{\delta}-1}{b-2}$ points in a ball of radius δ in Y. For any $R > 2\delta$,

$$|N_{R-\delta}(S)\backslash S| - a|S| \leq |N_{R-\delta}(S)\backslash N_\delta(S)| \leq a|N_R^i(S)\backslash S|$$

so, for any $R > \delta$,

$$|N_R(S)| \leq (a+1)|N_{R+\delta}^i(S)|$$

Let $K = C(a+1)$, and let $r = r_K + \delta$. Notice that r_K depends only on b, δ, λ, and ϵ.

For any i and any $S \in FIN(Y_i) \subset FIN(Y)$,

$$\begin{aligned} C|S| &= \frac{K}{(a+1)}|S| \\ &\leq \frac{1}{(a+1)}\,|N_{r_K}(S)| \\ &\leq \frac{1}{(a+1)}\left((a+1)\left|N_{r_K+\delta}^i(S)\right|\right) \\ &= \left|N_r^i(S)\right| \end{aligned}$$

APPENDIX (Continued)

Define $\alpha_i : X_i \to FIN(Y_i)$ by $\alpha_i(x) = N_r^i(f_i(x))$. For any $S \in FIN(X_i)$,

$$|S| \leq C|f_i(S)| \leq |N_r^i(f_i(S))| = | \cup_{s\in S} \alpha_i(s)|$$

Hall's Selection Theorem says there is an injection $\phi_i : X_i \to Y_i$ with $\phi_i(x) \in N_r(f_i(x))$. The X_i are disjoint, so the collection of ϕ_i gives an injection $\phi : \cup X_i \to \cup Y_i$ which respects the partition and is distance at most $\beta = r + \delta$ from f.

A similar process for $g = f^{-1}$ yields an injection $\psi : \cup Y_i \to \cup X_i$ distance at most β from g and respecting the partitions.

The Schroeder-Bernstein Theorem then gives a bijection $\chi : \cup X_i \to \cup Y_i$, which, by construction, respects the partitions. For $a \in \cup X_i$, $\chi(x)$ is either $\phi(x)$ or $\psi^{-1}(x)$. On one hand, $\phi(x)$ is distance at most β from $f(x)$. On the other hand, if $\psi(y) = x$, and $\chi(x) = y$, then

$$\begin{aligned} d_Y(\chi(x), f(x)) &= d_Y(y, f(x)) \\ &\leq b^2 d_X(g(y), g(f(x))) \\ &= b^2 d_X(g(y), x) \\ &= b^2 d_X(g(y), \psi(y)) \\ &\leq b^2\beta \end{aligned}$$

So in either case, χ is within $b^2\beta$ of f. Furthermore, since χ is a bijective quasi-isometry, it is bilipschitz.

Take any $z \in X \setminus \cup X_i$. The set $\cup X_i$ is δ-coarsely dense in X, so there is some x in $\cup X_i$ with $d_X(x, z) \leq \delta$. Define $\chi(z) = \chi(x)$. This extends χ to be a quasi-isometry of all of X, and

$$d_Y(\chi(z), f(z)) \leq d_Y(f(z), f(x)) + d_Y(f(x), \chi(x)) \leq \lambda\delta + \epsilon + b^2(r_K + 2\delta)$$

□

Let G be a finite, connected graph with directed edges. G may have edges with the same initial and terminal vertex, and may have multiple edges between a pair of vertices. Associate a height change to each edge. Suppose there is a loop in G such that the sum of the height changes across edges of the loop is strictly positive.

Consider a bounded valence tree with directed edges covering G, $V \xrightarrow{p} G$, such that for every vertex $v \in V$, the edges coming in to v cover the edges coming in to $p(v)$ at least two to one, and similarly for the outgoing edges. If G is not just a single vertex with a single edge, then this condition can be relaxed slightly. If there is an edge e and a vertex v in G such that v is the initial and terminal vertex for e, then at a vertex in $p^{-1}(v)$ there need be only one incoming and one outgoing edge covering e.

The height change of an edge in V is the height change of its image in G.

APPENDIX (Continued)

The following Lemma and its proof are a generalization of Whyte's proof [13] of the quasi-isometry classification of graphs of $\mathbb{Z}$'s. The situation described above excludes the cases of a tree with constant height function and the Bass-Serre tree of a solvable Baumslag-Solitar group.

Whyte's techniques will apply to show that there is a coarsely height preserving quasi-isometry between V and a tree, V', where every vertex has at least two incident edges which increase height, and at least two incident edges which increase height.

If V' were homogeneous then we would take a lamination by lines of constant slope, and build a quasi-isometry line-by-line between our tree and the $(2, 2)$-homogeneous tree.

Since V' need not be homogeneous, we will instead take a lamination of V' by lines with a fixed relationship between height change along the line and number of incident edges along the line. Such lines will still have slope bounded above and below, so we can still build a line-by-line quasi-isometry between V' and the $(2, 2)$-homogeneous tree.

The idea here is that if a line in V' starts accumulating too many incident edges, decrease its slope. Then a height preserving quasi-isometry to a line in the homogeneous tree will stretch distance, thus spreading out the incident edges.

Lemma 5. *Let $V \xrightarrow{p} G$ and $W \xrightarrow{q} H$ be two trees as described above. There is a coarsely height preserving quasi-isometry between V and W. Furthermore, the quasi-isometry and coarseness constants can be bounded in terms of information from G and H and the valence bounds for V and W.*

Proof. Assume W is the $(2,2)$-homogeneous tree, that is, the tree which has at every vertex two edges which increase height by one, and two edges which decrease height by one. For the general statement of the Lemma it suffices to compose two instances of this special case.

Let M be the maximum height change across an edge of G.

Let δ be the diameter of G.

Let N be the number of vertices of G.

Let L be the maximum valence of a vertex in V.

As in Lemma 3.1 of [13], take a maximal subtree of G and a family of lifts of this subtree to V such that every vertex of V belongs to exactly one lift in the family. Collapsing these subtrees is a (δ, δ)-quasi-isometry which is $2\delta M$-coarsely height preserving. By assumption, there was a loop in G which strictly increased height, so every vertex of the tree has at least one edge which increases height and one which decreases height. By construction, either there are, at every vertex, at least two edges which increase height and two edges which decrease height, or there are edges with zero height change which can be collapsed to make this true. The resulting tree is V', which has valence less than NL.

Let $b = NL - 2$. Let $0 < \alpha \leq 1$ be some number such that every vertex of V' has at least two edges which increase height by at least α and two edges which decrease height by at least α. Such an α exists since there were only finitely many possible height changes across an edge of G.

Claim. There exists a constant $J > 0$ and a lamination of V' by lines r such that

$$\left| h(r(0), r(n)) - \frac{\alpha}{b} E(0, n) \right| \leq J$$

where $E(0, n)$ denotes the number of edges incident to the segment $[r(0), r(n)]$.

Assuming the claim, let $\beta = 2\frac{\alpha}{b}$ and take a lamination of W by lines of constant slope $(\beta, 2)$ as in Theorem 2.4 of [13].

For a line r in the lamination of V',

$$2(n + 1) \leq E(0, n) \leq b(n + 1)$$

so

$$2\frac{\alpha}{b}n + 2\frac{\alpha}{b} - J \leq h(r(0), r(n)) \leq \alpha n + \alpha + J$$

Thus, the slope of r is bounded above and also away from zero, so there is a height preserving quasi-isometry between r and a line of constant slope in W.

Let $w(x, y)$ be the number of vertices adjacent to a line of the lamination of W with height between x and y. Define $v(x, y)$ similarly for V.

$$\left| w(x, y) - \frac{b}{\alpha}(y - x) \right| \leq (4\frac{b}{\alpha} + 2)$$

$$\left| v(x,y) - \frac{b}{\alpha}(y-x) \right| \leq 2\frac{b}{\alpha}(M+J)$$

For sufficiently large K we have $v(x,y) \leq w(x-K, y+K)$, and vice versa. This is precisely the estimate that we need to complete Whyte's proof.

It is sufficient to take $K = 4M + 3$. The quasi-isometry $V \to W$ constructed in this way is therefore $(4M + 3 + 2\delta M)$-coarsely height preserving. The quasi-isometry constants depend on M, δ, N, and L.

Proof of the Claim

It is sufficient to take $J = M + 2\alpha$.

Let $r(0)$ be some vertex. There is an edge incident to $r(0)$ which decreases height by at least α. Follow this edge to $r(1)$.

$$-J = -M - 2\alpha \leq h(r(0), r(1)) - \frac{\alpha}{b}E(0,1) \leq -\alpha - 4\frac{\alpha}{b} < 0$$

Continue by induction. Suppose that r has been extended to $r(n)$. If

$$0 \leq h(r(0), r(n)) - \frac{\alpha}{b}E(0,n) \leq J$$

APPENDIX (Continued)

then follow an edge which decreases height by at least α to get to $r(n+1)$.

$$\begin{aligned}-J &\leq h(r(0), r(n)) - \frac{\alpha}{b}E(0,n) - M - \alpha \\ &\leq h(r(0), r(n+1)) - \frac{\alpha}{b}E(0,n+1) \\ &\leq h(r(0), r(n)) - \alpha - \frac{\alpha}{b}(E(0,n)+2) < J\end{aligned}$$

Conversely, suppose

$$-J \leq h(r(0), r(n)) - \frac{\alpha}{b}E(0,n) < 0$$

Follow an edge which increases height by at least α to get to $r(n+1)$.

$$\begin{aligned}-J &\leq h(r(0), r(n)) + \alpha - \frac{\alpha}{b}(E(0,n)+b) \\ &\leq h(r(0), r(n+1)) - \frac{\alpha}{b}E(0,n+1) \\ &\leq h(r(0), r(n)) + M - \frac{\alpha}{b}(E(0,n)+2) < J\end{aligned}$$

Repeat this process to build a ray from $r(0)$ such that

$$\left| h(r(0), r(n)) + \frac{\alpha}{b}E(0,n) \right| \leq J$$

These two rays give the desired line.

Take some vertex adjacent to this line. This vertex has at least two edges which increase slope, and two edges which increase slope. Only one of these edges is the one that connects the vertex to the line, so we can repeat the construction to get another line through this vertex disjoint from the first line. We can continue to extend in this way to laminate all of V'. $\square$

CITED LITERATURE

1. Behrstock, J. A. and Neumann, W. D.: Quasi-isometric classification of graph manifold groups. preprint, 2006. `arXiv:math.GT/0604042`.

2. Brady, N. and Bridson, M. R.: There is only one gap in the isoperimetric spectrum. Geom. Funct. Anal., 10(5):1053–1070, 2000.

3. Brady, N., Bridson, M., Forester, M., and Shankar, K.: Snowflake groups, Perron-Frobenius eigenvalues, and isoperimetric spectra. preprint, 2006. `arXiv:math.GR/0608155`.

4. Bridson, M. R. and Gersten, S. M.: The optimal isoperimetric inequality for torus bundles over the circle. Q. J. Math., 2(47):1–23, 1996.

5. Bridson, M. R. and Haefliger, A.: Metric Spaces of Non-Positive Curvature, volume 319 of Grundlehren der mathematischen Wissenschaften. Berlin, Springer, 1999.

6. Croke, C. B. and Kleiner, B.: Spaces with nonpositive curvature and their ideal boundaries. Topology, 39:549–556, 2000.

7. Mosher, L., Sageev, M., and Whyte, K.: Quasi-actions on trees I: Bounded valence. Annals of Mathematics, (158), 2003. `arXiv:math.GR/0010136`.

8. Mosher, L., Sageev, M., and Whyte, K.: Quasi-actions on trees II: Finite depth Bass-Serre trees. preprint, 2004. `arXiv:math.GR/0405237`.

9. Papasoglu, P. and Whyte, K.: Quasi-isometries between groups with infinitely many ends. Commentarii Mathematici Helvetici, 77, 2002. `arXiv:math.GT/0405274`.

10. Scott, P. and Wall, C. T. C.: Topological methods in group theory. In Homological Group Theory, London Math. Soc. Lecture Notes, pages 137–203, Durham, September 1979.

11. Serre, J.-P.: Trees. Springer Monographs in Mathematics. Berlin, Springer, 2003. corrected second printing of the first English edition.

12. Whyte, K.: Amenability, bilipschitz equivalence, and the Von Neumann conjecture. Duke Mathematical Journal, 99(1):93–112, 1999. `arXiv:math.GR/9704202`.

13. Whyte, K.: The large scale geometry of the higher Baumslag-Solitar groups. Geom. Funct. Anal., 11:1327–1343, 2001. `arXiv:math.GT/0405272`.

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