Extensions of mixed Hodge structures

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1. Introduction

According to Deligne, the cohomology groups of a complex algebraic variety carry a generalized Hodge structure, or, in precise terms, a mixed Hodge structure [2]. The purpose of this paper is to introduce an absract theory of extensions of mixed Hodge structures which has proved useful in the study of low-dimensional varieties [1]. To give the theory meaning, we will give one simple, but illustrative application:

Theorem A. *Let X be an irreducible projective algebraic curve whose singularities are ordinary and whose normalization is non-hyperelliptic (no two-sheeted cover of the Riemann sphere exists). Then X is determined by the polarized mixed Hodge structure on $H^1(X)$.*

In very rough outline, the proof is as follows: Let $\pi : \tilde{X} \to X$ be the normalization, and observe that $H^1(X)$ is an extension of a Hodge structure of pure weight one by a Hodge structure of pure type $(0,0)$:

$$0 \longrightarrow W_0 \longrightarrow H^1(X) \xrightarrow{\pi^*} H^1(\tilde{X}) \longrightarrow 0.$$

The classical Torelli theorem applied to the canonical quotient of weight one yields $\tilde{X}$. The geometric information contained in the extension then tells how to contract the fibers of π inside $\tilde{X}$ to obtain X. In technical language, this information is carried by the so-called one-motif of $H^1(X)$, a canonical homomorphism from a lattice to an abelian variety,

$$u : Gr_0 H_1(X, \mathbf{Z}) \longrightarrow J^0 W_{-1} H_1(X)$$

and by the polarizing form on the weight zero lattice.

As an illustration of the applications in higher dimensions, we mention the following result on singular K-3 surfaces:

Theorem B. *Let X be the union of a plane and a cubic surfaces in $\mathbb{P}_3$, simply tangent at no more than one point, and elsewhere in general position. Then X is determined by the polarized mixed Hodge structure on the primitive part of $H^2(X)$.*

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An account of these and other applications will appear in the future; for a preliminary report, see [1].

2. Extensions of mixed Hodge structures

a. Preliminary definitions

A *Hodge structure H of weight l* is a pair consisting of a lattice $H_{\mathbf{Z}}$ and a decreasing filtration $F^\bullet$ of $H_{\mathbf{C}} = H_{\mathbf{Z}} \otimes \mathbf{C}$ such that

$$H_{\mathbf{C}} = F^p \otimes \overline{F}^q$$

when $p+q = l+1$. Barring denotes conjugation with respect to $H_{\mathbf{R}} = H_{\mathbf{Z}} \otimes \mathbf{R}$. The relation above implies that the subspaces

$$H^{p,q} = F^p \cap \overline{F}^q$$

give a decomposition

$$H_{\mathbf{C}} = \bigoplus_{p+q=l} H^{p,q}$$

where

$$H^{p,q} = \overline{H^{q,p}}.$$

A *polarization* of H is a nonsingular, rational bilinear form S on $H_{\mathbf{Q}} = H_{\mathbf{Z}} \otimes \mathbf{Q}$ such that

(i) $S(x,y) = (-1)^l S(y,x)$

(ii) $S(x,y) = 0$ if $x \in F^p$, $y \in F^q$, and $p+q > l$

(iii) $i^{p-q} S(x, \overline{y})$ is hermitian positive definite on $H^{p,q}$.

A *mixed Hodge structure* H is a triple consisting of

(i) a lattice $H_{\mathbf{Z}}$

(ii) an increasing filtration W of $H_{\mathbf{Q}}$

(iii) a decreasing filtration $F^\bullet$ of $H_{\mathbf{C}}$.

These, the weight and Hodge filtrations, are compatible in the sense that $F^\bullet$ induces a Hodge structure of weight l on each of the graded pieces

$$Gr_l^W = W_l / W_{l-1}.$$

For the sake of brevity, we write

$$H^{p,q} = (Gr_{p+q}^W \otimes \mathbf{C})^{p,q}$$

and we introduce the lattice

$$L^p H = H^{p,q} \cap Gr^W_{2p,\mathbf{Z}}.$$

A *polarization* of H is a set of bilinear forms $\{S_l\}$ which polarize the graded pieces individually.

Two measures of the complexity of a mixed Hodge structure, the *length* and *level*, are useful: To define the first, let $[a, b]$ be the smallest interval such that $W_l/W_{l-1} = 0$ for $l \notin [a, b]$. Then a and b are the lowest and highest weights, and $b - a$ is the length. A mixed Hodge structure of length zero is a Hodge structure of pure weight. To define the second, let $[c, d]$ be the smallest interval such that $F^p/F^{p+1} = 0$ for $p \notin [c, d]$. Then $d - c$ is the level. A level zero structure is necessarily of pure type (p, p). As such, it is rigid: there is a unique admissible Hodge filtration, namely $F^p = H_{\mathbf{C}}$. The unique structure of type (p, p) on the lattice of integers will be denoted by $T\langle p\rangle$.

A *morphism* of mixed Hodge structures,

$$\phi : A \longrightarrow B$$

is given by a homomorphism of lattices which preserves both filtrations: there is a fixed integer m such that

$$\phi(F^p A) \subset F^{p+m} B$$
$$\phi(W_l A) = W_{l+2m} B$$

for all p and l. The *weight* of ϕ is, by definition, the integer $2m$. The category of mixed Hodge structures is abelian, and it admits both duals and tensor products. Finally, the lattice

$$\mathrm{Hom}(B, A)_{\mathbf{Z}} = \mathrm{Hom}(B_{\mathbf{Z}}, A_{\mathbf{Z}})$$

supports a canonical mixed Hodge structure with

$$W_l \, \mathrm{Hom}_{\mathbf{Q}} = \{\phi : \phi(W_r A) \subseteq W_{r+l} B \text{ for all } r\}$$
$$F^p \, \mathrm{Hom}_{\mathbf{C}} = \{\phi : \phi(F^r A) \subseteq F^{r+p} B \text{ for all } r\}$$

b. The group of extension classes

We can now introduce the fundamental object of study: an *extension* is an exact sequence of mixed Hodge structures,

$$0 \longrightarrow A \xrightarrow{\ i\ } H \xrightarrow{\ \pi\ } B \longrightarrow 0.$$

By analogy with principal fiber bundles, we say that H is an extension of B by A, that a *section* is a morphism

$$s : B \longrightarrow H$$

such that $\pi \circ s = 1_B$, and that an extension which admits a section is *split*. A *morphism* of extensions is a commutative diagram of morphisms, as below:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & H' & \longrightarrow & B' & \longrightarrow & 0 \end{array}$$

An isomorphism for which α and β are each the identity is called a *congruence*; a split extension is congruent to the trivial one given by the direct sum

$$H = A \oplus B.$$

Finally, let us define

$$\mathrm{Ext}(B, A)$$

to be the set of congruence classes of extensions of B by A. Its group structure is given by the result below.

Proposition 1. *Baer summation imposes the structure of an abelian group on* $\mathrm{Ext}(B, A)$, *with zero given by the class of split extensions. Moreover,* $\mathrm{Ext}(*, *)$ *is a functor, contravariant in the first variable, and covariant in the second.*

The proof depends only on the fact that the category of mixed Hodge structures is abelian: see the proof of the same proposition for R-modules in [4, p.63].

Now the extensions which arise naturally from a fixed mixed Hodge structure,

$$0 \longrightarrow W_m \longrightarrow H \longrightarrow H/W_m \longrightarrow 0$$

are *separated* in the sense that the highest weight of A is less than the lowest weight of B. In this case Ext is naturally isomorphic to a generalized complex torus (the quotient of a complex vector space by a discrete subgroups). To describe it, we define the p-th Jacobian of a mixed Hodge structure to be

$$J^p H = H_{\mathbf{C}} / (F^p H + H_{\mathbf{Z}}).$$

Whenever

$$p > \frac{1}{2}(\text{highest weight of } H),$$

the lattice $H_{\mathbf{Z}}$ projects without kernel to a discrete subgroup of $H_{\mathbf{C}}/F^p$, so $J^p H$ is indeed a generalized torus. The fine structure is then given by the result below:

Proposition 2. *Let A and B be separated mixed Hodge structures. Then there is a canonical and functorial isomorphism*

$$\mathrm{Ext}(B, A) \cong J^0 \, \mathrm{Hom}(B, A).$$

Example. Consider extensions of $T\langle p\rangle$ by $T\langle q\rangle$, where $p > q$. There is a canonical isomorphism

$$\mathrm{Hom}\,(T\langle p\rangle, T\langle q\rangle)_{\mathbf{C}} \longrightarrow T\langle r\rangle_{\mathbf{C}},$$

where $r = q - p$. Since $r < 0$, $F^0 T\langle r\rangle = 0$, and so

$$J^0\,\mathrm{Hom} = \frac{T\langle r\rangle_{\mathbf{C}}}{T\langle r\rangle_{\mathbf{Z}}} \cong \mathbf{C}^*.$$

We conclude that there is a nontrivial one-parameter family of extensions, even though $T\langle p\rangle$ and $T\langle q\rangle$ are themselves rigid.

Proof of Proposition 2. We shall construct a natural action of $J^0\,\mathrm{Hom}$ on Ext which is both transitive and effective. The resulting correspondence between $J^0\,\mathrm{Hom}$ and the orbit of the class of split extensions then gives the required identification. To this end, define a *normalized extension* of B by A to be one with underlying lattice.

$$L_{\mathbf{Z}} = A_{\mathbf{Z}} \oplus B_{\mathbf{Z}}.$$

Because of the separation hypothesis, the weigh filtration on $L_{\mathbf{Q}}$ is determined by that on A and B:

$$W_m L_{\mathbf{Q}} = W_m A_{\mathbf{Q}} \oplus W_m B_{\mathbf{Q}}$$

is the only choice which makes

$$0 \longrightarrow A_{\mathbf{Q}} \xrightarrow{\ i\ } L_{\mathbf{Q}} \xrightarrow{\ \pi\ } B_{\mathbf{Q}} \longrightarrow 0$$

an exact sequence of filtered objects. Consequently, a normalized extension is determined by a decreasing filtration on $L_{\mathbf{C}}$ such that

$$F^p L \cap A_{\mathbf{C}} = F^p A$$

$$\pi(F^P L) = F^p B.$$

We shall denote the set of normalized extensions by

$$\widetilde{\mathrm{Ext}}(B, A).$$

Next, we claim that an arbitrary extension is congruent to a normalized one. To see this, define an *integral retraction* to be a homomorphism

$$r_{\mathbf{Z}} : H_{\mathbf{Z}} \longrightarrow A_{\mathbf{Z}}$$

such that $r_{\mathbf{Z}} \circ i = \mathrm{id}$. Transport $F^p H$ to a filtration of $L_{\mathbf{C}}$ using the isomorphism

$$(r, \pi) : H_{\mathbf{Z}} \longrightarrow L_{\mathbf{Z}}.$$

This defines a normalized extension congruent to the given one, consequently determines a natural isomorphism

$$\mathrm{Ext}(B,A) \cong \widetilde{\mathrm{Ext}}(B,A)/\text{congruence}.$$

Therefore it suffices to define an action on the right-hand side. To do this, consider the automorphisms of $L_{\mathbf{C}} = L_{\mathbf{Z}} \otimes \mathbf{C}$ given by

$$g(\psi) = \begin{bmatrix} 1_A & \psi \\ 0 & 1_B \end{bmatrix}$$

where

$$\psi \in \mathrm{Hom}(B,A)_{\mathbf{C}}.$$

Then $\mathrm{Hom}_{\mathbf{C}}$ acts on $\widetilde{\mathrm{Ext}}$ by

$$F^\bullet \longrightarrow g(\psi)F^\bullet.$$

One easily verifies that the action is transitive and that the isotropy group of the trivial extension,

$$F_0^\bullet = F^\bullet A \oplus F^\bullet B,$$

is $F^0 \,\mathrm{Hom}$. Consequently there is a natural identification

$$\widetilde{\mathrm{Ext}}(B,A) \cong \mathrm{Hom}(B,A)_{\mathbf{C}}/F^0 \,\mathrm{Hom}(B,A).$$

To complete the proof, observe that congruences of normalized extensions are given by matrices $g(\psi)$ which are integrally defined. Consequently

$$\frac{\widetilde{\mathrm{Ext}}(B,A)}{\text{congruence}} \cong \frac{\mathrm{Hom}(B,A)_{\mathbf{C}}}{F^0 \,\mathrm{Hom}(B,A) + \mathrm{Hom}(B,A)_{\mathbf{Z}}}$$

as desired.

Remark. The group law agrees with that given by Baer summation. To see this, recall that in any abelian category, pushforwards and pullbacks of extensions exist, as do direct sums, and the diagonal and codiagonal maps:

$$\Delta(x) = (x,x)$$

$$\nabla(x,y) = x+y.$$

The baer sum of extensions H_1 and H_2 is gotten by pulling back the direct sum along the diagonal, and pushing it forward along the codiagonal:

$$H_1 + H_2 = \nabla_*\Delta^*(H_1 \oplus H_2)$$

But the direct sum is represented by the matrix $g(\psi_1) \oplus g(\psi_2)$, and application of $\nabla_*\Delta^*$ yields $g(\psi_1+\psi_2)$, as desired.

c. Motifs

Let us define a *weak motif* to be a homomorphism

$$u : L \longrightarrow J$$

of a lattice to a generalized complex torus. Morphisms of such are commutative diagrams of homomorphisms,

$$\begin{array}{ccc} L_1 & \longrightarrow & J_1 \\ {\scriptstyle f_L}\downarrow & & \downarrow{\scriptstyle f_J} \\ L_2 & \longrightarrow & J_2 \end{array}$$

in which f_J is holomorphic with closed image. The category of all such is abelian. Moreover, if L is torsion-free, then the set of weak motifs with values in J is itself a generalized complex torus: choose a basis $\lambda_1, \ldots, \lambda_m$ for L and send u to $(u(\lambda_1), \ldots, u(\lambda_m))$ to get

$$\mathrm{Hom}(L, J) \xrightarrow{\cong} (J)^m.$$

If in addition J is an extension of an abelian variety by complex multiplicative group (product of $\mathbf{C}^{*\prime}$s), then u is called a *one-motif.* By definition, a morphism of such objects preserves the multiplicative extensions.

Let us now describe the weak motif defined by an extension of mixed Hodge structures:

Proposition 3. *Let H be a separated extension. Then there is a functorially defined weak motif*

$$u_H : L^p B \longrightarrow J^p A$$

which depends only on the congruence class of H.

Proof. By proposition 2, there is a homomorphism

$$\psi_H : B_{\mathbf{C}} \longrightarrow A_{\mathbf{C}}$$

which represents H, unique up to addition of elements in

$$F^0 \mathrm{Hom}(B, A) \longrightarrow \mathrm{Hom}(B, A)_{\mathbf{Z}}.$$

Consequently the formula

$$u_H(\beta) = \psi_H(\beta) + F^p A + A_{\mathbf{Z}}$$

gives a homomorphism, independent of the choice of ψ, with the required properties.

In fact, the constructions yields more: First, the correspondence $H \to u_H$ determines a holomorphic homomorphism

$$\mathrm{Ext}(B, A) \longrightarrow \mathrm{Hom}(L^p B, J^p A).$$

Second, pullback of extensions along the inclusion of mixed Hodge structures

$$k : L^p B \longrightarrow B$$

yields a factorization

$$\begin{array}{ccc} \mathrm{Ext}(B, A) & \xrightarrow{k^*} & \mathrm{Ext}(L^p B, A) \\ & \searrow & \downarrow \cong \\ & & \mathrm{Hom}(L^p B, J^p A). \end{array}$$

Lifting homomorphism of $L^p B$ from $J^p A$ to $A_{\mathbf{C}}$, and making use of proposition 2, one easily verifies that the right-hand map is an isomorphism of generalized tori. In particular, we see that an extension is determined by its weak motif if and only if B is of pure type (p, p).

Remark. (1) The last statement must be interpreted with some care: it does not express an isomorphism of functors, because not all morphism of weak motifs are induced by morphisms of mixed Hodge structures. More precisely, there are complex homomorphisms $J^p A \to J^p A'$ which are not induced by morphisms $A \to A'$.

(2) For one-motifs, Deligne proved the stronger statement. Indeed, he exhibited an equivalence of categories between mixed Hodge structures of level one and one-motifs. (*cf.* [2, section10]).

(3) The dual extension

$$0 \longrightarrow \hat{B} \longrightarrow \hat{H} \longrightarrow \hat{A} \longrightarrow 0$$

defines a weak motif

$$\hat{u}_H : L^{-p} \hat{A} \longrightarrow J^{-p} \hat{B}.$$

Its value on any element γ is represented by a linear functional u_γ on $B_{\mathbf{C}}$. To describe it, let $\langle\ ,\ \rangle$ denote the pairing between a module and its dual, and let $\hat{\psi}_H$ be a homomorphism representing the dual extension. Then u_γ acts by

$$\begin{aligned} u_\gamma(\beta) &= \langle \beta, \hat{\psi}_H(\gamma) \rangle \\ &= \langle \psi_H(\beta), \gamma \rangle. \end{aligned} \tag{$*$}$$

This remark, although trivial, will be useful in the applications.

d. A formula for ψ

To calculate the extensions which arise in geometry, we need a formula for the representing homomorphism ψ. This will follow from the following direct characterization of ψ. To this end, we define a *section of the Hodge filtration* to be a homomorphism

$$s_F : B_{\mathbf{C}} \longrightarrow H_{\mathbf{C}}$$

such that $\pi \circ s_F = \mathrm{id}$ and such that

$$s_F(F^p B) \subseteq F^p H$$

for all p.

Lemma 4. *Fix a separated extension H of B by A. Then all homomorphisms which represent H are of the form*

$$\psi = r_{\mathbf{Z}} \circ s_F,$$

where $r_{\mathbf{Z}}$ is an integral retraction, where s_F is a section of the Hodge filtration.

There is a useful alternative form of the lemma which characterizes $\psi(\omega)$ as a linear functional on $\hat{A}$: let $s_{\mathbf{Z}} = \hat{r}_{\mathbf{Z}}$ be the adjoint section of the dual lattice structure. Then $\psi(\omega)$ acts by

$$\gamma \longrightarrow \langle s_F(\omega), s_{\mathbf{Z}}(\gamma) \rangle.$$

In a typical geometric situation, H is the k-th cohomology of an algebraic variety, $s_F(\omega)$ is represented by a suitable deRham element, and the canonical pairing is represented by integration:

$$\psi(\omega)(\gamma) = \int_{S_{\mathbf{Z}}(\gamma)} S_F(\omega). \tag{$*$}$$

Thus the extension is calculated in terms of the period matrix of a singular variety.

Proof of Lemma 4. We must show that $g(\psi)$ carries the reference filtration on L to the filtration of a normalized extension congruent to H. To do this efficiently, view elements of L as column vectors ${}^t(a, b)$. Then the column-vector of homomorphisms ${}^t(r, \pi)$ defines a congruence between H and a normalized extension with filtration $F^p L$. Moreover, the row vector of homomorphisms (i, s) sends $L_{\mathbf{C}}$ isomorphically to $H_{\mathbf{C}}$, carrying the reference filtration to the given one on H, since

$$i(F^p A) + s(F^p B) \subseteq F^p H.$$

But

$$\begin{pmatrix} r \\ \pi \end{pmatrix} (i, s) = \begin{bmatrix} ri & rs \\ \pi i & \pi s \end{bmatrix} = \begin{bmatrix} 1_A & \psi \\ 0 & 1_B \end{bmatrix} = g(\psi).$$

Consequently

$$\begin{aligned} g(\psi)(F^pA \oplus F^pB) &= \binom{r}{\pi}(i,s)(F^pA \oplus F^pB) \\ &= \binom{r}{\pi} F^pH \\ &= F^pL. \end{aligned}$$

This complete the proof.

With the lemma in hand, we can give the formula. First, choose a basis $\{\gamma^i\}$ of $A_{\mathbf{Z}}$, let $\{\gamma_i\}$ be the dual basis in $\hat{A}_{\mathbf{Z}}$, and let $\{\Gamma_i\}$ be a lifting to $\hat{H}_{\mathbf{Z}}$. Then

$$r_{\mathbf{Z}}(\Omega) = \sum \gamma^i \langle \Omega, \Gamma_i \rangle$$

defines an integral retraction, and in fact all such are obtained in this way. Second, let $\omega \in F^pB$, and let Ω be a lifting to F^pH. Since there is always a section of the Hodge filtration which sends ω to Ω, there is a representing homomorphism with

$$\psi(\omega) = \sum \gamma^i \langle \Omega, \Gamma_i \rangle.$$

This is the desired formula. In the geometric case it becomes

$$\psi(\omega) = \sum \gamma^i \int_{\Gamma_i} \Omega,$$

where $\Omega = s_F(\omega)$, $\Gamma_i = s_{\mathbf{Z}}(\gamma_i)$.

e. Jacobians

We conclude this section with a few remarks on Jacobians, beginning with a proof that J^pH is indeed a generalized torus.

Lemma 5. *Let H be a mixed Hodge structure of highest weight m. If $p > m/2$ then the natural map*

$$\rho : H_{\mathbf{R}} \longrightarrow H_{\mathbf{C}}/F^p$$

is an injection of real vector spaces.

Proof. An element of the kernel of ρ lies in $F^p \cap H_{\mathbf{R}}$, hence in $F^p \cap \overline{F}^p$. Let l be the weight of x, and consider the projection of x in Gr_l^W. An element of $F^p \cap \overline{F}^p Gr_l$ is a sum of components with type (a,b), where $a \geq p$ and $b \geq p$. Therefore $l \geq 2p > m$, and consequently $Gr_l^W = 0$. It follows that $x = 0$, as desired.

Lemma 6. *If $p > m/2$, then the p-th Jacobian of H is a generalized torus.*

Proof. Let K be a complement of $\rho(H_{\mathbf{R}})$ in the real vector space of $H_{\mathbf{C}}/F^p$. Then the decomposition

$$H_{\mathbf{C}}/F^p \cong H_{\mathbf{R}} \oplus K$$

yields an isomorphism of real manifolds

$$\begin{aligned} J^pH &\cong (H_{\mathbf{R}}/H_{\mathbf{Z}}) \oplus K \\ &\cong (S^1)^a \times \mathbf{R}^b. \end{aligned}$$

Since $H_{\mathbf{Z}}$ acts discretely in $H_{\mathbf{R}}$, $\rho(H_{\mathbf{Z}})$ acts discretely in $H_{\mathbf{C}}/F^p$, as desired.

Remarks. (1) If $F^p = 0$, then J^pH is of multiplicative type:

$$J^pH = H_{\mathbf{C}}/H_{\mathbf{Z}} \cong (\mathbf{C}^*)^m.$$

This is always the case when H has level zero and weight less than $2p$.

(2) If $H_{\mathbf{C}} = F^p \oplus \overline{F}^p$, then a dimension count shows that $b = 0$ in the above lemma, so that J^pH is compact. In particular, J^pH is compact when H is a Hodge structure of weight $2p-1$.

(3) Let U be a complex vector space, $L \subset U$ a lattice, $J = U/L$ a generalized torus. The dual torus is then the quotient of the dual vector space by the dual lattice:

$$\hat{J} = \hat{U}/\hat{L}.$$

An integral unimodular bilinear form on U gives an isomorphism $U \to \hat{U}$ which carries L to $\hat{L}$, hence an isomorphism between J and $\hat{J}$.

To apply this to Jacobians, recall that the annihilator of F^pH in $\hat{H}$ is $J^{1-p}\hat{H}$. Consequently the torus dual to J^pH is $J^{1-p}\hat{H}$. Moreover, if H is a Hodge structure of weight $2p-1$ with a unimodular polarization, then there is a canonical isomorphism

$$J^pH \longrightarrow J^{1-p}\hat{H}.$$

Note that the polarization makes J^pH and abelian variety when H is a level one structure.

Next, we claim that the Jacobian is filtered by the subobjects

$$W_lJ^pH = (W_l \otimes \mathbf{C} + F^p + H_{\mathbf{Z}})/(F^p + H_{\mathbf{Z}}).$$

The lemma below shows that they are also generalized tori because it implies that

$$W_lJ^pH = J^pW_lH.$$

Consequently there are holomorphic fibrations

$$0 \longrightarrow W_{l-1}J^pH \longrightarrow W_lJ^pH \longrightarrow Gr_lJ^pH \longrightarrow 0.$$

Because of the strictness property of morphisms ([3, p. 39]), we also have

$$Gr_lJ^pH \cong J^pGr_lH.$$

Example. Let H be a polarized mixed Hodge structure of level one and highest weight $2p$. Then the canonical fibration gives

$$0 \longrightarrow J^p W_{2p-2} \longrightarrow J^p W_{2p-1} \longrightarrow J^p Gr^W_{2p-1} \longrightarrow 0,$$

where the quotient is an abelian variety and the subobject is a torus of multiplicative type. The weak motif

$$u : L^p H \longrightarrow J^p W_{2p-1}$$

then carries the structure of a one-motif.

Lemma 7. *Let H be a mixed Hodge structure of highest weight $2p-1$. Then*

$$W_l \cap (F^p + H_{\mathbf{Z}}) = (W_l \cap F^p) + (W_l \cap H_{\mathbf{Z}}).$$

Proof. Write $x \in W_l \cap (F^p + H_{\mathbf{Z}})$ as $y+z$, where $y \in F^p$ and $z \in H_{\mathbf{Z}}$. Suppose $z \in W_m$, and consider the congruence $x \equiv y+z$ modulo W_{m-1}. If $m > l$, then $x \equiv 0$, hence $y+z \equiv 0$. Consider now the components of x, y and z in the I^{rs} decomposition (see [3, p.37]), where $r+s = m$. Because $y_{rs} \equiv 0$ for $r < p$, we have also $z_{rs} \equiv 0$ for $r < p$. Conjugating the last relation, we obtain $z_{rs} \equiv 0$ for $s < p$ as well. But since $r+s < 2p$, one of these alternatives must hold for each z_{rs}, so that z itself is congruent to zero modulo W_{m-1}. But then $y+z \equiv 0$ implies $y \equiv 0$. But now both y and z lie in W_{m-1}, provided that $m > l$. We conclude that both elements lie in W_l, as desired.

3. Applications to curves

a. Topology

To prove the Torelli theorem for curves mentioned in the introduction, we need a very explicit description of the mixed Hodge structure. We begin by discussing the topology, an understanding of which will yield both the weight filtration and the polarizations.

Thus, let X be a curve, Σ its singular locus,

$$\pi : \tilde{X} \longrightarrow X$$

its normalization, $\tilde{\Sigma} = \pi^{-1}(\Sigma)$, and

$$i : \tilde{\Sigma} \longrightarrow \tilde{X}$$

the natural inclusion.

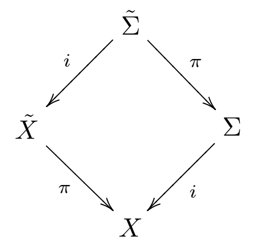
Lemma 8. *There is an exact sequence of singular homology,*

$$0 \longrightarrow H_1(\tilde{X}) \xrightarrow{\pi_*} H_1(X) \xrightarrow{\partial} H_0(\tilde{\Sigma}) \xrightarrow{\delta_*} H_0(\tilde{X}) \oplus H_0(\Sigma),$$

where $\delta_* = i_* \oplus (-\pi_*)$.

Proof. Consider the diagram

(∗)



From a formal point of view, this represents the nerve of a generalized cover, and the sequence above is its Mayer-Vietoris sequence. To make this remark precise, consider the space $|X|$ obtained by glueing $\tilde{\Sigma} \times [0,1]$ to $\tilde{X}$ and Σ along the maps

$$i : \tilde{\Sigma} \times \{0\} \longrightarrow \tilde{X}$$

$$\pi : \tilde{\Sigma} \times \{1\} \longrightarrow \Sigma.$$

There are natural projections

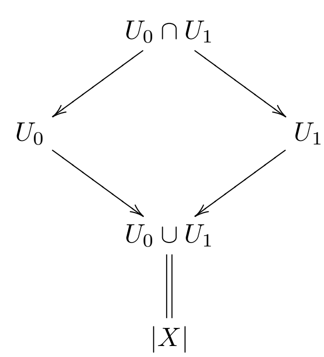
$$p : |X| \longrightarrow [0,1]$$

$$q : |X| \longrightarrow X,$$

and the fiber of q over $x \in X - \Sigma$ is a single point, whereas the fiber of q over $x \in \Sigma$ is cone over $\pi^{-1}(x)$. In either case the fiber is contractible, so q is a homotopy equivalence. Now cover $|X|$ by the open sets

$$U_0 = p^{-1}(0 \le t < 1)$$
$$U_1 = p^{-1}(0 < t \le 1).$$

The nerve of this cover is



which is homotopy equivalent to the diagram (∗). Applying these equivalences to the usual Mayer-Vietoris sequence of the cover, we obtain the Mayer-Vietoris sequence of the generalized cover.

Now each term in the generalized Mayer-Vietoris sequence corresponding to a smooth variety carries natural Hodge structure. Moreover, since the category of Hodge structures is abelian, the group

$$\mathscr{Z}_X = \text{kernel}(\delta_*)$$

carries a natural structure of pure type $(0,0)$. Thus, we have an exact sequence

$$0 \longrightarrow H_1(\tilde{X}, \mathbf{Z}) \xrightarrow{\pi_*} H_1(X, \mathbf{Z}) \xrightarrow{\partial} \mathscr{Z}_X \longrightarrow 0$$

with Hodge structures of weights -1 and 0 on the end terms. Consequently it is natural to define a weight filtration (over $\mathbf{Z}$) by

$$\begin{aligned} W_{-1} &= \pi_* H_1(\tilde{X}, \mathbf{Z}) \\ W_0 &= H_1(X, \mathbf{Z}). \end{aligned}$$

Next, let us define natural polarizations. On the weight -1 term this is clear: transport the intersection product of one-cycles on $\tilde{X}$ to X by means of the isomorphism

$$\pi_* : H_1(\tilde{X}) \xrightarrow{\cong} W_{-1}.$$

On the weight zero part, we proceed as follows: Let S_0 be the unique symmetric bilinear form on $H_0(\tilde{\Sigma}, \mathbf{Z})$ for which the classes of points form an orthonormal basis. Then carry S_0 back by the isomorphism

$$\partial : W_0/W_{-1} \xrightarrow{\cong} \mathscr{Z}_X.$$

Implicit in this construction is the identification of $\mathscr{Z}_X$ with a special group of zero cycles of degree zero on $\tilde{X}$. Such a cycle ξ is characterized by the following conditions:

(i) the support of ξ is in $\tilde{\Sigma}$

(ii) the degree of ξ on each component of $\tilde{X}$ is zero

(iii) the degree of ξ on each fiber of π is zero.

In general, the form S_0 is complicated. However, let us calculate it when X is an irreducible curve with ordinary singularities. To this end, let y be a singular point, and order the points of $\pi^{-1}(y)$ as $x_0, \ldots, x_{m-1}$, where m is the multiplicity. Define

$$\mathscr{Z}(y) = \left\{ \sum n_i X_i \middle| \sum n_i = 0 \right\}.$$

An easy calculation shows that

$$\mathscr{Z}_X = \bigoplus_{y \in \Sigma} \mathscr{Z}(y).$$

Moreover, since distinct summands are orthogonal under S_0, it suffices to describe S_0 on one such. To do this, introduce the basis

$$e_i = x_i - x_{i-1},$$

where $i = 1, \ldots, m-1$. Then

$$\begin{aligned} \langle e_i, e_i \rangle &= 2 \\ \langle e_i, e_{i-1} \rangle &= -1, \text{ and} \\ \langle e_i, e_j \rangle &= 0 \end{aligned}$$

if $|i-j| > 1$. Consequently the matrix of S_0 in this basis is

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & \ddots & \end{bmatrix}$$

the matrix of the Dynking diagram A_{m-1}. Note that the vectors e_i form a system of primitive, positive roots, and that any root vector is of the form $x_i - x_j$. These vectors are characterized by the fact that their length is minimal among all nonzero lattice vectors. Note also that any system of primitive, positive roots is conjugate to the given one by a general element of the orthogonal group of S_0 on $\mathscr{X}(y)$. Such elements are faithfully represented by permutations of the set $\pi^{-1}(y)$.

b. The mixed Hodge structure

Let us now describe the mixed Hodge structure on cohomology. The dual of our basic sequences gives

$$0 \longrightarrow \hat{\mathscr{X}}_X \xrightarrow{\hat{\partial}} H^1(X, \mathbf{Z}) \xrightarrow{\pi^*} H^1(\tilde{X}, \mathbf{Z}) \longrightarrow 0,$$

hence a weight filtration

$$\begin{aligned} W_0 &= \text{ kernel}(\pi^*) \\ W_1 &= H^1(X). \end{aligned}$$

It therefore remains to define an interpolating Hodge filtration on $H^1(X)$, compatible with the given filtrations on the end terms. To do this, we define an *abelian differential* on X to be a holomorphic form on $X - \Sigma$ which is square-integrable:

$$\int_{X-\Sigma} \omega \wedge \overline{\omega} < \infty.$$

The line integral of such a form over a closed path in X is defined and depends only on its homology class. Consequently the abelian differentials span a subspace of the complex cohomology, which we denote by F^1. This gives the desired filtration:

$$F^1 = \{\text{classes of abelian differentials}\}$$
$$F^0 = H^1(X, \mathbf{C}).$$

Let us now verify the necessary compatibility statements:

(a) $F^1 \cap W_0 = 0$

(b) $W_1/W_0 = F^1(W_1/W_0) \oplus \overline{F}^1(W_1/W_0)$.

These are jointly equivalent to the assertion that F induces Hodge structures of weight zero and one on subspace and quotient. Equivalently, we must verify

(a′) $F^1 \cap \ker \pi^* = 0$

(b′) $\pi^* F^1 H^1(X) = F^1 H^1(\tilde{X})$.

(Use (b′) and its conjugate statement to recover (b)). Consequently it suffices to prove that

$$\pi^* : F^1 H^1(X) \longrightarrow F^1 H^1(\tilde{X})$$

is an isomorphism. But this is easy in view of the relation of improper integrals

$$\int_{\tilde{X}-\tilde{\Sigma}} \pi^*\omega \wedge \overline{\pi^*\omega} = \int_{X-\Sigma} \omega \wedge \tilde{\omega}.$$

Because square-integrable holomorphic forms on $\tilde{X} - \tilde{\Sigma}$ extend to abelian differentials on $\tilde{X}$, π^* has the asserted range. Because π is a homeomorphism on the complement of $\tilde{\Sigma}$, abelian differentials on $\tilde{X}$ define abelian differentials on X. This establishes surjectivity. Finally, suppose that $\pi^*\omega$ vanishes in cohomology. Then the L^2-norm of $\pi^*\omega$ vanishes, and so it vanishes as a differential form. Consequently ω also vanishes as a differential, and this establishes injectivity.

c. Comparison of motifs

The final tool which we need for the Torelli theorem is a comparison between the Hodge-theoretic motif and the Abel-Jacobi homomorphism

$$\mathscr{A} : \mathscr{Z}_X \longrightarrow \mathrm{Jac}(X).$$

Proposition 9. *Let X be a projective algebraic curve. Then there is a natural isomorphism of one-motifs*

$$\hat{u}_H \xrightarrow{\cong} \mathscr{A}$$

Proof. The isomorphism is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} L^0H_1(X) & \xrightarrow{\hat{u}_H} & J^0W_{-1}H_1(X) \\ \Big\downarrow \partial & & \Big\downarrow \cong \\ \mathscr{Z}_X & \xrightarrow{\mathscr{A}} & \mathrm{Jac}(\tilde{X}), \end{array}$$

where the right hand isomorphism is the composition of the map induced by projection,

$$\pi_* : J^0H_1(\tilde{X}) \xrightarrow{\cong} J^0W_{-1}H_1(X),$$

and the duality induced by integration,

$$J^0H_1(\tilde{X}) \cong H^{1,0}(\tilde{X})^*/H_1(\tilde{X}, \mathbf{Z}).$$

According to the prescriptions of section 2c the motivic functional is given by

$$u_\gamma : \omega \longmapsto \langle \psi(\omega), \gamma \rangle$$

where $\gamma \in L^0H_1(X)$, $\omega \in F^1Gr_1H^1(X)$. According to section 2d, this can be written

$$u_\gamma : \omega \longmapsto \int\limits_{S_{\mathbf{Z}}(\gamma)} S_F(\omega).$$

But the right-hand side is just the integral of ω, viewed as an abelian differential on $\tilde{X}$, over a chain whose boundary is the zero-cycle $\xi = \partial\gamma$. In other words, u_γ acts, under the appropriate identifications, as does the Abel-Jacobi functional,

$$\mathscr{A}_\xi : \omega \longrightarrow (\text{abelian sum of } \omega \text{ with respect to } \xi).$$

This complete the proof.

Remarks. (1) Since $H^1(X)$ has level one, it is completely determined by its one-motif.

(2) This in turn implies that the group of extensions can be represented as a sum of Jacobians:

$$\begin{aligned} \mathrm{Ext}(H^1(\tilde{X}), W_0) &\cong \mathrm{Ext}(W_0/W_{-1}, H_1(\tilde{X}) \\ &\cong \mathrm{Hom}(\mathscr{Z}_X, H_1(\tilde{X})) \\ &\cong (\mathrm{Jac}(\tilde{X}))^m, \end{aligned}$$

where $m = \operatorname{rank} \mathscr{Z}_X$. All isomorphisms are canonical except the last, which depends on a choice of basis for $\mathscr{Z}_X$

d. Proof of the Torelli theorem

We will now prove theorem A by using the one-motif to construct a diagram isomorphic to

$$\begin{array}{ccccc} & & \tilde{\Sigma} & & \\ & \stackrel{i}{\swarrow} & & \stackrel{\pi}{\searrow} & \\ \tilde{X} & & & & \Sigma. \end{array}$$

This suffices, since X is the variety obtained by contracting $\tilde{\Sigma} \subset \tilde{X}$ along π to Σ:

$$X \cong \tilde{X} \bigcup_{\tilde{\Sigma}} \Sigma.$$

To get $\tilde{X}$, we apply the classical Torelli theorem to the polarized Hodge structure on W_1/W_0. To get Σ, we observe thath ∂ places the indecomposable summands on $L^0 H_1(X)$ in one-to-one correspondence with the points of Σ:

$$y \longleftrightarrow \mathscr{L}(y).$$

Finally, to construct $\tilde{\Sigma}$ and its associated maps, we employ the motivic homomorphism connecting the pieces of pure weight: Let Λ be an indecomposable summand of $L^0 H_1(X)$, corresponding to a point $y \in \Sigma$, and let ξ be a nonzero vector of minimal length in Λ. Then

$$\partial \xi = b - a,$$

with $a, b \in \pi^{-1}(y)$. Comparing the Hodge-theoretic motif with the Abel-Jacobi map, we see that $\hat{u}(\xi)$ determines the class of $b - a$ in the Jacobian of $\tilde{X}$. Now consider the Abel-Jacobi map on the Cartesian square which sends (u, v) to the class of $v - u$:

$$\mathscr{A} : \tilde{X} \times \tilde{X} \longrightarrow \mathrm{Jac}(\tilde{X}).$$

An easy argument using the fact that $\tilde{X}$ is nonhyperelliptic shows that $\mathscr{A}$ descends to an imbedding of the symmetric square of $\tilde{X}$, modulo the diagonal, which maps to zero. Consequently $\hat{u}(\xi)$ in fact determines the point-set $\{a, b\}$ in $\tilde{X}$. The union of these point-sets for all minimal vectors in Λ is just the fiber $\pi^{-1}(y)$, as a subset of $\tilde{X}$. Thus we have built the diagram

$$\begin{array}{ccccc} & & \pi^{-1}(y) & & \\ & \swarrow & & \searrow & \\ \tilde{X} & & & & \{y\} \end{array}$$

from the one-motif

$$\hat{u} : \Lambda \longrightarrow J^0 W_{-1} H_1(X),$$

together with the polarization of Λ. Amalgamating these diagrams in the obvious way, we obtain the desired presentation of X.

Remarks. (1) Suppose that $\tilde{X}$ is hyperelliptic, but of genus at least two. In general, the points of $\pi^{-1}(y)$ are determined only up to the canonical involution, so that finitely many curves correspond to a given mixed Hodge structure.

(2) Suppose that $\tilde{X}$ is elliptic. Let $|\Sigma|$ and $|\tilde{\Sigma}|$ denote the number of points in the given set. Then X defines a marking of $\tilde{X}$ by $|\tilde{\Sigma}|$ points. Since $\tilde{X}$ has a one-parameter family of automorphisms, the number of moduli of X is $|\tilde{\Sigma}|$. On the other hand, the number of moduli of $H^1(X)$ is the number of moduli of its one-motif, which is

$$\text{genus}(\tilde{X}) + \text{rank}(\mathscr{X}_X) = 1 + |\tilde{\Sigma}| - |\Sigma|.$$

This equals the number of moduli of X precisely when there is one singular point. If there are more, then infinitely many curves correspond to the polarized mixed Hodge structure on $H^1(X)$.

(3) Suppose X is an elliptic curve with one ordinary singular point. Then Torelli holds: Pick a base point x_0 and identify $\tilde{X}$ with its Jacobian by $x \mapsto$ class of $x - x_0$. Then choose a system of positive simple roots and define points $x_1, \ldots, x_n$ on $\tilde{X}$ inductively by

$$x_i = \hat{u}(\xi_i) + x_{i-1}.$$

Contraction of $\{x_0, \ldots, x_n\} \subset \tilde{X}$ to a point produces X.

(4) If $\tilde{X}$ is rational, then $H^1(X)$ is of level zero, hence deprived of moduli. On the other hand, X marks $|\tilde{\Sigma}|$ points on $\tilde{X}$, and so depends on $|\tilde{\Sigma}| - 3$ moduli. The Torelli theorem for $H^1(X)$ therefore fails completely.

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