Discriminant Complements and Kernels of Monodromy Representations

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Abstract

Let $\Phi_{d,n}$ be the fundamental group of the space of smooth projective hypersurfaces of degree d and dimension n and let ρ be its natural monodromy representation. Then the kernel of ρ is *large* for $d \geq 3$ with the exception of the cases $(d,n) = (3,0),\ (3,1)$. For these and for $d < 3$ the kernel is finite. A large group is one that admits a homomorphism to a semisimple Lie group of noncompact type with Zariski-dense image. By the Tits alternative a large group contains a free subgroup of rank two.

1. Introduction

A hypersurface of degree d in a complex projective space $\mathbb{P}^{n+1}$ is defined by an equation of the form

$$F(x) = \sum a_L x^L = 0, \tag{1.1}$$

where $x^L = x_0^{L_0} \cdots x_{n+1}^{L_{n+1}}$ is a monomial of degree d and where the a_L are arbitrary complex numbers, not all zero. Viewed as an equation in both the a's and the x's, (1.1) defines a hypersurface $\mathbf{X}$ in $\mathbb{P}^N \times \mathbb{P}^{n+1}$, where $N+1$ is the dimension of the space of homogeneous polynomials of degree d in $n+2$ variables, and where the projection p onto the first factor makes $\mathbf{X}$ into a family with fibers $X_a = p^{-1}(a)$. This is the universal family of hypersurfaces of degree d and dimension n. Let Δ be the set of points a in $\mathbb{P}^N$ such that the corresponding fiber is singular. This is the *discriminant locus*; it is well-known to be irreducible and of codimension one. Our aim is to study the fundamental group of its complement, which we write as

$$\Phi = \pi_1(\mathbb{P}^N - \Delta).$$

When we need to make precise statements we will sometimes write $\Phi_{d,n} = \pi_1(U_{d,n}, o)$, where d and n are as above, $U_{d,n} = \mathbb{P}^N - \Delta$, and o is a base point.

The groups Φ are almost always nontrivial and in fact are almost always *large*. By this we mean that there is a homomorphism of Φ to a non-compact semi-simple real algebraic group which has Zariski-dense image. Large groups are infinite, and, moreover, always contain a free group of rank two. This follows from the Tits alternative [35], which states that in characteristic zero a linear group either has a solvable subgroup of finite index or contains a free group of rank two.

To show that $\Phi = \Phi_{d,n}$ is large we consider the image $\Gamma = \Gamma_{d,n}$ of the monodromy representation

$$\rho : \Phi \longrightarrow G. \tag{1.2}$$

Here and throughout this paper $G = G_{d,n}$ denotes the group of automorphisms of the primitive cohomology $H^n(X_o, \mathbb{R})_o$ which preserve the cup product. When n is odd the primitive cohomology is the same as the cohomology, and when n is even it is the orthogonal complement of $h^{n/2}$, where h is the hyperplane class. Thus G is either a symplectic or an orthogonal group, depending on the parity of n, and is an almost simple real algebraic group.

About the image of the monodromy representation, much is known. Using results of Ebeling [17] and Janssen [25], Beauville in [4] established the following:

1.1. Theorem. *Let $G_{\mathbb{Z}}$ be the subgroup of G which preserves the integral cohomology. Then the monodromy group $\Gamma_{d,n}$ is of finite index in $G_{\mathbb{Z}}$. Thus it is an arithmetic subgroup.*

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The result in [4] is much more precise: it identifies Γ as a specific subgroup of finite (and small) index in $G_{\mathbb{Z}}$. Now suppose that $d > 2$ and that $(d, n) \neq (3, 2)$. Then G is noncompact, and the results of Borel [5] and Borel-Harish-Chandra [6] apply to show that Γ is (a) Zariski-dense and (b) a lattice. Thus (a) the smallest algebraic subgroup of G which contains Γ is G itself and (b) G/Γ has finite volume.

Consider now the kernel of the monodromy representation, which we denote by K and which fits in the exact sequence

$$1 \longrightarrow K \longrightarrow \Phi \xrightarrow{\rho} \Gamma \longrightarrow 1. \tag{1.3}$$

The purpose of this paper is to show that in almost all cases it is also large:

1.2. Theorem. *The kernel of the monodromy representation* (1.2) *is large if* $d > 2$ *and* $(d, n) \neq (3, 1),\ (3, 0)$.

The theorem is sharp in the sense that the remaining groups are finite. When $d = 2$, the case of quadrics, Φ is finite cyclic. When $(d, n) = (3, 0)$, the configuration space U parametrizes unordered sets of three distinct points in the projective line and so Φ is the braid group for three strands in the sphere. It has order 12 and can be faithfully represented by symmetries of a regular hexagon.

When $(d, n) = (3, 1)$ the configuration space U parametrizes smooth cubic plane curves and the above sequence can be written as

$$1 \longrightarrow K \longrightarrow \Phi_{3,1} \xrightarrow{\rho} SL(2, \mathbb{Z}) \longrightarrow 1,$$

where K is the three-dimensional Heisenberg group over the field $\mathbb{Z}/3$, a finite group of order 27. Moreover, $\Phi_{3,1}$ is a semi-direct product, where $SL(2, \mathbb{Z})$ acts on K in the natural way. This result, due to Dolgachev and Libgober [16], is to our knowledge the only one which determines the exact sequence (1.3) for hypersurfaces of positive dimension and degree larger than two. Note that in this case Φ is large but K is finite.

Note also that there are two kinds of groups for which the natural monodromy representation has finite image but large kernel. These are the braid groups $\Phi_{d,0}$ for $d > 3$ and the group $\Phi_{3,2}$ for the space of cubic surfaces. Thus all of them are large. For the braid groups this result is classical, but for $\Phi_{3,2}$ it is new. Since $\Phi_{3,2}$ is large it is infinite, a fact which answers a question left open by Libgober in [26].

Concerning the proof of Theorem 1.2, we would like to say first of all that it depends, like anything else in this subject, on the Picard-Lefschetz formulas. We illustrate their importance by sketching how they imply the non-triviality of the monodromy representation (1.2). Consider a smooth point c of the discriminant locus. For these X_c has a exactly one node: an isolated singularity defined in suitable local coordinates by a nondegenerate sum of squares. Consider also a loop $\gamma = \gamma_c$ defined by following a path α from the base point to the edge of a complex disk normal to Δ and centered at c, traveling once around the circle bounding this disk, and then returning to the base point along α reversed. By analogy with the case of knots, we call these loops (and also their homotopy classes) the *meridians* of Δ. Then $T = \rho(\gamma)$ is a *Picard-Lefschetz* transformation, given by the formula

$$T(x) = x \pm (x, \delta)\delta. \tag{1.4}$$

Here (x, y) is the cup product and δ is the *vanishing cycle* associated to γ. When n is odd, $(\delta, \delta) = 0$ and the sign in (1.4) is $-$. When n is even and $(\delta, \delta) = \pm 2$, the sign in (1.4) is $\mp$ (see [11], paragraph 4.1). Thus when n is even δ is automatically nonhomologous to zero, and so T must be nontrivial. Since vanishing cycles exist whenever the hypersurface X_o can degenerate to a variety with a node, we conclude that ρ is nontrivial for n even and $d > 1$. Slightly less elementary arguments show that the homology class of the vanishing cycle, and hence the monodromy representation, is nontrivial for all $d > 1$ except for the case $(d, n) = (2, 1)$.

The proofs of theorem 1.1, an earlier result of Deligne asserting the Zariski density of $\Gamma_{d,n}$, and the main result of this paper are based on the Picard-Lefschetz formulas (1.4). Our proof begins with the construction of a universal family of cyclic covers of $\mathbb{P}^{n+1}$ branched along the hypersurfaces X. From it we define a second monodromy representation $\bar{\rho}'$ of Φ. Suitable versions of the Picard-Lefschetz formulas and Deligne's theorem apply to show that $\bar{\rho}'$ has Zariski-dense image. Finally, we apply Margulis' super-rigidity theorem to show that $\bar{\rho}'(K)$, where K is the kernel of the natural monodromy representation, is Zariski-dense. Thus K is large.

We mention the paper [27] as an example of the use of an associated family of cyclic covers to construct representations (in this case for the braid groups of the sphere). We also note the related results of the article [14] which we learned of while preparing the final version of this manuscript. The main theorem is that the complement of the dual $\widehat{C}$ of an immersed curve C of genus at least one, or of an immersed rational curve of degee at least four, is *big* in the sense that it contains a free group of rank two. When C is smooth, imbedded, and of even degree at least four this follows from a construction of Griffiths [21]: consider the family of hyperelliptic curves obtained as double covers of a line L not tangent to C which is branched at the points $L \cap C$. It defines a monodromy representation of $\Phi = \pi_1(\widehat{\mathbb{P}}^2 - \widehat{C})$ with Zariski-dense image. Consequently Φ is large, and, *a fortiori*, big. Such constructions have inspired the present paper. By using cyclic covers of higher degree one can treat the case of odd degree greater than four in the same way.

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2. Outline of the proof

As noted above, the proof of the main theorem is based on the construction of an auxiliary representation ρ' defined via a family of cyclic covers Y of $\mathbb{P}^{n+1}$ branched along the hypersurfaces X. To describe it, let k be a divisor of d and consider the equation

$$F(a, x) = y^k + \sum a_L x^L = 0, \tag{2.1}$$

which for the moment we view as defining a set $\widehat{\mathbf{Y}}$ in $(\mathbb{C}^{N+1} - \{ 0 \}) \times \mathbb{C}^{n+3}$ with coordinates a_L for $\mathbb{C}^{N+1}$ and coordinates $x_0, \cdots, x_{n+2}$ and y for $\mathbb{C}^{n+3}$. Construct an action of $\mathbb{C}^*$ on it by multiplying the coordinates x_i by t and by multiplying y by $t^{d/k}$. View the quotient $\mathbf{Y}$ in $(\mathbb{C}^{N+1} - \{ 0 \}) \times \mathbb{P}^{n+2}$, where we use $\mathbb{P}^{n+2}$ to denote the weighted projective space for which the x_i have weight one and for which y has weight d/k.

The resulting universal family of cyclic covers $\mathbf{Y}$ is defined on $\mathbb{C}^{N+1} - \{ 0 \}$ and has smooth fibers over $\widetilde{U} = \mathbb{C}^{N+1} - \widetilde{\Delta}$, where $\widetilde{\Delta}$ is the pre-image of Δ. Since $\mathbb{C}^{N+1} - \{ 0 \}$ is a principal $\mathbb{C}^*$ bundle over $\mathbb{P}^N$, the same holds over $\widetilde{U}$ and $\widetilde{\Delta}$. It follows that one has a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Phi} \longrightarrow \Phi \longrightarrow 1,$$

where $\widetilde{\Phi} = \pi_1(\widetilde{U})$. We introduce $\widetilde{U}$ and $\widetilde{\Phi}$ purely for the technical reason that the universal family of cyclic branched covers need not be defined over U itself.

The family $\mathbf{Y}|\widetilde{U}$ has a monodromy representation which we denote by $\tilde{\rho}$ and which takes values in a real algebraic group $\widetilde{G}$ of automorphisms of $H^{n+1}(Y_{\tilde{o}}, \mathbb{C})$ which commute with the cyclic group of covering transformations (and which preserve the hyperplane class and the cup product). Here $\tilde{o}$ is a base point in $\widetilde{U}$ which lies above the previously chosen base point o of U, and $Y_{\tilde{o}}$ denotes the k-fold cyclic cover of $\mathbb{P}^{n+1}$ branched over X_o.

The group $\widetilde{G}$ is semisimple but in general has more than one simple factor. Let G' be one of these and let

$$\rho' : \widetilde{\Phi} \longrightarrow G',$$

denote the composition of $\tilde{\rho}$ with the projection to G'. Then we must establish the following:

2.1. Technical point. *The factor G' can be chosen to be a non-compact almost simple real algebraic group. The image of ρ' is Zariski-dense in G'.*

Suppose that this is true. Then we can argue as follows. First, the group of matrices which commute with $\rho'(\mathbb{Z})$ contains a Zariski-dense group. Consequently $\rho'(\mathbb{Z})$ lies in the center of G'. Therefore there is a quotient representation

$$\bar{\rho}' : \Phi \longrightarrow \bar{G}',$$

where $\bar{G}'$ is the adjoint group of G' (that is, G' modulo its center). Moreover, the representation $\bar{\rho}'$ also has Zariski-dense image.

Now consider our original representation (1.2). Replacing Φ by a normal subgroup of finite index we may assume that the image of ρ lies in the identity component of G in the analytic topology and that the image of $\bar{\rho}'$ lies in the identity component of $\bar{G}'$ in the Zariski topology. Let $\bar{G}$ denote the identity component (in the analytic topology) of G modulo its center, and let $\bar{\rho} : \Phi \longrightarrow \bar{G}$ denote the resulting representation. We still have that $\bar{\rho}(\Phi)$ is a lattice in $\bar{G}$ and that $\bar{\rho}'(\Phi)$ is Zariski-dense in $\bar{G}'$.

Now let $\bar{K}$ be the kernel of $\bar{\rho}$, and let L be the Zariski-closure of $\bar{\rho}'(\bar{K})$. Since $\bar{K}$ is normal in Φ and $\bar{\rho}'(\Phi)$ is Zariski-dense in $\bar{G}'$, L is normal in $\bar{G}'$. Since $\bar{G}'$ is a *simple* algebraic group, either $L = \bar{G}'$ or $L = \{ 1 \}$.

If the first of the two alternatives holds, then $\bar{\rho}'(K)$ is Zariski dense, and so K is large. This is because K has finite index in $\bar{K}$ and so $\bar{\rho}'(K)$ and $\bar{\rho}'(\bar{K})$ have the same Zariski closure.

We now show that the second alternative leads to a contradiction, from which it follows that K must be large. Indeed, if $\bar{\rho}'(\bar{K}) = \{ 1 \}$, then the expression $\bar{\rho}' \circ \bar{\rho}^{-1}$ defines a homomorphism from the lattice $\bar{\rho}(\Phi)$ in $\bar{G}$ to the Zariski-dense subgroup $\bar{\rho}'(\Phi)$ in $\bar{G}'$. If the real rank of $\bar{G}$ is at least two, the Margulis rigidity theorem [28], [38] Theorem 5.1.2, applies to give an extension of $\bar{\rho}'$ to a homomorphism of $\bar{G}$ to $\bar{G}'$. Since $\bar{\rho}'(\Phi)$ is Zariski-dense, the extension is surjective. Since $\bar{G}$ is simple, it is an isomorphism. Thus the complexified lie algebras $\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}}$ must be isomorphic. However, one easily shows that $\mathfrak{g}_{\mathbb{C}} \not\cong \mathfrak{g}'_{\mathbb{C}}$, and this contradiction completes the proof.

We carry out the details separately in two cases. First, for the simpler case where d is even and $(d,n) \neq (4,1)$, we use double covers ($k=2$). Then G' is the full group of automorphisms of the primitive (or anti-invariant) part of $H^{n+1}(Y,\mathbb{R})$ and so is again an orthogonal or symplectic group. The technical point 2.1 follows from a density result of Deligne that we recall in section 3. Deligne's result gives an alternative between Zariski density and finite image, and the possibility of finite image is excluded in section 4. Finally the Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}'_{\mathbb{C}}$ are not isomorphic, since when one of them is symplectic (type C_ℓ), the other is orthogonal (type B_ℓ or D_ℓ). By lemma 8.1 the rank ℓ is at least three, so there are no accidental isomorphisms, e.g., $B_2 \cong C_2$.

For the remaining cases, namely d odd or $(d,n) = (4,1)$ we use d-fold covers, i.e., $k=d$. For these we must identify the group $\widetilde{G}$ of automorphisms of $H^{n+1}(Y,\mathbb{R})_0$ which preserve the cup product and which commute with the cyclic automorphism σ. This is the natural group in which the monodromy representation $\tilde{\rho}$ takes its values. Now a linear map commutes with σ if and only if it preserves the eigenspace decomposition of σ, which we write as

$$H^{n+1}(Y,\mathbb{C})_0 = \bigoplus_{\mu \neq 1} H(\mu).$$

As noted in (8.3), the dimension of $H(\mu)$ is independent of μ. Now let $\widetilde{G}(\mu)$ be subgroup of $\widetilde{G}$ which acts by the identity on $H(\lambda)$ for $\lambda \neq \mu,\ \bar{\mu}$. It can be viewed as a group of transformations of $H(\mu) + H(\bar{\mu})$. Thus there is a decomposition

$$\widetilde{G} = \prod_{\mu \in S} \widetilde{G}(\mu), \tag{2.2}$$

where

$$S = \{ \mu \mid \mu^k = 1,\ \mu \neq 1,\ \Im\mu \geq 0 \}.$$

When μ is non-real, $\widetilde{G}(\mu)$ can be identified via the projection $H(\mu) \oplus H(\bar{\mu}) \longrightarrow H(\mu)$ with the group of transformations of $H(\mu)$ which are unitary with respect to the hermitian form $h(x,y) = i^{n+1}(x,\bar{y})$, where (x,y) is the cup product. This form may be (and usually is) indefinite. When $\mu = -1$, $\widetilde{G}(\mu)$ is the group of transformations of $H(-1)$ which preserve the cup product. It is therefore an orthogonal or symplectic group.

We will show that at least one of the components $\widetilde{\rho}_\mu(\Phi) \subset \widetilde{G}(\mu)$ is Zariski-dense, and we will take $G' = \widetilde{G}(\mu)$. The necessary Zariski density result, which is a straightforward adaptation of Deligne's, is proved in section 7 after some preliminary work on complex reflections in section 6. Again, the possibility of finite image has to be excluded, and the argument for this is in section 5. Finally, to prove that $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}'_{\mathbb{C}}$ are not isomorphic one observes that $\mathfrak{g}_{\mathbb{C}}$ is of type B_ℓ, C_ℓ or D_ℓ while $\mathfrak{g}'_{\mathbb{C}}$ is of type A_ℓ (since G' is of type $SU(r,s)$. One only needs to avoid the isomorphism $D_3 \cong A_3$, which follows from the lower bound of the rank of $\mathfrak{g}_{\mathbb{C}}$ in lemma 8.1.

In order to apply Margulis' theorem we also need to verify that the real rank of G is at least two. This is done in section 8.

To summarize, we have established the following general criterion, and our proof of Theorem 1.2 is an application of it.

2.2. Criterion. *The kernel K of a linear representation $\rho : \Phi \longrightarrow G$ is large if*

1. *$\rho(\Phi) \subset G$ is a lattice in a simple Lie group G of real rank at least two.*

2. *There exist a non-compact, almost simple real algebraic group G', a central extension $\widetilde{\Phi}$ of Φ and a linear representation $\rho' : \widetilde{\Phi} \longrightarrow G'$ with Zariski-dense image.*
3. *G and G' are not locally isomorphic.*

An immediate consequence is the following:

2.3. Corollary. *Let Φ be a group which admits a representation $\rho : \Phi \longrightarrow G$ to a simple Lie group of real rank greater than 1 with image a lattice. Suppose further that there exist an almost simple real algebraic group G', a central extension $\widetilde{\Phi}$ of Φ, and a representation $\rho' : \widetilde{\Phi} \longrightarrow G'$ with Zariski-dense image. Suppose in addition that G and G' are not locally isomorphic. Then Φ is not isomorphic to a lattice in any simple Lie group of real rank greater than 1.*

Proof: Suppose that $\tau : \Phi \longrightarrow \Sigma$ is an isomorphism of Φ with a lattice Σ in a Lie group H of real rank greater than one. If H is not locally isomorphic to G', then apply the criterion with τ in place of ρ to conclude that τ has large kernel, hence cannot be an isomorphism. Suppose next that H is locally isomorphic to G'. Apply the criterion with τ in place of ρ and with ρ in place of ρ' to conclude as before that the kernel of τ is large.

For most families of hypersurfaces the natural monodromy representation and the representation for the associated family of cyclic covers satisfy the hypotheses of the corollary to give the following:

2.4. Theorem. *If $d > 2$, $n > 0$, and $(d, n) \neq (3, 1)$, $(3, 2)$, the group $\Phi_{d,n}$ is not isomorphic to a lattice in a simple Lie group of real rank greater than one.*

It seems reasonable that the preceding theorem holds with "semisimple" in place of "simple." However, we are unable show that this is the case. Indeed, our results so far are compatible with an isomorphism $\Phi \cong \Gamma \times \Gamma'$. We can exclude this in certain cases (see section 9), but not for an arbitrary subgroup of finite index, which is what one expects.

2.5. Remarks.

(a) Suppose that $d \geq 3$ and let γ be a meridian of $\Phi_{d,n}$. When n is odd, $\rho(\gamma)$ is a nontrivial symplectic transvection. Since it is of infinite order, so is the meridian γ. When n is even, $\rho(\gamma)$ is a reflection, hence of order two. Now suppose that d is even and consider the monodromy representation of the central extension $\widetilde{\Phi}$ constructed from double covers. Let $\tilde{\gamma}$ be a lift of γ to an element of $\widetilde{\Phi}$. Then $\rho'(\tilde{\gamma})$ is a nontrivial symplectic transvection, no power of which is central. Thus $\bar{\rho}'(\gamma)$ is of infinite order, and, once again, we conclude that γ is of infinite order.

(b) M. Kontsevich informs us that he can prove that for any $d > 2$ (and at least for $n = 2$) the local monodromy corresponding to a meridian is of infinite order in the group of connected components of the symplectomorphism group of X_o. This implies that the meridians are of infinite order for all $d > 2$, not necessarily even as above. The symplectic nature of the monodromy for a meridian (for $n = 2$) is studied in great detail by P. Seidel in his thesis [32].

(c) For the case of double covers the image Γ' of the fundamental group under the second monodromy representation $\rho'(\widetilde{\Phi})$ is a lattice. This follows from the argument given by Beauville to prove theorem 1.1. It is enough to be able to degenerate the branch locus X to a variety which has an isolated singularity of the form $x^3 + y^3 + z^4 +$ a sum of squares $= 0$. Then the roles of the kernels K and K' are symmetric and one concludes that K' is also large.

3. Zariski Density

The question of Zariski-density for monodromy groups of Lefschetz pencils was settled by Deligne in [11] and [12]. We review these results here in a form convenient for the proof of the main theorem in the case of even degree and also for the proof of a density theorem for unitary groups (section 7). To begin, we have the following purely group-theoretic fact: [12](4.4):

3.1. Theorem. (Deligne) *Let V be a vector space (over $\mathbb{C}$) with a non-degenerate bilinear form $(\ ,\ )$ which is either symmetric or skew-symmetric. Let Γ be a group of linear transformations of V which preserves the bilinear form. Assume the existence of a subset $E \subset V$ such that Γ is generated by the Picard-Lefschetz transformations (1.4) with $\delta \in E$. Suppose that E consists of a single Γ-orbit and that it spans V. Then Γ is either finite or Zariski-dense.*

To apply this theorem in a geometric setting, consider a family of n-dimensional varieties $p : \mathbf{X} \longrightarrow S$ with discriminant locus Δ and monodromy representation $\rho : \pi_1(S - \Delta) \longrightarrow \mathrm{Aut}(H^n(X_o))$. Assume that S is either $\mathbb{C}^{N+1} - \{0\}, N \geq 1$ or $\mathbb{P}^N$, so that S is simply connected and hence that $\pi_1(S - \Delta)$ is generated by *meridians* (cf. §1 for the definition). Assume also that for each meridian there is a class $\delta \in H^n(X_o)$ such that the corresponding monodromy transformation is given by the Picard-Lefschetz formula (1.4). Let E denote the set of these classes (called the *vanishing cycles*). Let $V^n(X_o) \subset H^n(X_o)$ be the span of E, called the *vanishing cohomology*.

A cycle orthogonal to $V = V^n(X_o)$ is invariant under all Picard-Lefschetz transformations, hence is invariant under the action of monodromy. Consequently its orthogonal complement $V^\perp$ is the space of invariant cycles. The image of $H^n(\mathbf{X})$ in $H^n(X_o)$ also consists of invariant cycles. By theorem 4.1.1 (or corollary (4.1.2)) of [13], this inclusion is an equality. One concludes that $V^\perp$ is the same as the image of $H^n(\mathbf{X})$, which is a sub-Hodge structure, and so the bilinear form restricted to it is nondegenerate. Therefore the bilinear form restricted to $V = V^n(X_o)$ is also nondegenerate. Consequently $V^n(X_o)$ is an orthogonal or symplectic space, and the monodromy group acts on $V^n(X_o)$ by orthogonal or symplectic transformations.

When the discriminant locus is irreducible the argument of Zariski [37] or [11], paragraph preceding Corollary 5.5, shows that the meridians of $\pi_1(S - \Delta)$ are mutually conjugate. Writing down a conjugacy $\gamma' = \kappa^{-1}\gamma\kappa$ and applying it to (1.4), one concludes that $\delta' = \rho(\kappa^{-1})(\delta)$. Thus the vanishing cycles constitute a single orbit. To summarize, we have the following, (c.f. [11], Proposition 5.3, Theorem 5.4, and [12], Lemma 4.4.2):

3.2. Theorem. *Let $\mathbf{X} \longrightarrow S$, with $S = \mathbb{C}^{N+1} - \{0\}$ or $\mathbb{P}^N$ and $N \geq 1$, be a family with irreducible discriminant locus and such that the monodromy transformations of meridians are Picard-Lefschetz transformations. Then the monodromy group is either finite or is a Zariski-dense subgroup of the (orthogonal or symplectic) group of automorphisms of the vanishing cohomology.*

To decide which of the two alternatives holds, consider the period mapping

$$f : U \longrightarrow D/\Gamma,$$

where D is the space [23] which classifies the Hodge structures $V^n(X_a)$ and where Γ is the monodromy group. Then one has the following well-known principle:

3.3. Lemma. *If the monodromy group is finite, then the period map is constant.*

Proof: Let f be the period map and suppose that the monodromy representation is finite. Then there is an unramified cover $\widetilde{S}$ of the domain of f for which the monodromy representation is trivial. Consequently there is lift $\tilde{f}$ to $\widetilde{S}$ which takes values in the period domain D. Let $\bar{S}$ be a smooth compactification of $\widetilde{S}$. Since D acts like a bounded domain for horizontal holomorphic maps, $\tilde{f}$ extends to a holomorphic map of $\bar{S}$ to D. Any such map with compact domain is constant [24].

As a consequence of the previous lemma and theorem, we have a practical density criterion:

3.4. Theorem. *Let $\mathbf{X}$ be a family of varieties over $\mathbb{C}^{N+1} - \{0\}$ or $\mathbb{P}^N$, $N \geq 1$, whose monodromy group is generated by Picard-Lefschetz transformations* (1.4), *which has irreducible discriminant locus, and whose period map has nonzero derivative at one point. Then the monodromy group is Zariski-dense in the (orthogonal or symplectic) automorphism group of the vanishing cohomology.*

Irreducibility of the discriminant locus for hypersurfaces is well known, and can be proved as follows. Consider the Veronese imbedding v of $\mathbb{P}^{n+1}$ in $\mathbb{P}^N$. This is the map which sends the homogeneous coordinate vector $[x_0, \ldots, x_{n+1}]$ to $[x^{M_0}, \ldots, x^{M_N}]$ where the x^{M_i} are an ordered basis for the monomials of degree d in the x_i. If H is a hyperplane in $\mathbb{P}^N$, then $v^{-1}(H)$ is a hypersurface of degree d in $\mathbb{P}^{n+1}$. All hypersurfaces are obtained in this way, so the dual projective space $\widehat{\mathbb{P}}^N$ parametrizes the universal family. A hypersurface is singular if and only if H is tangent to the Veronese manifold $\mathcal{V} = v(\mathbb{P}^{n+1})$. Thus the discriminant is the variety $\widehat{\mathcal{V}}$ dual to $\mathcal{V}$. Since the variety dual to an irreducible variety is also irreducible, it follows that the discriminant is irreducible.

Finally, we observe that in the situations considered in this paper, vanishing cohomology and primitive cohomology coincide. This can easily be checked by computing the invariant cohomology using a suitable compactification and appealing to (4.1.1) of [13]. Since this is not essential to our arguments we omit further details.

4. Rational differentials and the Griffiths residue calculus

Griffiths' local Torelli theorem [22] tells us that the period map for hypersurfaces of degree d and dimension n is is nontrivial for $d > 2$ and $n > 1$ with the exception of the case $(d, n) = (3, 2)$. In fact, it says more: the kernel of the differential is the tangent space to the orbit of the natural action of the projective linear group. The proof is based on the residue calculus for rational differential forms and some simple commutative algebra (Macaulay's theorem).

What we require here is a weak (but sharp) version of Griffiths' result for the variations of Hodge structures defined by families of cyclic covers of hypersurfaces. For double covers this is straightforward, since such covers can be viewed as hypersurfaces in a weighted projective space [15]. For higher cyclic covers the variations of Hodge structure are complex, and in general the symmetry of Hodge numbers, $h^{p,q} = h^{q,p}$ is broken. Nonetheless, the residue calculus still gives the needed result. Since this last part is nonstandard, we sketch recall the basics of the residue calculus, how it applies to the case of double covers, and how it extends to the case of higher cyclic covers.

To begin, consider weighted projective space $\mathbb{P}^{n+1}$ where the weights of x_i are w_i. Fix a weighted homogeneous polynomial $P(x)$ and let X be the variety which it defines. We assume that it is smooth. Now take a meromorphic differential ν on $\mathbb{P}^{n+1}$ which has a pole of order $q+1$ on X. Its residue is the cohomology class on X defined by the formula

$$\int_\gamma \operatorname{res} \nu = \frac{1}{2\pi} \int_{\partial T(\gamma)} \nu,$$

where $T(\gamma)$ is a tubular neighborhood of an n-cycle γ. The integrand can be written as

$$\nu(A, P, q) = \frac{A\Omega}{P^{q+1}}. \tag{4.1}$$

where

$$\Omega = \sum (-1)^i \, w_i x_i \, dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_{n+1}.$$

The "volume form" Ω has weight $w_0 + \cdots + w_{n+1}$ and the degree of A, which we write as $a(q)$, is such that ν is of weight zero. The primitive cohomology of X is spanned by Poincaré residues of rational differentials, and the space of residues with a pole of order $q+1$ is precisely $F^{n-q} H^n_o(X)$, the $(n-q)$-th level of the Hodge filtration on the primitive cohomology. When the numerator polynomial is a linear combination of the partial derivatives of P, the residue is cohomologous in $\mathbb{P}^{n+1} - X$ to a differential with a pole of order one lower. Let $J = (\partial P/\partial x_0 \, , \ldots , \, \partial P/\partial x_{n+1})$ be the Jacobian ideal and let $R = \mathbb{C}[x_0 \, , \ldots , \, x_{n+1}]/J$ be the quotient ring, which we note is graded. Then the residue maps $R^{a(q)}$ to F^q/F^{q+1}. By a theorem of Griffiths [22], this map is an isomorphism. For a smooth variety the "Jacobian ring" R is finite-dimensional, and so there is a least integer

$$t = (n + 2)(d - 2) \tag{4.2}$$

such that $R^i = 0$ for $i > t$. Moreover, and R^t is one-dimensional and the bilinear map

$$R^i \times R^{t-i} \longrightarrow R^t \cong \mathbb{C}.$$

is a perfect pairing (Macaulay's theorem). When R^i and R^{t-i} correspond to graded quotients of the Hodge filtration, the pairing corresponds to the cup product [9].

The derivative of the period map is given by formal differentiation of the expressions (4.1). Thus, if $P_t = P + tQ + \cdots$ represents a family of hypersurfaces and $\omega = \operatorname{res}(A\Omega/F^\ell)$ represents a family of cohomology classes on them, then

$$\frac{d}{dt} \operatorname{res} \frac{A\Omega}{P^{q+1}} = -(q+1) \operatorname{res} \frac{QA\Omega}{P^{q+2}}.$$

To show that the derivative of the period map is nonzero, it suffices to exhibit an A and a Q which are nonzero in R and such that the product QA is also nonzero. Here we implicitly use the identification $T \cong R^d$ of tangent vectors to the moduli space with the component of the Jacobian ring in degree d. Thus the natural components of the differential of the period map,

$$T \longrightarrow \operatorname{Hom}(H^{p,q}(X), H^{p-1,q+1}(X)),$$

can be identified with the multiplication homomorphism

$$R^d \longrightarrow \mathrm{Hom}(R^a, R^{a+d}),$$

where a is the degree of the numerator polynomial used in the residues of the forms (4.1). All of these results, discovered first by Griffiths in the case of hypersurfaces, hold for weighted hypersurfaces by the results described in [15] and [36].

Consider now a double cover Y of a hypersurface X of even degree d. If X is defined by $P(x_0, \dots, x_n) = 0$ then Y is defined by $y^2 + P(x_0, \dots, x_n) = 0$, where y has weight $d/2$ and where the x's have weight one. This last equation is homogeneous of degree d with respect to the given weighting, and Ω has weight $d/2 + n + 2$. Thus $\nu(A, y^2 + P, q)$ is of weight zero if $a(q) = (q + 1/2)d - (n + 2)$. Since y is in the Jacobian ideal, we may choose A to be a polynomial in the x's, and we may consider it modulo the Jacobian ideal of P. Thus the classical considerations of the residue calculus apply. If we choose $a(q)$ maximal subject to the constraints $p > q$ and $a \geq 0$ then

$$q = \left\{ \frac{n+1}{2} \right\},$$

where $\{\ x\ \}$ is the greatest integer *strictly less* than x. Both conditions are satisfied for $d \geq 4$ except that for $n = 1$ we require $d \geq 6$. Thus we have excluded the case $(d, n) = (4, 1)$ in which the resulting double cover is rational and the period map is constant.

Now let A be a polynomial of degree a which is nonzero modulo the Jacobian ideal. We must exhibit a polynomial Q of degree d such that AQ nonzero modulo J. By Macaulay's theorem there is a polynomial B such that AB is congruent to a generator of R^t, hence satisfies $AB \not\equiv 0 \bmod J$. Write B as a linear combination of monomials B_i and observe that there is an i such that $AB_i \not\equiv 0$. If B_i is of degree at least d, we can factor it as QB_i' with Q of degree d. Since $AQB_i' \not\equiv 0$, $AQ \not\equiv 0$, as required.

The condition that B have degree at least d reads $a + d \leq t$. Using the formulas (4.2) for t and the optimal choice for a, we see that this inequality is satisfied for the range of d and n considered. This computation completes the proof of the main theorem in the case d even, $d \geq 4$, except for the case $(d, n) = (4, 1)$.

5. Rational differentials for higher cyclic covers

To complete the proof of the main theorem we must consider arbitrary cyclic covers of $\mathbb{P}^{n+1}$ branched along a smooth hypersurface of degree d. Since the fundamental group of the complement of X is cyclic of order d, the number of sheets k must be a divisor of d. As mentioned in the outline of the proof, there is an automorphism σ of order k which operates on the universal family $\mathbf{Y}$ of such covers. Consequently the local system $\mathbb{H}$ of vanishing cohomology (cf. §3) splits over $\mathbb{C}$ into eigensystems $\mathbb{H}(\mu)$, where $\mu \neq 1$ is a k-th root of unity. Therefore the monodromy representation, which we now denote by ρ, splits as a sum of representations ρ_μ with values in the groups $\widetilde{G}(\mu)$ introduced in (2.2). As noted there we can view ρ_μ as taking values in a group of linear automorphisms of $H(\mu)$. This group is unitary for the hermitian form $h(x, y) = i^{n+1}(x, \bar{y})$ if μ is non-real, and that is the case that we will consider here.

Although the decomposition of $\mathbb{H}$ is over the complex numbers, important Hodge-theoretic data survive. The hermitian form $h(x, y)$ is nondegenerate and there is an induced Hodge decomposition, although $h^{p,q}(\mu) = h^{q,p}(\mu)$ may not hold. However, Griffiths' infinitesimal period relation,

$$\frac{d}{dt} F^p(\mu) \subset F^{p-1}(\mu)$$

remains true. Thus each $\mathbb{H}(\mu)$ is a *complex variation of Hodge structure*, c.f. [10], [34]. The associated period domains are homogeneous for the groups $\widetilde{G}(\mu)$.

To extend the arguments given above to the unitary representations ρ_μ we must extend Deligne's density theorem to this case. The essential point is that the monodromy groups $\Gamma(\mu)$ are generated not by Picard-Lefschetz transformations, but by their unitary analogue, which is a *complex reflection* [31], [18], [30]. These are linear maps of the form

$$T(x) = x \pm (\lambda - 1) h(x, \delta)\delta,$$

where h is the hermitian inner product defined above, $h(\delta, \delta) = \pm 1$, where $\pm$ is the same sign as that of $h(\delta, \delta)$, and where $\lambda \neq 1$ is a root of unity. The vector δ is an eigenvector of T with eigenvalue λ and T acts by the identity on the hyperplane perpendicular to δ. It turns out that the eigenvalue λ of T is, up to a fixed sign that depends only on the dimension of Y, equal to the eigenvalue μ of σ.

In section 7 we will prove an analogue of Deligne's theorem (3.1) for groups of complex reflections. It gives the usual dichotomy: either the monodromy group is finite, or it is Zariski-dense. In section 6 we will show that the monodromy groups $\Gamma(\mu)$ are indeed generated by complex reflections. It remains to show that the derivative of the period map for the complex variations of Hodge structure $\mathbb{H}(\mu)$ are nonzero given appropriate conditions on d, k, n, and μ.

For the computation fix $\zeta = e^{2\pi i/k}$ as a primitive k-th *root of unity* and let the cyclic action on the universal family (2.1) be given by $y \circ \sigma = \zeta y$. Then the "volume form" $\Omega(x,y)$ is an eigenvector with eigenvalue ζ and the rational differential

$$\frac{y^{i-1}A(x)\Omega(x,y)}{(y^k + P(x))^{q+1}} \tag{5.1}$$

has eigenvalue $\mu = \zeta^i$, as does its residue. Thus we will sometimes write $\mathbb{H}(i)$ for $\mathbb{H}(\zeta^i)$ and will use the corresponding notations $\widetilde{G}(i)$, $\tilde{\rho}_i$, etc. Residues with numerator $y^{i-1}A(x)$ and denominator $(y^k + P(x))^{q+1}$ span the spaces $H_0^{p,q}(i)$, where i ranges from 1 to $k-1$. Moreover, the corresponding space of numerator polynomials, taken modulo the Jacobian ideal of P, is isomorphic via the residue map to $H_0^{p,q}(i)$. Since P varies by addition of a polynomial in the x's, the standard unweighted theory applies to computation of the derivative map.

Let us illustrate the relevant techniques by computing the Hodge numbers and period map for triple covers of $\mathbb{P}^3$ branched along a smooth cubic surface. (This period map is studied in more detail in [1].) A triple cover of the kind considered is a cubic hypersurface in $\mathbb{P}^4$, and the usual computations with rational differentials show that $h^{3,0} = 0$, $h^{2,1} = 5$. The eigenspace $H^{2,1}(i)$ is spanned by residues of differentials with numerator $A(x)\Omega(x,y)$ and denominator $(y^3 + P(x))^2$. Since the degree of $\Omega(x_0, x_1, x_2, x_3, y)$ is 5, A is must be linear in the variables x_i. Thus $h^{2,1}(1) = 4$. The space $H^{1,2}(1)$ is spanned by residues of differentials with numerator $A(x)\Omega(x,y)$ and denominator $(y^3 + P(x))^3$. Thus the numerator is of degree four, but must be viewed modulo the Jacobian ideal. For dimension counts it is enough to consider the Fermat cubic, whose Jacobian ideal is generated by squares of variables. The only square-free quartic in four variables is $x_0x_1x_2x_3$, so $h^{1,2}(1) = 1$. Similar computations show that the remaining Hodge numbers for $H^3(1)$ are zero and yield in addition the numbers for $H^3(2)$. One can also argue that $H^3(1) \oplus H^3(2)$ is defined over $\mathbb{R}$, since the eigenvalues are conjugate. A Hodge structure defined over $\mathbb{R}$ satisfies $h^{p,q} = h^{q,p}$. From this one deduces that $h^{2,1}(2) = 1$, $h^{1,2}(2) = 4$. Since there is just one conjugate pair of eigenvalues of σ, there is just one component in the decomposition (2.2), $\widetilde{G} = \widetilde{G}(\zeta)$, and this group is isomorphic to $U(1,4)$. Since the coefficients of the monodromy matrices lie in the ring $\mathbb{Z}[\zeta]$, where ζ is a primitive cube root of unity, the representation $\tilde{\rho}$ takes values in a discrete subgroup of $\widetilde{G}$. Therefore the complex variation of Hodge structures define period mappings

$$p : U_{3,2} \longrightarrow B_4/\Gamma',$$

where B_4 is the unit ball in complex 4-space and Γ' a discrete group acting on it.

To show that the period map p_i is nonconstant it suffices to show that its differential is nonzero at a single point. We do this for the Fermat variety. A basis for $H^{2,1}(1)$ is given by the linear forms x_i, and a basis for $H^{1,2}$ is given by their product $x_0x_1x_2x_3$. Let m_i be the product of all the x_k except x_i. These forms constitute a basis for the tangent space to moduli. Since $m_i x_i = x_0x_1x_2x_3$, multiplication by m_i defines a nonzero homomorphism from $H^{2,1}(1)$ to $H^{1,2}(1)$. Thus the differential of the period map is nonzero at the Fermat. In fact it is of rank four, since the homomorphisms defined by the m_i are linearly independent. Similar considerations show that the period map for $\mathbb{H}(2)$ is of rank four. The relevant bases are $\{\ y\ \}$ for $H^{2,1}(2)$ and $\{\ ym_0, ym_1, ym_2, ym_3\ \}$ for $H^{1,2}(2)$.

For the general case it will be enough to establish the following.

5.1. Proposition. *Let* $\mathbf{Y}$ *be the universal family of d-sheeted covers of* $\mathbb{P}^{n+1}$ *branched over smooth hypersurfaces of degree d. The derivative of the period map for* $\mathbb{H}^{n+1}(1)$ *is nontrivial if* $n \geq 2$ *and* $d \geq 3$ *or if* $n = 1$ *and* $d \geq 4$.

Proof: Elements of $H^{p,q}(1)$ with $p+q=n+1$ are given by rational differential forms with numerator $A(x)\Omega(x,y)$ and denominator $(y^d+P(x))^{q+1}$. The numerator must have degree $a=(q+1)d-(n+3)$. As before choose q so that a is maximized subject to the constraints $p>q$ and $a\geq 0$. Then $q=\{\ n/2+1/d\ \}$. If $n\geq 2$ and $d\geq 3$ or if $n=1$ and $d\geq 4$, then $a\geq 0$. Thus numerator polynomials $A(x)$ which are nonzero modulo the Jacobian ideal exist. One establishes the existence of a polynomial $Q(x)$ of degree d such that QA is nonzero modulo the Jacobian ideal using the same argument as in the case of double covers.

A different component of the period map is required if the branch locus is a finite set of points, which is the case for the braid group of $\mathbb{P}^1$:

5.2. Proposition. *For $n=0$ the period map for $\mathbb{H}^1(i)$ is non-constant if $d\geq 4$ and $i\geq 2$.*

Proof: An element of $H^{1,0}(i)$ is the residue of a rational differential with numerator $y^{i-1}A(x_0,x_1)\Omega$ and denominator $y^d+P(x_0,x_1)$. The degree of A is $a=d-2-i$. The top degree for the Jacobian ideal is $2d-4$. Thus we require $a+d\leq 2d-4$, which is satisfied if $i\geq 2$. Since $a\geq 0$, one must also require $d\geq 4$.

We observe that the local systems which occur as constituents for k-sheeted covers, where k divides d, also occur as constituents of d-sheeted covers.

5.3. Remark. *Let $\mathbb{H}(k,\mu)$ be the complex variation of Hodge structure associated to a k-sheeted cyclic cover of $\mathbb{P}^{n+1}$ branched along a hypersurface of degree d, belonging to the eigenvalue μ, where k is a divisor of d. Then $\mathbb{H}(k,\mu)$ is isomorphic to $\mathbb{H}(d,\mu)$.*

Proof: Consider the substitution $y=z^{d/k}$ which effects the transformation

$$\frac{y^iA(x)\Omega(x,y)}{(y^k+P(x))^{q+1}}\mapsto (d/k)\frac{z^{(i+1)(d/k)-1}A(x)\Omega(x,z)}{(z^d+P(x))^{q+1}}.$$

These differentials are eigenvectors with the same eigenvalue. The map which sends residues of the first kind of rational differential to residues of the second defines the required isomorphism.

6. Complex Reflections

We now review some known facts on how complex reflections arise for degenerations of cyclic covers. When the branch locus acquires a node, the local equation is

$$y^k+x_1^2+\cdots+x_{n+1}^2=t, \tag{6.1}$$

which is a special case of the situation studied by Pham in [31], where the left-hand side is a sum of powers. Our discussion is based on Chapter 9 of [29] and Chapter 2 of [2].

Consider first the case $y^k=t$. It is a family of zero-dimensional varieties $\{\ \xi_1(t)\ ,\ldots,\ \xi_k(t)\ \}$ whose vanishing cycles are successive differences of roots,

$$\xi_1-\xi_2,\ \ldots,\ \xi_{k-1}-\xi_k, \tag{6.2}$$

and whose monodromy is given by cyclically shifting indices to the right:

$$T(\xi_i-\xi_{i+1})=\xi_{i+1}-\xi_{i+2},$$

where i is taken modulo k. Thus T acts on the $(k-1)$-dimensional space of vanishing cycles as a transformation of order k. Over the complex numbers it is diagonalizable, and the eigenvalues are the k-th roots of unity $\mu\neq 1$. Note that $T=\sigma_0$ where σ_0 is the generator for the automorphism group of the cyclic cover $y^k=t$ given by $y\longrightarrow\zeta y$, where $\zeta=e^{2\pi i/k}$ is our chosen primitive k-th root of unity.

The intersection product B defines a possibly degenerate bilinear form on the space of vanishing cycles. For the singularity $y^k=t$ it is $(\xi_i,\xi_j)=\delta_{ij}$, so relative to the basis (6.2) it is the negative of the matrix for the Dynkin diagram A_{k-1} — the positive-definite matrix with two's along the diagonal, one's immediately above and below the diagonal, and zeroes elsewhere.

Now suppose that $f(x)=t$ and $g(y)=t$ are families which acquire an isolated singularity at $t=0$. Then $f(x)+g(y)=t$ is a family of the same kind; we denote it by $f\oplus g$. The theorem of Sebastiani and Thom [33], or [2], cf. Theorem 2.1.3, asserts that vanishing cycles for the sum of two singularities are given as the join of vanishing cycles for f and g. Thus, if a and b are vanishing cycles of dimensions m and n, then the join $a * b$ is a vanishing cycle of dimension $m + n + 1$, and, moreover, the monodromy acts by $T(a * b) = T(a) * T(b)$. From an algebraic standpoint the join is a tensor product, so one can write $V(f \oplus g) = V(f) \otimes V(g)$ where $V(f)$ is the space of vanishing cycles for f, and one can write the monodromy operator as $T_{f\oplus g} = T_f \otimes T_g$.

The *suspension* of a singularity $f(x) = t$ is by definition the singularity $y^2 + f(x) = t$ obtained by adding a single square. If a is a vanishing cycle for f then $(y_0 - y_1) \otimes a$ is a vanishing cycle for the suspension, and the suspended monodromy is given by

$$T((y_0 - y_1) \otimes a) = -(y_0 - y_1) \otimes T(a).$$

In particular, the local monodromy of a singularity and its double suspension are isomorphic.

The intersection matrix B' of a suspended singularity (relative to the same canonical basis) is a function of the intersection matrix B for the given singularity, cf. Theorem 2.14 of [2]. When the bilinear form for B is symmetric, the rule for producing B' from B is: make the diagonal entries zero and change the sign of the above-diagonal entries. When B' has an even number of rows of columns, the determinant is one, and when the number of rows and columns is odd, it is zero. Thus the intersection matrix for $x^2 + y^k = t$ is nondegenerate if and only if k is odd. In addition, the intersection matrix of a double suspension is the negative of the given matrix. Thus the matrix of any suspension of $y^k = t$ is determined. It is nondegenerate if the dimension of the cyclic cover (6.1) is even or if the dimension is odd and k is also odd. Otherwise it is degenerate.

It follows from our discussion that the space of vanishing cycles V for the singularity (6.1) is $(k-1)$-dimensional and that the local monodromy transformation is $T = \sigma_0 \otimes (-1) \otimes \cdots \otimes (-1)$ where σ_0 is the covering automorphism $y \longrightarrow \zeta y$ for $y^k = t$. Thus T is a cyclic transformation of order k or $2k$, depending on whether the dimension of the cyclic cover is even or odd. In any case, T is diagonalizable with eigenvectors η_i and eigenvalues λ_i, where $\lambda_i = \pm\mu_i$ with $\mu_i = \zeta^i$ where ζ is our fixed primitive k-th root of unity and $i = 1, \cdots, k-1$. Note that the cyclic automorphism σ of the universal family (2.1), given by $y \mapsto \zeta y$ acts as $\sigma_0 \otimes (+1) \otimes \cdots \otimes (+1)$ on the vanishing homology of (6.1). Thus the eigenspaces of σ and T coincide, and their respective eigenvalues differ by the fixed sign $(-1)^{n+1}$. Since the eigenvalues μ_i are distinct, the eigenvectors η_i are orthogonal with respect to the hermitian form. Thus $h(\eta_i, \eta_i) \neq 0$. Moreover the sign of $h(\eta_i, \eta_i)$ depends only on the index i, globally determined on (2.1), independently of the particular smooth point on the discriminant locus whose choice is implicit in (6.1). We conclude that on the space of vanishing cycles,

$$T(x) = \sum_{i=1}^{k-1} \lambda_i \frac{h(x, \eta_i)}{h(\eta_i, \eta_i)} \eta_i, \tag{6.3}$$

where $\lambda_i = (-1)^{n+1}\mu_i$.

Now consider a cycle x in $H^{n+1}(Y_{\bar{o}})$, and *suppose that k is odd*. Then the intersection form on the space V of local vanishing cycles for the degeneration (6.1) is *nondegenerate*. Consequently $H^{n+1}(Y_{\bar{o}})$ splits orthogonally as $V \oplus V^\perp$. The action on $H^{n+1}(Y_{\bar{o}})$ of the monodromy transformation T for the meridian corresponding to the degeneration (6.1) is given by (6.3) on V and by the identity on $V^\perp$. Thus it is given for arbitrary x by the formula

$$T(x) = x + \sum_{i=1}^{k-1} (\lambda_i - 1) \frac{h(x, \eta_i)}{h(\eta_i, \eta_i)} \eta_i. \tag{6.4}$$

Finally, for each $i = 1, \cdots, k-1$ we can normalize the eigenvector η_i to an eigenvector δ_i satisfying $h(\delta_i, \delta_i) = \epsilon_i = \pm 1$. To summarize, we have proved the following:

6.1. Proposition. *Consider the family (2.1) of k-fold cyclic covers of $\mathbb{P}^{n+1}$ branched over a smooth hypersurface of degree d, where both k and d are odd. Let T be the monodromy corresponding to a generic degeneration of the branch locus, as in (6.1). Then T acts on the i-th eigenspace of the cyclic automorphism σ (defined by $y \mapsto \zeta y$ in (2.1)) by a complex reflection with eigenvalue $\lambda_i = (-1)^{n+1}\zeta^i$. Thus*

$$T(x) = x + \epsilon_i(\lambda_i - 1)h(x, \delta_i)\delta_i$$

holds for all $x \in \mathbb{H}(i)$.

6.2. Remark. In remark 2.5.a we observed that the meridians of $\Phi_{d,n}$ are of infinite order for n odd and for n even, $d \geq 4$ even. Consider now the case in which n is even and d is odd, let $\zeta = \exp(2\pi i/d)$, and let $\bar{\rho}'$ be the corresponding representation, in which meridians of $\widetilde{\Delta}$ correspond to complex reflections of order $2d$. These complex reflections and their powers different from the identity are non-central if the ζ eigenspace has dimension at least two, which is always the case for $d \geq 3$, $n \geq 2$. Thus $\bar{\rho}'(\gamma)$ has order $2d$. By this simple argument we conclude that in the stated range of (n, d), meridians always have order greater than two. However, our argument does not give the stronger result 2.5.b asserted by Kontsevich.

7. Density of unitary monodromy groups

We now show how the argument Deligne used in [12], section 4.4, to prove Theorem 3.1 can be adapted to establish a density theorem for groups generated by complex reflections on a space $\mathbb{C}(p,q)$ endowed with a hermitian form h of signature (p,q). If A is a subset of $\mathbb{C}(p,q)$ or of $U(p,q)$, we use PA to denote its projection in $\mathbb{P}(\mathbb{C}(p,q))$ or $PU(p,q)$.

7.1. Theorem. *Let $\epsilon = \pm 1$ be fixed, and let Δ be a set of vectors in a hermitian space $\mathbb{C}(p,q)$ which lie in the unit quadric $h(\delta,\delta) = \epsilon$. Fix a root of unity $\lambda \neq \pm 1$ and let Γ be the subgroup of $U(p,q)$ generated by the complex reflections $s_\delta(x) = x + \epsilon(\lambda - 1)h(x,\delta)\delta$ for all δ in Δ. Suppose that $p + q > 1$, that Δ consists of a single Γ-orbit, and that Δ spans $\mathbb{C}(p,q)$. Then either Γ is finite or $P\Gamma$ Zariski-dense in $PU(p,q)$.*

Let $\bar{\Gamma}$ be the Zariski closure of a subgroup Γ of $U(p,q)$ which contains the λ-reflections for all vectors δ in a set Δ. Then $\bar{\Gamma}$ also contains the λ-reflections for the set $R = \bar{\Gamma}\Delta$. Indeed, if g is an element of $\bar{\Gamma}$, then

$$g^{-1} s_\delta g = s_{g^{-1}(\delta)}. \tag{7.1}$$

Thus it is enough to establish the following result in order to prove our density theorem:

7.2. Theorem. *Let $\epsilon = \pm 1$ be fixed, and let R be a set of vectors in a hermitian space $\mathbb{C}(p,q)$ which lie in the unit quadric $h(\delta,\delta) = \epsilon$. Fix a root of unity $\lambda \neq \pm 1$ and let M be the smallest algebraic subgroup of $U(p,q)$ which contains the complex reflections $s_\delta(x) = x + \epsilon(\lambda - 1)h(x,\delta)\delta$ for all δ in R. Suppose that $p + q > 1$, that R consists of a single M-orbit, and that R spans $\mathbb{C}(p,q)$. Then either M is finite or $PM = PU(p,q)$.*

We begin with a special case of the theorem for groups generated by a pair of complex reflections.

7.3. Lemma. *Let $\lambda \neq \pm 1$ be a root of unity, and let U be the unitary group of a nondegenerate hermitian form on $\mathbb{C}^2$. Let δ_1 and δ_2 be independent vectors with nonzero inner product, and let Γ be the group generated by complex reflections with common eigenvalue λ. Then either Γ is finite or its image in the projective unitary group is Zariski-dense. In the positive-definite case Γ is finite if and only if the inner products (δ_1, δ_2) lie in a fixed finite set S which depends only on λ and h. In the indefinite case Γ is never finite.*

We treat the definite case first. To begin, note that the group U acts on the Riemann sphere $\mathbb{P}^1$ via the natural map $U \longrightarrow PU$, where PU is the projectivized unitary group. Let PR be the image of $R \subset \mathbb{C}^2$ in $\mathbb{P}^1$. Since λ is a root of unity, the projection $P\Gamma$ is a finite group if and only if Γ is. The finite subgroups of rotations of the sphere are well known. There are two infinite series: the cyclic groups, where the vectors δ are all proportional, and the dihedral groups where $\lambda = -1$. There are three additional groups, given by the symmetries of the five platonic solids, and S is the set of possible values of $h(\delta_1, \delta_2)$ that can arise for these three groups.

We suppose that (δ_1, δ_2) lies outside S, so that $P\Gamma$ is infinite. Then its Zariski closure PM is either PU or a group whose identity component is a circle. In this case PR contains a great circle α. However, PR is stable under the action of PM, hence under the rotations corresponding to axes in PR. Since $\lambda \neq \pm 1$, the orbit PR contains additional great circles which meet α in an angle $0 < \phi \leq \pi/2$. The union of these, one for each point of the given circle, forms a band about the equator, hence has nonempty interior. Such a set is Zariski-dense in the Riemann sphere viewed as a real algebraic variety. Since PR is a closed real algebraic set, $PR = S^2$. Since $PR \cong PM/H$, where H is the isotropy group of a point on the sphere, $PM = PU$.

In the case of an indefinite hermitian form, the group $U = U(1,1)$, acts on the hyperbolic plane via the projection to PU, and $P\Gamma$ is a group generated by a pair of elliptic elements of equal order but with distinct fixed points. One elliptic element moves the fixed point of the other, and so their commutator γ is hyperbolic (c.f. Theorem 7.39.2 of [3]). The Zariski closure of the cyclic group $\{ \gamma^n \}$ is a one-parameter subgroup of PU. Consequently the orbit PR contains a geodesic α through one of the elliptic fixed points. By (7.1) the other points of α are fixed points of other elliptic transformations in PM. Now the orbit PR contains the image of α under each of these transformations, and so PR contains an open set of the hyperbolic plane. This implies that either $PM = PU$ or PM is contained in a parabolic subgroup. Since PM contains non-trivial elliptic elements that last possibility cannot occur, and so $PM = PU$.

Next we show that if the set R which defines the reflections is large, then so is the group containing those reflections.

7.4. Lemma. *Fix a root of unity $\lambda \neq \pm 1$ and $\epsilon = \pm 1$. Let R be a semi-algebraic subset of the unit quadric $h(\delta, \delta) = \epsilon$. Let M be the smallest algebraic subgroup of $U(p, q)$ containing the complex reflections $s_\delta(x) = x + \epsilon(\lambda - 1)h(x, \delta)\delta$, $\delta \in R$. If $p + q > 1$ and if PR is Zariski-dense in $\mathbb{P}(\mathbb{C}(p, q))$, then $M = PU(p, q)$.*

The proof is by induction on $n = p + q$. For $n = 2$ the result follows from the proof of lemma 7.3. Let $n > 2$ and assume $p \leq q$. Then $q \geq 2$. Fix a codimension two subspace of $\mathbb{C}(p, q)$ of signature $(p, q - 2)$ and let W_t be the pencil of hyperplanes of $\mathbb{C}(p, q)$ containing this codimension two subspace. Then the restriction of h to each W_t is a non-degenerate form of signature $(p, q - 1)$.

Consider a subgroup M of $U(p, q)$ which satisfies the hypotheses of the lemma, and let $R_t = R \cap W_t$. Since PR, respectively PR_t is semi-algebraic in $\mathbb{P}(\mathbb{C}(p, q))$, respectively in PW_t, it is Zariski dense if and only if it has non-empty interior in the analytic topology. Thus R has non-empty interior in $\mathbb{P}(\mathbb{C}(p, q))$, and so for dimension reasons PR_t has non-empty interior in PW_t for generic t. Thus PR_t is Zariski dense in PW_t for generic t.

Fix one such value of t, let $W = W_t$ and let $M'(R \cap W) \subset M$ denote the smallest algebraic subgroup of M containing $R \cap W$. Let $M(R \cap W)$ denote the set of restrictions of elements of $M'(R \cap W)$ to W. Then $R \cap W$ and $M(R \cap W)$ satisfy the induction hypothesis, thus $PM(R \cap W) = PU(W)$. Now the orthogonal complement of W is a Zariski closed set, as is $W \cup W^\perp$. Since R is Zariski-dense there is a δ in $R - W$ and a δ' in W such that $h(\delta, \delta') \neq 0$. Consider the function $f_\delta(x) = h(x, \delta')$. If it is constant on the Zariski closure C of $R \cap W$, then the derivative df_δ vanishes on C. Therefore C lies in the intersection of the hyperplane $df(x) = 0$ with W, which is a proper algebraic subset of W. Consequently $R \cap W$ is not Zariski-dense, a contradiction. Thus f_δ is nonconstant and so we can choose δ in $R \cap W$ such that $h(\delta', \delta)$ lies outside the fixed set S. Then lemma 7.3 implies that the unitary group of the plane F spanned by δ and δ' is contained in M. But $U(W)$ and $U(F)$ generate $U(p, q)$ and the proof of the lemma is complete.

To complete the proof of Theorem (7.2) we must show that either R is sufficiently large or that M is finite. Observe that since R is an M-orbit, it is a semi-algebraic set. Let W be a subspace of $\mathbb{C}(p, q)$ which is maximal with respect to the property "$W \cap R$ is Zariski-dense in the unit quadric of W." Our aim is to show that either $W = \mathbb{C}(p, q)$ or that M is finite. Consider first the case $W = 0$. Then the inner products $h(\delta, \delta')$ for any pair of elements in R lie in the fixed finite set S of lemma 7.3. Now let $\delta_1, \ldots, \delta_n$ be a basis of $\mathbb{C}(p, q)$ whose elements are chosen from R. Then the inner products $h(\delta, \delta_i)$ lie in S for all δ and i. Consequently R is a finite set and M, which is faithfully represented as a group of permutations on R, is finite as well.

Henceforth we assume that W is nonzero. If it is not maximal there is a vector δ in $R - W$ and we may consider the function $f_\delta(x) = h(x, \delta)$ on the set $R \cap W$. If f_δ is identically zero for all δ in $R - W$, then $R \subset W \cup W^\perp$. Therefore $\mathbb{C}(p, q) = W + W^\perp$, from which one concludes that $W = W \oplus W^\perp$ and so M is a subgroup of $U(W) \times U(W^\perp)$. But R consists of a single M-orbit and contains a point of W, which implies that $R \subset W$, a contradiction.

We can now assume that there is a $\delta \in R - W$ such that the function f_δ is not identically zero. If one of these functions is not locally constant, then it must take values outside the set S. Then the inner product (x, δ) lies outside S for an open dense set of x in $R \cap W$. For each such x, R is dense in the span of x and δ. We conclude that R is dense in $W + \mathbb{C}\delta$. Thus W is not maximal, a contradiction.

At this point we are reduced to the case in which all the functions f_δ are locally constant, with at least one which is not identically zero. To say that f_δ is locally constant on a dense subset of the unit quadric in W is to say that its derivative is zero on that quadric. Equivalently, tangent spaces to the quadric are contained in the kernel of df_δ, that is, in the hyperplane $\delta^\perp$. But if all tangent spaces to the quadric are contained in that hyperplane, then so is the quadric itself. Then the function in question is identically zero, contrary to hypothesis. The proof is now complete.

To apply the density theorem we need to show that the "complex vanishing cycles" contain a basis for the vanishing cohomology and form a single orbit. These cycles are by definition the eigencomponents of ordinary vanishing cycles. Consider now a generalized Picard-Lefschetz transformation given by (6.4). It can be rewritten as

$$\rho(\gamma)(x) = x + \sum \epsilon_i(\lambda_i - 1)h(x, \delta_i)\delta_i,$$

where the δ_i are complex vanishing cycles and the λ_i are suitable complex numbers. Let

$$\rho(\gamma')(x) = x + \sum \epsilon_i(\lambda_i - 1)h(x, \delta_i')\delta_i'$$

be another generalized Picard-Lefschetz tranformation. If $\gamma' = \kappa^{-1}\gamma\kappa$ then the two preceding equations yield

$$\sum \epsilon_i(\lambda_i - 1)h(x, \delta_i')\delta_i' = \sum \epsilon_i(\lambda_i - 1)h(\kappa.x, \delta_i)\kappa^{-1}.\delta_i,$$

where $\kappa.x$ stands for $\rho(\kappa)(x)$. Comparing eigencomponents on each side we find

$$\delta_i' = \kappa^{-1}.\delta_i,$$

as required. By the same argument as used in §3, one sees that the complex vanishing cycles span $H(i)$.

8. Bounds on the real and complex rank

In this section we derive lower bounds for the complex and real ranks of the groups $G_{d,n}$ of automorphisms of the primitive cohomology $H^n_o(X_{d,n}, \mathbb{R})$ where $X_{d,n}$ is a hypersurface of degree d and dimension n. Recall that for a field k, the k-rank is the dimension of the largest subgroup that can be diagonalized over k. These bounds complete the outline of proof. We also show that all the eigenspaces of the cyclic automorphism σ have the same dimension.

The main result is the following:

8.1. Lemma. *The complex rank of $G_{d,n}$ is at least five for $d \geq 3$, $n \geq 1$, with the exception of $(d,n) = (3,1)$, for which it is one, and $(d,n) = (4,1), (3,2)$ for which it is three. Under the same conditions the real rank is at least two with the exception of the cases $(d,n) = (3,1)$, $(3,2)$ for which the real ranks are one and zero, respectively.*

To prove the first assertion we note that the complex rank is given by $\text{rank}_{\mathbb{C}}\, G_{d,n} = [B_{d,n}/2]$ where $[x]$ is the greatest integer in x and where $B_{d,n} = \dim\, H^n_o(X_{d,n})$ is the primitive middle Betti number. To compute it we compute the Euler characteristic $\chi_{d,n}$ recursively using the fact that a d-fold cyclic cover of $\mathbb{P}^n$ branched along a hypersurface of degree d is a hypersurface of degree d in $\mathbb{P}^{n+1}$. Thus, mimicking the proof of Hurwitz's formula for Riemann surfaces, we have

$$\chi_{d,n} = d\,\chi(\mathbb{P}^n - B) + \chi(B) = d(n+1) + (1-d)\chi_{d,n-1}.$$

Since $\chi_{d,0} = d$, the Euler characteristics of all hypersurfaces are determined. Rewriting this recursion relation in terms of the n-th primitive Betti number we obtain

$$B_{d,n} = (d-1)\left(B_{d,n-1} + (-1)^n\right), \tag{8.1}$$

From it we deduce an expression in closed form:

$$B_{d,n} = (d-1)^n\,(d-2) + \frac{(d-1)^n - (-1)^n}{d} + (-1)^n. \tag{8.2}$$

The preceding two formulas imply that $B_{d,n}$ is an increasing function of n and of d. Now assume $d \geq 3$, $n \geq 1$. Then $d + n \geq 4$. If $d + n \leq 6$, then $(d,n) = (3,3), (4,2), (5,1)$ and $B_{3,3} = 10, B_{4,2} = 21, B_{5,1} = 12$. Thus $B_{d,n} \geq 10$ except when $d + n = 4$ or 5. These are the cases $(d,n) = (3,1), (3,2), (4,1)$ where $B_{d,n} = 2, 6, 6$ respectively. The inequalities on the complex rank are now established.

Let us now turn to the proof of the second assertion of the lemma. For n odd the group $G_{d,n}$ is a real symplectic group. Its real and complex ranks are the same, and so the bound follows from the first assertion.

For n even the group $G_{d,n}$ is the orthogonal group of the cup product on the primitive cohomology. This bilinear form has signature (r, s), and the real rank of G is the minimum of r and s. The signature is computed from the Hodge decomposition: r, the number of positive eigenvalues, is the sum of the $h^{p,q}$ for p even, while s is the sum for p odd. According to the first inequality of lemma 8.2, the Hodge numbers $h^{p,q}(d, n)$ of $X_{d,n}$ satisfy $h^{p,q}(d+1, n) > h^{p,q}(d, n)$. Thus the real rank is an increasing function of the degree. Consequently it is enough to show that it is at least two for quartic surfaces and for cubic hypersurfaces of dimension four or more. For quartic hypersurfaces $h^{2,0} = 1$ and $h^{1,1} = 19$, so $(r, s) = (2, 19)$. For cubic hypersurfaces there is a greatest integer $p \leq n$ such that $h^{p,q} \neq 0$, where $p + q = n$. We will compute this "first" Hodge number and see that under the hypotheses of the lemma, $p > q$. Since n is even, $h^{p,q}$ and $h^{q,p}$ have the same parity. Thus one of r, s is at least two. According to the second inequality of lemma 8.2, $h^{p-1,q+1}(d, n) > h^{p,q}(d, n)$ if $p > q$. Thus $h^{p-1,q+1}(d, n) > h^{p,q}(d, n) > 0$. We conclude that the other component of the signature, s or r, must be at least two. For the Hodge numbers of cubic hypersurfaces of dimension $n = 3k + r$ where $r = 0$, 1, or 2, one uses the calculus of [22] to show the following: (a) if $n \equiv 0 \bmod 3$ then the first Hodge number is $h^{2k,k} = n + 2$, (b) if $n \equiv 1 \bmod 3$ then it is $h^{2k+1,k} = 1$, (c) if $n \equiv 2 \bmod 3$ then it is $h^{2k+1,k+1} = (n + 1)(n + 2)/2$. When $k > 0$ these Hodge numbers satisfy $p > q$, and so the proof of the lemma is complete.

8.2. Lemma *Let $h^{p,q}(d, n)$ be the dimension of $H_o^{p,q}(X_{d,n})$. Then the inequalities below hold:*

$$h^{p,q}(d + 1, n) > h^{p,q}(d, n)$$
$$h^{p,q}(d, n) > h^{p+1,q-1}(d, n) \text{ if } p \geq q$$

Proof: It is enough to prove the inequalities when $X_{d,n}$ is the Fermat hypersurface defined by $F_d(x) = x_0^d + \cdots + x_{n+1}^d = 0$. Because of the symmetry $h^{p,q} = h^{q,p}$, it is also enough to prove the inequalities for $p \geq q$. To this end recall that $h^{p,q} = \dim R^a$, where R is the Jacobian ring for F_d and where $a = (q + 1)d - (n + 2)$ is the degree of the adjoint polynomial in the numerator of the expression

$$\text{res}\, \frac{A\Omega}{F_d^{q+1}}.$$

Now there is a map $\mu : R^{a(q,d)}(F_d) \longrightarrow R^{a(q,d+1)}(F_{d+1})$ defined by $\mu(P) = (x_0 \cdots x_q)P$. This makes sense because $q \leq n$. We claim that that resulting map from $H^{p,q}(X_{d,n})$ to $H^{p,q}(X_{d+1,n})$ is injective but not surjective.

To prove the claim, observe that the Jacobian ideal is generated by the powers x_i^{d-1} and so has a vector space basis consisting of monomials x^M. The same is true of the quotient ring $R(F_d)$. Indeed, a basis is given by (the classes of) those monomials not divisible by x_i^{d-1} for any i. Now consider a polynomial which represents an element of the kernel of μ. It can be be chosen to be a linear combination of monomials x^M which are not divisible by x_i^{d-1} for any i. Its image is represented by a linear combination of monomials $(x_0 \cdots x_q)x^M$. Each of these is divisible by some x_i^d. Thus either x^M is divisible by x_i^d, $i > q$, a contradiction, or by x_i^{d-1}, $i \leq q$, also a contradiction. Thus injectivity part the claim is established.

For the surjectivity part note that image of the map μ has a basis of monomials x^M which are divisible by x_i for $i = 0, \ldots, q$. Thus, to show that μ is not surjective it suffices to show that there is a monomial for $R^{a(q,d+1)}(F_{d+1})$ that is not divisible by x_0. Such a monomial has the form $x_1^{M_1} \cdots x_{n+2}^{M_{n+2}}$ where $M_i \leq d - 1$. It exists if $a(q, d + 1) \leq (n + 1)(d - 1)$. The largest relevant values of q and $a(q, d + 1)$ are $n/2$ and $(n/2 + 1)d - (n + 2)$. For these the preceding inequality holds and so the first inequality of the lemma holds strictly.

For the second inequality we use the fact that basis elements for the Jacobian ring of F_d correspond to lattice points of the cube in $(n + 2)$-space defined by the inequalities $0 \leq m_i \leq d - 2$. A basis for R^a corresponds to the set of lattice points which lie on the convex subset $C(a)$ of the cube obtained by slicing it with the hyperplane $m_0 + \cdots + m_{n+1} = a$. The volume of $C(a)$ is a strictly increasing function of a for $0 \leq a \leq t/2$, where $t = (n + 2)(d - 2)$. For $t/2 \leq a \leq t$ the volume function $V(a)$ is strictly decreasing, and in general its graph is symmetric around $a = t/2$. Let $L(a)$ be the number of lattice points in $C(a)$. If $L(a)$ satisfies the same monotonicity properties as does $V(a)$, then the second inequality follows. To show this, we prove the following result.

8.3. Lemma. *Let $L_{d,n}(k)$ be the number of points in the set $\mathcal{L}_{d,n}(k) = \{\ x \in \mathbb{Z}^n \mid 0 \leq x_i \leq d,\ x_1 + \cdots + x_n = k\ \}$. Assume that $n > 1$. Then $L_{d,n}(k)$ is a strictly increasing function of k for $k < dn/2$ and is symmetric around $k = dn/2$.*

Proof: Symmetry follows from the bijection $\mathcal{L}_{d,n}(k) \longrightarrow \mathcal{L}_{d,n}(dn-k)$ given by $x \mapsto \delta - x$ where $\delta = (d\ , \ldots,\ d)$. We shall say that these two sets are dual to eachother. For the inequality we argue by induction, noting first that $L_{d,2}(k) = k+1$ for $k \leq d$. Now observe that $\mathcal{L}_{d,n}(k)$ can be written as a disjoint union of sets $S_i = \{\ x \in \mathcal{L}_{d,n}(i) \mid x_n = k - i\ \}$ where i ranges from $k-d$ to k. Thus

$$L_{d,n}(k) = \sum_{i=k-d}^{k} L_{d,n-1}(i).$$

Consequently

$$L_{d,n}(k+1) - L_{d,n}(k) = L_{d,n-1}(k+1) - L_{d,n-1}(k-d).$$

By the induction hypothesis the right-hand side is positive if $k - d < (n-1)d/2$ and if $k+1$ is not greater than the index dual to $k-d$, namely $(n-1)d - (k-d)$. Thus we require also that $k+1 \leq (n-1)d - (k-d)$. Both inequalities hold if $k < nd/2$, which is what we assume. Thus the proof is complete.

Dimension of the eigenspaces.

We close this section by noting that the eigenspaces $H^n(X)(\lambda)$ for $\lambda \neq 1$ all have the same dimension, explaining why the primitive middle Betti number is divisible by $d-1$, where d is the degree. Indeed, we have the following,

$$\dim\ H^n(X, \mathbb{C})(\lambda) = \dim\ H^n(X, \mathbb{C})(\mu) = \dim\ H^n(\mathbb{P}^n - B, \mathbb{C}) + (-1)^n. \tag{8.3}$$

When the degree is prime there is a short proof: consider the field $k = \mathbb{Q}[\omega]$ where ω is a primitive d-th root of unity and observe that its Galois group permutes the factors $H^n(X,k)(\lambda)$ for $\lambda \neq 1$. For the general case let $p : X \longrightarrow \mathbb{P}^n$ be the projection and note that $H^n(X, \mathbb{C}) = H^n(\mathbb{P}^n, p_*\mathbb{C})$. The group of d-th roots of unity acts on $p_*\mathbb{C}$ and decomposes it into eigensheaves $\mathbb{C}_\lambda$, where $\lambda^d = 1$. Thus the λ-th eigenspace of $H^n(X, \mathbb{C})$ can be identified with $H^n(\mathbb{P}^n, \mathbb{C}_\lambda)$. The component for $\lambda = 1$ is one-dimensional and is spanned by the hyperplane class. For $\lambda \neq 1$ the sheaf $\mathbb{C}_\lambda$ is isomorphic to the extension by zero of its restriction to $\mathbb{P}^n - B$. Thus the eigenspace can be identified with $H^n(\mathbb{P}^n - B, \mathbb{C}_\lambda)$. By the argument of lecture 8 in [7] used in the proof of vanishing theorems, the groups $H^i(\mathbb{P}^n - B, \mathbb{C}_\lambda)$ vanish for $i \neq n,\ \lambda \neq 1$. Thus $\dim\ H^n(\mathbb{P}^n - B, \mathbb{C}_\lambda) = (-1)^n \chi(\lambda)$, where $\chi(\lambda)$ is the Euler characteristic of $\mathbb{C}_\lambda$. Fix a suitable open tubular neighborhood U of B and a good finite cell decomposition K of $\mathbb{P}^n - U$. Then $\chi(\lambda)$ is the Euler characteristic of the complex of $\mathbb{C}_\lambda$-valued cochains on K, which depends only on the number of cells in each dimension, not on λ. This establishes the first equality above. For the second use $\chi(\lambda) = \chi(1)$ and the vanishing of $H^i(\mathbb{P}^n - B, \mathbb{C})$ for $i \neq n, 0$.

9. Remarks and open questions

We close with some remarks on (a) the possiblity of an isomorphism $\Phi \cong \Gamma \times \Gamma'$, (b) the impossibility of producing additional representations by iterating the suspension (globally), and (c) generalizations of the main theorem.

(A) Products

So far everything that has been said is consistent with an isomorphism between Φ and the product $\Gamma \times \Gamma'$, where Γ' is the monodromy group $\bar{\rho}'(\Phi)$. This, however, is not the case, at least for surfaces, for we can show that *if k is a divisor of d and d is odd, then $\Phi_{d,2}$ and $\Gamma \times \Gamma'$ are not isomorphic.* The argument is based on the fact that the abelianization of Φ is a cyclic group of order equal to the degree of the discriminant, which we denote by r. This is because (a) the generators $g_1\ , \ldots,\ g_r$ of Φ are mutually conjugate, hence equal in the abelianization, (b) $g_1 \cdots g_r = 1$, (c) the additional relations are trivial when abelianized. See [37]. For the last point note that Φ is also the fundamental group of the complement of a generic plane section Δ' of Δ. This complement has nodes and cusps as its only singularities. The nodes yield relations of the form $gg' = g'g$ where g and g' are conjugates of the given generators. The cusps yield braid relations $gg'g = g'gg'$.

Both are trivial in the abelianization. Thus the abelianization is generated by a single element with relation $g^r = 1$. The degree of the discriminant is given in [16], page 6, line 2:

$$r = \deg(\Delta) = 4(d-1)^3.$$

If Φ is isomorphic to the Cartesian product, then there is a corresponding isomorphism of abelianizations. Let us therefore compute what we can of the abelianizations of Γ and Γ'. For Γ we note that the generators are the elements g_i as above satisfying additional relations which include $g_i^2 = 1$. Therefore Γ abelianized is a quotient of $\mathbb{Z}/2$. Consider next the case of Γ' for cyclic covers of degree k. Then Γ' is a product of groups $\Gamma'(i)$ for $i = 1, \ldots, k-1$. Generators and relations are as in the previous case except that among the additional relations are $g_i^{2k} = 1$ instead of $g_i^2 = 1$. Therefore the abelianization is a quotient of $\mathbb{Z}/2k$. Consequently the abelianization of the product $\Gamma \times \Gamma'$ is a quotient of the product of $\mathbb{Z}/2$ with a product of $\mathbb{Z}/2k$'s. But the largest the order of an element in such a quotient can be is $2k$, which is always less than the degree of the discriminant, provided that $d > 2$, which is the case.

(B) Suspensions

Since Φ is not in general isomorphic to $\Gamma \times \Gamma'$ it is natural to ask whether there are further representations with large kernels. One potential construction of new representations is given by iterating the suspension. By this we mean that we take repeated double covers. Unfortunately, this produces nothing new, since it turns out that the global suspension is periodic of period two. To make a precise statement, let $P(x)$ be a polynomial of degree $2d$ which defines a smooth hypersurface X in $\mathbb{P}^n$. Let $X(2)$ be the hypersurface defined by

$$P(x) + y_1^2 + y_2^2$$

in a weighted projective space $\mathbb{P}^{n+2}$ where the x_i have weight one and the y_i have weight d. *Then there is an isomorphism*

$$H_o^n(X) \otimes T \longrightarrow H_o^{n+2}(X(2)),$$

where T is a trivial Hodge structure of dimension one and type $(1,1)$ and where the subscript denotes primitive cohomology.

For the proof we note that the map

$$\frac{A\Omega(x)}{P^{q+1}} \mapsto \frac{A\Omega(x, y_1, y_2)}{(y_1^2 + y_2^2 + P)^{q+2}}$$

is well-defined and via the residue provides an isomorphism compatible with the Hodge filtrations which is defined over the complex numbers. However, it can be defined geometrically and so is defined over the integers. To see why, consider first the trivial case $g(x) = f(x) + y_1^2 + y_2^2 = 0$ in affine coordinates, where x is a scalar variable and f has degree $2d$. Thus $f(x) = 0$ defines a finite point set, and $g(x) = 0$ is its double suspension. Let p be one point of the given finite set. Then $f(p) = 0$, so the locus $\{ (p, y_1, y_2) \mid y_1^2 + y_2^2 = 0 \}$ lies on the double suspension. This locus is a pair of lines meeting in a point, and the statement remains true in projective coordinates. Thus we may associate to p a difference of lines $\ell_p - \ell'_p$. This map induces an isomorphism $H_0(X, \mathbb{Z}) \longrightarrow H_2(X(2), \mathbb{Z})$ which is in fact a morphism of Hodge structures. For the general case we parametrize the construction just made. The map in cohomology which corresponds to the previous construction is the dual of the inverse of the map in homology.

(C) Generalizations

The main theorem 1.2 can be generalized in a number of ways. First, using the techniques of [36], it is certainly possible to get sharp results for various kinds of weighted hypersurfaces, just as we have obtained sharp results for standard hypersurfaces. Second, one can prove a quite general (but not sharp) result that reflects the fairly weak hypothesis of criterion 2.2:

9.1. Theorem *Let L be a positive line bundle on a projective algebraic manifold M of dimension at least three. Let P be the projectivization of the space of sections of L^d, and let Δ be the discrimant locus defined by sections of L^d whose zero set Z is singular. Then for d sufficiently large the kernel of the monodromy representation of $\Phi = \pi_1(P - \Delta)$ is large and its image is a lattice.*

The monodromy representation has the primitive cohomology

$$H^{m-1}(Z)_0 = kernel \left[H^{m-1}(Z) \stackrel{Gysin}{\longrightarrow} H^{m+1}(M) \right]$$

as underlying vector space. The results needed for the proof are all in the literature. First, note that the condition that a section s of L^d have a singularity of type "$x^3 + y^3 + z^4 +$ sum of squares" at a given point is set of linear conditions and so can be satisfied for d sufficiently large. Consequently by the Beauville-Ebeling-Janssen argument, the image of the natural monodromy representation is a lattice. Second, by the results of Green [20], the local Torelli theorem for cyclic covers holds for d sufficiently large, so some component of the second monodromy representation has nonzero differential. The standard argument used just following Theorem 3.4 proves that the discriminant locus is irreducible, and so Theorem 7.2 applies to give Zariski-density for the second monodromy representation. Finally, the Hodge numbers, like the standard case of projective hypersurfaces, are polynomials in d with positive leading coefficient and of degree equal to the dimension of M. Consequently they are large for d large, and therefore both the real and complex rank of the relevant algebraic groups can be assumed sufficiently large by taking d large enough. Thus the hypotheses of criterion 2.2 are satisfied.

For a quick proof of the statement on the behavior of the Hodge numbers, consider first the Poincaré residue sequence

$$0 \longrightarrow \Omega^m_M \longrightarrow \Omega^m_M(L^d) \longrightarrow \Omega^{m-1}_Z \longrightarrow 0,$$

where Z is a smooth divisor of L^d and m is the dimension of M. From the Kodaira vanishing theorem we have

$$H^0(\Omega^{m-1}_Z)_0 \cong cokernel \left[H^0(\Omega^m_M) \longrightarrow H^0(\Omega^m_M(L^d)) \right].$$

By the Riemann-Roch theorem the dimension of the right-most term is a polynomial with leading coefficient Cd^m, while the dimension of the middle term is constant as a function of d. Therefore the Hodge number in question is a polynomial in d of the required form. For the other Hodge numbers we use the identification

$$H^q(\Omega^p_Z)_0 \cong cokernel \left[H^0(\Omega^{m-1}_Z \otimes \Theta_M \otimes N^{q-1}_Z) \longrightarrow H^0(\Omega^{m-1}_Z \otimes N^q_Z) \right],$$

where N is the normal bundle of Z in M, where Θ_M is the holomorphic tangent bundle of M, and where $p+q = m-1$. See Proposition 6.2, [8], a consequence of Green's Koszul cohomology formula (Theorem 4.f.1 in [19]) for d sufficiently large. Now tensor the Poincaré residue sequence with L^{qd} to get

$$0 \longrightarrow \Omega^m_M(L^{qd}) \longrightarrow \Omega^m_M(L^{(q+1)d}) \longrightarrow \Omega^{m-1}_Z \otimes N^q_Z \longrightarrow 0.$$

From the Kodaira vanishing theorem and the Riemann-Roch formula one finds that the dimension of the right-hand part of the cokernel formula is a polynomial with leading term $C((q+1)d)^m$, where $C = c_1(L)^m/m!$. A similar argument shows that the dimension of the left-hand part is a polynomial with leading coefficient $C(qd)^m$. Thus the leading term of dim $H^q(\Omega^p)_0$ is bounded below by a positive constant depending on L, q, and M, times d^m.

(D) Questions.

We close with some open questions. The main problem is, of course, to understand the nature of the groups $\Phi_{d,n}$. Are they linear? Are they residually finite? It seems reasonable to conjecture that in general they are not linear groups, and, in particular, are not lattices in Lie groups. We settle this last question for $\Phi_{3,2}$ in the note [1].

The structure of $\Phi_{d,n}$ is closely related to the structure of the kernel K of the natural monodromy representation. For $n = 0$, K is the pure d-strand braid group the sphere and so is finitely generated. For $(d, n) = (3, 2)$, the case of cubic surfaces, K is not finitely generated (see [1]). It is therefore natural to ask when K is finitely generated and when it is not.

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