Generic Integral Manifolds for Weight Two Period Domains

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Abstract

We define the notion of a generic integral element for the Griffiths distribution on a weight two period domain, draw the analogy with the classical contact distribution, and then show how to explicitly construct an infinite-dimensional family of integral manifolds tangent to a given element.

1 Introduction

In this note we shall study a class of integral manifolds for a generalization of the classical contact distribution. This distribution arises naturally in algebraic geometry as the infinitesimal restriction satisfied by the period maps of algebraic surfaces. To be more precise, consider the group $G = SO(2p, q)$, the compact subgroup $U(p) \times SO(q)$, and the homogeneous space

$$D = G/V.$$

The homogeneous space parametrizes Hodge decompositions

$$H_{\mathbf{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

on a fixed complex vector space endowed with a nondegenerate, symmetric bilinear form defined over an underlying real vector space $H_{\mathbf{R}}$ whose complexification is $H_{\mathbf{C}}$. The models for these spaces are the primitive second cohomology of an algebraic surface with real or complex coordinates. If one has a family of algebraic surfaces S_t parametrized by t, then one has a family of Hodge decompositions which are determined by the subspaces $H^{2,0}(S_t)$. As Griffiths showed in [4], these spaces satisfy

$$\frac{d}{dt} H^{2,0}(S_t) \subset H^{2,0}(S_t) \oplus H^{1,1}(S_t).$$

This condition, which asserts that a famly of subspaces $H_t^{2,0}$ cannot vary arbitrarily, but rather like a Frenet frame of a space curve, defines the *Griffiths distribution* on the homogeneous space D. Thus families of algebraic surfaces define integral manifolds of this distribution.

In [2] it was shown that for $p > 1$ even and $q > 1$, integral manifolds which are of maximal dimension are quite special: as germs they are congruent under the action of G; moreover, they correspond to the germ of an imbedding $U(p, q/2) \longrightarrow SO(2p, q)$. Thus germs of integral manifolds of this kind are rigid and depend on finitely many parameters.

Here we shall study certain integral manifolds which are maximal with respect to inclusion: they do not lie in a larger dimensional integral manifold. There is natural class of these which we call "generic," characterized by the existence of a family of vectors $v(t)$ in $H_t^{2,0}$ such that the partial derivatives $\partial v/\partial t_i$ span $H_t^{1,1}$. (See definition 1). These integral manifolds behave quite differently: they are not rigid, and they depend on infinitely many parameters. Moreover, as we shall see in Theorem 1 and the paragraph which follows it, generic integral manifolds are given quite explicitly by a system of generating functions. This is in strict analogy to the case of the contact distribution, where maximal integral manifolds are flexible and determined explicity by a generating function. The function(s) constitute the infinite-dimensional parameters.

2 Generalized Contact Distributions

Recall that the contact distribution is the annihilator E in the tangent bundle of $\mathbf{R}^{2n+1}$ or $\mathbf{C}^{2n+1}$ of the one-form

$$\omega = dz - y \cdot dx = dz - \sum_{i=1}^{n} y_i dx_i,$$

and that integral manifolds are by definition submanifolds tangent to E. They are of dimension at most n and those of maximum dimension — the Legendre manifolds (see [1]) — are given, up to certain admissible changes of coordinates, by a generating function f through the prescription

$$z = f(x) \tag{1}$$

$$y = \nabla f. \tag{2}$$

Thus maximal integral manifolds are graphs of one-jets of functions, and so constitute an infinite-dimensional family.

After a suitable change of variables, any distribution of codimension one can be reduced locally to a product of the trivial distribution and a contact distribution. Consequently the local nature of their maximal integral manifolds is completely understood. By "maximal" we mean "maximal with respect to inclusion." The situation for distributions of codimension greater than one is completely different. There is no general theory which answers the basic questions and the nature of the integral manifolds is in general very complicated. There is always a maximum dimension for integral manifolds and manifolds of that dimension are obviously maximal with respect to inclusion. However, maximal integral manifolds may not be of maximal dimension. The distribution which we shall study here is of codimension greater than one and exhibits the just-mentioned behavior. It is a natural one to study for a number of reasons. First, it is defined by a matrix-valued analogue of the contact distribution. Second, it arises as a local model U for the Griffiths distribution on a period domain D of weight two [4] discussed in the introduction.

We shall now describe this local model U, then give the main result of the paper. To this end, consider the group G of matrices of the form

$$g = \begin{pmatrix} 1_p & 0 & 0 \\ X & 1_q & 0 \\ Z & Y & 1_p \end{pmatrix}$$

where 1_n denotes the $n \times n$ identity matrix. A basis for the left-invariant one-forms on G is given by the Maurer-Cartan form

$$\Omega \underset{def}{=} g^{-1}dg = \begin{pmatrix} 0 & 0 & 0 \\ dX & 0 & 0 \\ \omega & dY & 0 \end{pmatrix} \tag{3}$$

where

$$\omega = dZ - YdX \tag{4}$$

On G define a distribution E as the set of tangent vectors which annihilate the entries of ω. When $p = 1$, the group is the Heisenberg group and ω is the contact form. Our local model U is the unipotent subgroup defined by the equations

$$Y = {}^tX \tag{5}$$

and

$$Z + {}^tZ = {}^tXX. \tag{6}$$

A neighborhood of the identity in U is isomorphic to a neighborhood of a fixed but arbitrary point of D. Under this identification the restriction of the Griffiths distribution is the same as the restriction of the distribution E to U.

Since X determines Y and the symmetric part of Z, coordinates on U are given by the entries of X and the skew-symmetric part of Z. Therefore U has dimension $pq + p(p-1)/2$. From (4) and (5), one finds that

$$\omega = dZ - {}^tXdX, \tag{7}$$

and from the exterior derivative of (6) one finds that

$$\omega^+ = 0, \tag{8}$$

where $\omega = \omega^+ + \omega^-$ is the decomposition into symmetric and antisymmetric parts. Thus E has dimension pq and codimension $p(p-1)/2$. Our main interest will be in the case $p > 2$, i.e., codimension greater than one.

Consider now a subspace S of the tangent space to U at some point. If it is tangent to an integral manifold it annihilates not only ω but also $d\omega$. An arbitrary subspace satisfying these two conditions — a potential tangent space to an integral manifold — is called an *integral element*. For the contact distribution all integral elements are integrable. For the Griffiths distribution "integrability" holds for integral elements of maximal dimension ($pq/2$ when q even, $p(q-1)/2+1$ when q odd, [**?**]). For other integral elements, e.g., those which are maximal with respect to inclusion but not of maximal dimension, one could presumably answer the integrability question using the Cartan-Kähler theory. What we do instead is to solve the integrability problem explictly for generic integral elements, which are easily shown to be maximal [2]:

Theorem 1 *Let S be a generic q-dimensional integral element for a period domain with Hodge numbers $p = h^{2,0}$, $q = h^{1,1}$, where $p > 1$. Then S is tangent to an integral manifold. Such integral manifolds are determined in a canonical way by holomorphic functions $f_2, \cdots, f_p$ of a complex variable $u = (u_1, \cdots, u_q)$ which satisfy the system of partial differential equations*

$$[H_{f_i}, H_{f_j}] = 0 \tag{9}$$

where H_f is the Hessian matrix of f. The space of solutions to this equation for fixed S is infinite-dimensional.

The set of generic integral elements is open in the set of all q-dimensional integral elements. We shall formulate and prove this result in the next section. To explain the canonical construction, recall that the entries of the matrices X and the skew-symmetric part of Z give coordinates on U. Thus an integral manifold will be specified by giving these coordinates in terms of the functions f_i. To do so, let $[a_1, \cdots, a_p]$ denote the matrix with column vectors a_i, set

$$X(u) = [\,u, \nabla f_2, \cdots, \nabla f_p\,],$$

and put

$$Z_{j1}(u) = f_j(u),$$

where $j > 1$. For the entries Z_{jk} with $j > k$, choose arbitrary solutions of the equations

$$dZ_{jk} = \sum_a X_{aj} dX_{ak}$$

deduced from the jk entry of

$$\omega = dZ - {}^tX dX = 0.$$

Use these to determine the antisymmetric part of Z. For the symmetric part use the quadratic equation (6). The analogy with the contact system — both in the form of the equations and the form of the solutions, is clear. Indeed, when $p = 2$ the equation $\omega = 0$ is the same as the equation

$$dZ_{12} - \sum_a X_{a1} dX_{a2} = 0$$

Thus, if we set $z = Z_{12}$, $x = (x_{11}, \cdots, x_{1q})$. and $y = (x_{21}, \cdots, x_{2q})$, then both equations and solutions coincide with those of the contact case. Note, however, that the relations (9) are a new feature of the case $p > 2$.

3 Integral elements

In order to give a precise definition of "generic" we describe in some detail the tangent space of D at a fixed point of reference and the integral elements it contains. To this end we choose the local correspondence between D and U so that identity matrix of U is mapped to the reference point. Thus the Lie algebra $\mathfrak{u}$ — the tangent space at the identity of U — corresponds to the tangent space of D at the reference point. Now consider a curve $g(t)$ based at the identity matrix with arbitrary initial tangent $\tau = g'(0)$. It has the form

$$g(t) = \begin{pmatrix} 1_p & 0 & 0 \\ a(t) & 1_q & 0 \\ b(t) & {}^t a(t) & 1_p \end{pmatrix}$$

where

$$b = c(t) + \frac{1}{2} {}^t a(t) a(t)$$

with $c(t)$ is skew-symmetric, and where $a(0) = 0$, $c(0) = 0$. Differentiating, we find

$$\tau = \begin{pmatrix} 0 & 0 & 0 \\ \phi & 0 & 0 \\ \psi & {}^t\phi & 0 \end{pmatrix}$$

where $a'(0) = \phi$ and $c'(0) = \psi$ are arbitrary matrices subject to the condition that ψ be skew-symmetric. We can read this as saying $\Omega(\tau) = \tau$, where Ω is the Maurer-Cartan form (3). In more detail,

$$dX(\tau) = \phi \tag{10}$$

$$dZ(\tau) = \psi. \tag{11}$$

Thus a matrix $\tau(\phi, \psi)$ is an element of E if an only if $\psi = 0$. Consequently the map

$$\phi \mapsto \tau(\phi) \underset{def}{=} \tau(\phi, 0)$$

defines an isomorphism of E at the identity with $q \times p$ matrices, i.e., with linear maps

$$\phi : \mathbf{C}^p \longrightarrow \mathbf{C}^q$$

Now let $\mathfrak{a} \subset \mathfrak{u}$ be an integral element and let $\tau_i = \tau(\phi_i)$ be vectors in $\mathfrak{a}$. By definition $\mathfrak{a}$ annihilates

$$d\omega = -{}^t dX \wedge dX.$$

But

$$\begin{aligned} d\omega(\tau_1, \tau_2) &= -{}^t dX(\tau_1) dX(\tau_2) + {}^t dX(\tau_2) dX(\tau_1) \\ &= -{}^t\phi_1\phi_2 + {}^t\phi_2\phi_1 \end{aligned} \tag{12}$$

so that the commutator

$$(\phi_1, \phi_2) \underset{def}{=} {}^t\phi_1\phi_2 - {}^t\phi_2\phi_1 \tag{13}$$

vanishes. Equivalently, the Lie bracket $[\tau_1, \tau_2]$ vanishes. Thus we may regard $\mathfrak{a}$, either as a subspace of $\mathfrak{u}$ or of the linear maps from $\mathbf{C}^p$ to $\mathbf{C}^q$, as an *abelian subspace*. To summarize:

Lemma 1 *A subspace of $\mathfrak{u}$ is an integral element if and only if it is an abelian subspace.*

We can now define what me mean by generic:

Definition 1 *A q-dimensional abelian subspace $\mathfrak{a}$ of $\mathfrak{u}$ is generic if there is a vector $v \in \mathbf{C}^p$ such that $\mathfrak{a}(v) = \mathbf{C}^q$.*

By $\mathfrak{a}(v)$ we mean the space $\{ \phi(v) \mid \phi \in \mathfrak{a} \}$. To justify the terminology we claim that (a) there are such spaces and (b) the condition that a space be generic is an open one. For the first point let $\{ e_i \}$ denote the standard basis for $\mathbf{C}^n$ and let $v \cdot w$ denote the complex dot product. Then the commutator as defined in (13) of $q \times p$ matrices $a = [a_1, \cdots, a_p]$ and $b = [b_1, \cdots, b_p]$, is the skew-symmetric matrix with entries

$$(a, b)_{ij} = a_i \cdot b_j - b_i \cdot a_j.$$

Set

$$M_i = [e_i, 0, \cdots, 0]$$

for $i = 1..q$ and let $\mathfrak{a}_0$ be their span. It is clear that $(M_i, M_j) = 0$, so that $\mathfrak{a}_0$ is abelian. For the second point consider the space A of framed abelian subspaces, that is, abelian subspaces endowed with a basis $\{ M_i \}$. Consider the function F_v on A defined by

$$(M_1, \cdots, M_q) \mapsto M_1(v) \wedge \cdots \wedge M_q(v)$$

The union $\mathcal{G}$ of the sets $\mathcal{G}(v) = \{ \ (M_i) = \text{basis for an abelian space} \ \mid F_v(M_i) \neq 0 \ \}$ is open in A and contains any framing of $\mathfrak{a}_0$, whence the claim.

Now consider the f transformation defined by

$$\begin{aligned} X &\longrightarrow BXA \\ Z &\longrightarrow {}^tAZA \end{aligned} \tag{14}$$

where X is a $q \times p$ matrix, Z is a $p \times p$ matrix, where A is invertible and where B is complex orthogonal. The set of such transformations constitutes a complex Lie group $H \cong GL(p, \mathbf{C}) \times O(p, \mathbf{C})$ which acts on the local model U. This action fixes the identity and acts on the form ω by

$$\omega \longrightarrow {}^tA\omega A$$

Consequently it preserves the distribution E and so maps integral manfifolds to integral manifolds. Therefore in studying integral manifolds of E we may do so up to the action of H. Note also that for commutators,

$$(BX_1A, BX_2A) = {}^tA(X_1, X_2)A,$$

so that the transform of an abelian space is an abelian space. From this one sees that the orbit of $\mathcal{G}(e_1)$ is $\mathcal{G}$; consequently, we may, without loss of generality, reason about $\mathcal{G}(e_1)$ in place of $\mathcal{G}$. But an element of $\mathcal{G}(e_1)$ is a q-dimensional abelian space with a basis elements of the form

$$M_j = [\, e_j, * \,, \, \cdots \,, *],$$

where "$*$" stands for a column vector. We shall call such bases "distinguished."

Distinguished bases for abelian spaces can be characterized as follows. Given a matrix A, let $(A)_k$ denote the k-th column. Consider next a system of $q \times q$ matrices $\{ \ A_j \ \}$, where $j = 2 \dots p$, and construct a new system of $q \times p$ matrices

$$M_k = [e_k, \ (A_2)_k \,, \, \cdots \,, \, (A_p)_k],$$

where $k = 1 \dots q$. The correspondence $\{ \ A_j \ \} \longrightarrow \{ \ M_k \ \}$ is one-to-one, and a routine computation shows the following:

Proposition 1 *The span of a distinguished basis $\{ \ M_k \ \}$, $k = 1 \dots q$, is an abelian space if and only if $\{ \ A_j \ \}$, $j = 2 \dots p$, is a commuting set of symmetric matrices. The span of the M_k is then a maximal abelian space.*

Moreover, it is not hard to show that the natural relation between Hessians and tangent spaces holds:

Proposition 2 *Let $\{ \ A_j \ \}$ be a commuting set of symmetric matrices and let $\{ \ f_j \ \}$ be a solution to (9) such that $f_j(0) = 0$, $\nabla f_i(0) = 0$, and $H_{f_j}(0) = A_j$. Then the tangent space at the identity to the associated integral manifold is the abelian space associated to $\{ \ A_j \ \}$ which has basis $\{ \ M_k \ \}$.*

4 The canonical construction

We will now show that an integral manifold of E whose tangent space at the identity is in $\mathcal{G}(e_1)$ is given locally by the canonical construction. To this end, note that the genericity hypothesis (the condition $\mathfrak{a}(e_1) = \mathbf{C}^q$) is equivalent to the condition that the components of $(dX)_1$ are independent. In this case

$$\eta = dX_{11} \wedge \cdots \wedge dX_{r1}$$

is nonzero. By shrinking M we may assume that the product η is nonzero on all of M, so that the functions X_{a1} are independent on it. Then (7) implies that

$$dZ_{ij} = \sum X_{ai} dX_{aj}.$$

Consider in particular the case $j = 1$, $i > 1$, for which we obtain the equation

$$dZ_{i1} = \sum X_{ai} dX_{a1}.$$

Since no dX_{ab} for $b \neq 1$ occur, we see that Z_{i1} may be viewed as function f_i of the entries of $(X)_1$, the first column of X. Moreover, the X_{ai} are functions of these same entries, namely,

$$X_{ai} = \frac{\partial f_i}{\partial X_{a1}}$$

So far we have used just some of the equations $\omega = 0$ determined by (7). These equations assert that there exist certain functions Z_{ij} for $i > j > 1$ of $(X)_j$ which are in turn functions of $(X)_1$. Consequently the already-determined forms $\phi_{ij} = dZ_{ij}$ must be closed. Now

$$\phi_{ij} = \sum_a \frac{\partial f_i}{\partial X_{a1}} d\left(\frac{\partial f_j}{\partial X_{a1}}\right) = \sum_{ab} \frac{\partial f_i}{\partial X_{a1}} \frac{\partial^2 f_j}{\partial X_{a1} \partial X_{b1}} dX_{b1}$$

and so

$$d\phi_{ij} = \sum_{abc} \left[\frac{\partial^2 f_i}{\partial X_{a1} \partial X_{c1}} \frac{\partial^2 f_j}{\partial X_{a1} \partial X_{b1}} dX_{c1} \wedge dX_{b1} + \frac{\partial f_i}{\partial X_{a1}} \frac{\partial^3 f_j}{\partial X_{a1} \partial X_{b1} \partial X_{c1}} dX_{c1} \wedge dX_{b1} \right]$$

The third partial derivative is symmetric in b and c whereas the product $dX_{c1} \wedge dX_{b1}$ is antisymmetic in these indices. Consequently the second sum vanishes. Therefore the first sum must vanish. Since the coefficient of $dX_{c1} \wedge dX_{b1}$ is symmetric in b and c we conclude that it must vanish. But inspection reveals that coeffient to be the bc entry of the commutator

$$[H_{f_i}, H_{f_j}],$$

where H denotes the Hessian matrix. Consequently the consistency of our overdetermined system is just the set of partial differential equations (9).

5 Existence results

Let us now consider the problem of the existence and nature of solutions to the system of partial differential equations $[H_{f_i}, H_{f_j}] = 0$. The most basic question is whether enough there are enough solutions to pass an integral manifold through an arbitrary integral element. By Proposition (1) this is equivalent to the problem of constructing functions whose Hessians commute and whose values at the origin are given commuting symmetric matrices $\{ A_\ell \}$. For these it is enough to take the quadratic functions

$$f_\ell = \frac{1}{2}\sum_{ij}(A_\ell)_{ij}u_i u_j$$

We have therefore shown the following and with it part of (1):

Proposition 3 *Any generic element is tangent to an integral manifold.*

Let us consider now the problem of finding additional solutions to the equations (9). Note first that if two functions f and g have diagonal Hessians then they automatically satisfy $[H_f, H_g] = 0$. The condition that the Hessians be diagonal is the overdetermined system of equations

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = 0 \qquad \text{for } i \neq j, \tag{$*$}$$

In the case of two variables there is just one equation, a form of the wave equation, which has solutions of the form $h_1(x_1) + h_2(x_2)$. The solutions in the general case have the same form,

$$g(x_1\,, \cdots, x_n) = \sum_i h_i(x_i),$$

where the h_i are arbitrary functions of one variable. Therefore an integral manifold is specified by a set of functions $\{ h_{ij}(u) \}$, where

$$f_i(x_1\,, \cdots, x_s) = \sum_j h_{ij}(x_j).$$

This does not give a complete set of solutions, but it does give an infinite-dimensional set. Moreover, we can choose the h_{ij} in such a way that so that the Hessian of f_i at the origin is an arbitrary diagonal quadratic form. In fact, we can do somewhat more. Let $\{ A_\ell \}$ be a set of commuting complex symmetric matrices at least one of which has distinct eigenvalues. Then there is a set of common eigenvectors which form a basis for $\mathbf{C}^q$ and which are orthogonal relative to the complex dot product. Consequently there is a complex orthogonal matrix C which simultaneously diagonalizes the A_ℓ. Let D_ℓ be the diagonal matrix corresponding to A_ℓ, where $CA_\ell{}^tC = D_\ell$. There is an infinite dimensional family of solutions (f'_ℓ) to (9) such that $H_{f'_\ell}(0) = D_\ell$. Let $f_\ell(x) = f'_\ell(Cx)$. Then

$$H_{f_\ell} = {}^tCH_{f'_\ell}C$$

and, since C is complex orthogonal,

$$[H_{f_\ell}, H_{f_m}] = {}^tC[H_{f'_\ell}, H_{f'_m}]C$$

Thefore the infinite-dimensional family of functions (f_ℓ) is also a set of solutions to (9), and each member has the specified initial Hessians (A_ℓ). Consequently our previous integration result (5) can be strengthened:

Theorem 2 *Any generic abelian space of dimension r is tangent to an integral manifold. Moroever, the set of germs of integral manifolds tangent to this space is infinite dimensional.*

Remark.

Consider the case $q = 2$ with $p > 2$ arbitrary. Fix f_2 arbitrarily but generically in the sense that the Hessian generically has distinct eigenvalues. Consider the equations $[H_{f_2}, H_{f_j}] = 0$, for $j > 2$. For given j one has a single non-trivial partial differential equation which is linear of second order in f_2. By the Cauchy-Kowaleska Theorem [5] the solution space is infinite-dimensional. Now let $V(u)$ be the matrix of eigenvectors of $H_{f_2}(u)$. Then the transformation $A \longrightarrow {}^tV(u)AV(u)$ simultaneously diagonalizes all of the matrices $H_{f_i}(u)$. Consequently all of these matrices commute with each other, i.e., the functions f_i solve (9). The case $q > 2$ is more complicated because the partial differential equations constitute an overdetermined system, somewhat like the system $\nabla f = \xi$ for a given vector field ξ.

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