

HYPERBOLIC GEOMETRY AND MODULI OF REAL CUBIC SURFACES

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ABSTRACT. Let $\mathcal{M}_0^{\mathbb{R}}$ be the moduli space of smooth real cubic surfaces. We show that each of its components admits a real hyperbolic structure. More precisely, one can remove some lower-dimensional geodesic subspaces from a real hyperbolic space H^4 and form the quotient by an arithmetic group to obtain an orbifold isomorphic to a component of the moduli space. There are five components. For each we describe the corresponding lattices in $\mathrm{PO}(4, 1)$. We also derive several new and several old results on the topology of $\mathcal{M}_0^{\mathbb{R}}$. Let $\mathcal{M}_s^{\mathbb{R}}$ be the moduli space of real cubic surfaces that are stable in the sense of geometric invariant theory. We show that this space carries a hyperbolic structure whose restriction to $\mathcal{M}_0^{\mathbb{R}}$ is that just mentioned. The corresponding lattice in $\mathrm{PO}(4, 1)$, for which we find an explicit fundamental domain, is nonarithmetic.

1. INTRODUCTION

In [3] and [2] we showed that the moduli space \mathcal{M}_s of stable complex cubic surfaces is the quotient $P\Gamma \backslash \mathbb{C}H^4$ of complex hyperbolic 4-space by the lattice $P\Gamma = PU(4, 1, \mathcal{E})$ in $PU(4, 1)$, where \mathcal{E} is the ring of integers in $\mathbb{Q}(\sqrt{-3})$. We also showed that there is an infinite hyperplane arrangement \mathcal{H} in $\mathbb{C}H^4$ which is $P\Gamma$ -invariant and corresponds to the discriminant. Thus there is an identification of the moduli space \mathcal{M}_0 of smooth cubic surfaces with the quotient $P\Gamma \backslash (\mathbb{C}H^4 - \mathcal{H})$. The identification is given by a period map that associates to a cubic surface a suitable ‘‘Hodge structure with symmetry.’’ This is the Hodge structure on the cohomology of a cyclic triple cover of $\mathbb{C}P^3$ branched along the cubic surface. The symmetry, given by the branched covering transformation σ , provides the integer cohomology with the structure of an \mathcal{E} -module endowed with a natural hermitian form of signature 4,1. In the associated complex vector space there is a distinguished

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negative line generated by a form of type 2,1. The map that assigns this line to a cubic surface defines a period map $\mathcal{M}_0 \longrightarrow PT \setminus \mathbb{C}H^4$ that gives the asserted isomorphisms.

In this article we show that an analogous theorem holds for the moduli space of real cubic surfaces. We use the same Hodge structure but exploit the additional symmetry given by complex conjugation.

To state the main results, write $PC_s^{\mathbb{R}} \subseteq \mathbb{R}P^{19}$ for the set of real cubic surfaces in four variables that are stable in the sense of geometric invariant theory. Such surfaces have at most nodal singularities. Write $PC_0^{\mathbb{R}}$ for the subspace of surfaces whose complex points are smooth, and note that the group $\mathrm{PGL}(4, \mathbb{R})$ acts properly on $PC_s^{\mathbb{R}}$. Our moduli space is the real analytic orbifold $\mathcal{M}_s^{\mathbb{R}} = PC_s^{\mathbb{R}}/\mathrm{PGL}(4, \mathbb{R})$. It contains $\mathcal{M}_0^{\mathbb{R}} = PC_0^{\mathbb{R}}/\mathrm{PGL}(4, \mathbb{R})$ as an open subset. As known classically, $PC_0^{\mathbb{R}}$, as well as $\mathcal{M}_0^{\mathbb{R}}$, consists of five connected components. Within a component the topology of the real cubic surfaces is constant: it is a real projective plane with n handles attached, where $n = -1, 0, 1, 2, 3$. The case $n = -1$ is the disjoint union of a projective plane and the 2-sphere; adding a handle between the two connected components, one obtains the case $n = 0$.

We have two lines of results. The first concerns the uniformization of the separate components $\mathcal{M}_{0,j}^{\mathbb{R}}$ of $\mathcal{M}_0^{\mathbb{R}}$ by a real 4-dimensional hyperbolic space H^4 . We number the components so that a surface in $\mathcal{M}_{0,j}^{\mathbb{R}}$ is (topologically) $\mathbb{R}P^2$ with $3 - j$ handles, so $j = 0, \dots, 4$.

Theorem 1.1. *There is a union \mathcal{H}_j of two- and three-dimensional geodesic subspaces of H^4 and an isomorphism of real analytic orbifolds*

$$\mathcal{M}_{0,j}^{\mathbb{R}} \cong P\Gamma_j^{\mathbb{R}} \setminus (H^4 - \mathcal{H}_j).$$

Here $P\Gamma_j^{\mathbb{R}}$ is the projectivized group of matrices with integer coefficients which are orthogonal with respect to the quadratic form obtained from the diagonal form $[-1, 1, 1, 1, 1]$ by replacing the last j of the 1's by 3's.

This theorem follows quite easily from the theorem on complex cubic surfaces and an analysis of how complex conjugation acts on $\mathbb{C}H^4$. For each j , the H^4 in question is totally geodesically embedded as the fixed point set of an anti-holomorphic involution of $\mathbb{C}H^4$, and $\mathcal{H}_j = H^4 \cap \mathcal{H}$. We also describe the $P\Gamma_j^{\mathbb{R}}$ in terms of Coxeter groups.

The other main result of the paper concerns the uniformization of the whole moduli space $\mathcal{M}_s^{\mathbb{R}}$ of real stable surfaces. It turns out, rather unexpectedly for us, that it can also be uniformized by H^4 .

Theorem 1.2. *There is a nonarithmetic lattice $P\Gamma^{\mathbb{R}} \subset PO(4, 1)$ and a homeomorphism*

$$\mathcal{M}_s^{\mathbb{R}} \cong P\Gamma^{\mathbb{R}} \setminus H^4.$$

This homeomorphism restricts to an isomorphism of real analytic orbifolds,

$$\mathcal{M}_0^{\mathbb{R}} \cong P\Gamma^{\mathbb{R}} \backslash (H^4 - \mathcal{H}'),$$

where \mathcal{H}' is a $P\Gamma^{\mathbb{R}}$ -invariant union of two- and three-dimensional geodesic subspaces of H^4 .

We emphasize that the H^4 in this theorem is an ‘abstract’ H^4 , not one embedded in $\mathbb{C}H^4$. Thus the two theorems are different in nature, despite the similarity of their statements. To our knowledge, this is the first appearance of a nonarithmetic lattice in a moduli problem for real varieties. We can describe $P\Gamma^{\mathbb{R}}$ using the language of Coxeter diagrams, and give an explicit fundamental polyhedron. Thus $P\Gamma^{\mathbb{R}}$ can be described as explicitly as the $P\Gamma_j^{\mathbb{R}}$.

Now we discuss the ideas surrounding theorem 1.1. A remarkable feature of the real case allows us to go far beyond what is currently known for the complex case: each of the groups $P\Gamma_j^{\mathbb{R}}$ is essentially a Coxeter group. More precisely, $P\Gamma_j^{\mathbb{R}}$ is a Coxeter group for $j = 0, 3, 4$ and contains a Coxeter group of index two for $j = 1, 2$. The Coxeter diagrams can be derived by applying an algorithm of Vinberg, the results of which are presented in Figure 1.1. See (4.1) for the meanings of the edge labels; the vertex labels are explained in the caption. These diagrams are the heart of the paper. From them a wealth of information can be read. For example, one can write down fundamental domains and presentations for the groups $P\Gamma_j^{\mathbb{R}}$. In the complex case the situation is quite different. Although Falbel and Parker [13] succeeded in giving a fundamental domain and presentation for the group $PU(2, 1, \mathcal{E})$, such results for the larger group $P\Gamma = PU(4, 1, \mathcal{E})$ at present seem out of reach.

Much of the classical theory of real cubic surfaces, as well as new results, are encoded in the diagrams. The new results are our computation of the groups $\pi_1^{\text{orb}}(\mathcal{M}_{0,j}^{\mathbb{R}})$ (see table 1.1) and our proof that each $\mathcal{M}_{0,j}^{\mathbb{R}}$ has contractible universal cover. These appear in section 6, where we describe the topology of the spaces $\mathcal{M}_{0,j}^{\mathbb{R}}$. As an application to the classical theory, we re-compute the monodromy representation of $\pi_1(PC_{0,j}^{\mathbb{R}})$ on the configuration of lines on a cubic surface. The monodromy, computed in Segre’s classical treatise [24], is one of the most elaborate parts of his book. In our approach, we derive the monodromy groups from the Coxeter diagrams. As a result, we are able to correct an error of Segre. The results are summarized in table 1.1. Here S_n , A_n and D_∞ denote symmetric, alternating and infinite dihedral groups. Segre’s error was in the $j = 2$ case; see section 7 for details.

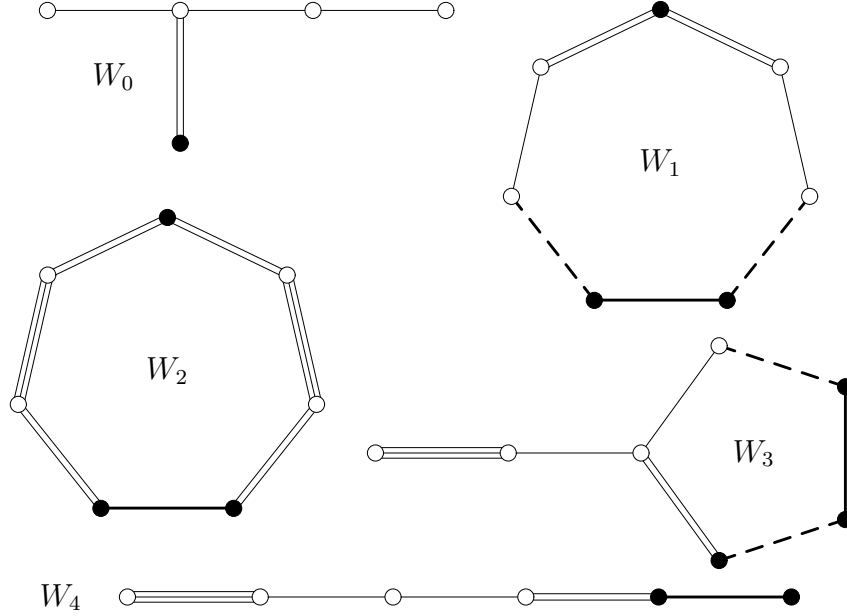


FIGURE 1.1. Coxeter polyhedra for the reflection subgroups W_j of $P\Gamma_j^{\mathbb{R}}$. The blackened nodes and triple bonds correspond to faces of the polyhedra that represent singular cubic surfaces.

j	$\pi_1^{\text{orb}}(\mathcal{M}_{0,j}^{\mathbb{R}})$	Monodromy on lines
0	S_5	A_5
1	$(S_3 \times S_3) \rtimes \mathbb{Z}/2$	$S_3 \times S_3$
2	$(D_\infty \times D_\infty) \rtimes \mathbb{Z}/2$	$(\mathbb{Z}/2)^3 \rtimes \mathbb{Z}/2$
3	} ∞	S_4
4		

TABLE 1.1. The orbifold fundamental groups of the moduli spaces $\mathcal{M}_{0,j}^{\mathbb{R}}$ and the monodromy actions of $\pi_1(PC_{0,j}^{\mathbb{R}})$ on the lines of a cubic surface.

The ideas surrounding theorem 1.2 are different. The starting point is the fact that one can glue the five spaces $P\Gamma_j^{\mathbb{R}} \setminus H^4$ together in a natural way. This is because whenever a nodal real surface appears in the boundaries of two components of $PC_0^{\mathbb{R}}$, certain points of the various $P\Gamma_j^{\mathbb{R}} \setminus H^4$ can be identified. The gluing is complicated but can be carried out, resulting in a space isometric to $P\Gamma^{\mathbb{R}} \setminus H^4$ for a suitable lattice $P\Gamma^{\mathbb{R}} \subseteq \text{PO}(4, 1)$. We emphasize that something special is happening

here, since in other cases application of the same ideas leads to a hyperbolic cone manifold instead of an orbifold. See Chu's theorem [9] on the moduli of 8-tuples in P^1 . What seems to distinguish the present case is that when two components of \mathcal{H} meet, they meet orthogonally.

The proof of theorem 1.2 proceeds in three steps. First, in section 9 we determine the identifications among the $P\Gamma_j^{\mathbb{R}} \backslash H^4$ that are necessary to obtain $\mathcal{M}_s^{\mathbb{R}}$. The result of the gluing is metrically complete and locally modeled on H^4 modulo finite groups. Then orbifold uniformization implies $\mathcal{M}_s^{\mathbb{R}} \cong P\Gamma^{\mathbb{R}} \backslash H^4$ for some lattice $P\Gamma^{\mathbb{R}}$. In section 10 we compute $P\Gamma^{\mathbb{R}}$. It turns out that it is not a Coxeter group, and indeed its reflection subgroup has infinite index. Nevertheless, an index 2 subgroup has a fundamental domain which happens to be a Coxeter polyhedron. Thus we obtain a simple concrete description of the group.

The final step is nonarithmeticity. Philosophically, this derives from the theorem of Gromov and Piatetski-Shapiro [16] that gluing two arithmetic lattices in $\mathrm{PO}(n, 1)$ in a suitable way yields a nonarithmetic lattice. However, their theorem does not quite apply because of the complexity of our gluing and certain properties of the pieces. Therefore we use a nonarithmeticity criterion of Deligne and Mostow. Nevertheless, we regard $P\Gamma^{\mathbb{R}}$ as the first appearance 'in nature' of the Gromov-Piatetski-Shapiro construction. See section 11 for details.

The paper is organized as follows. Sections 2–5 prove theorem 1.1. In section 2 we review the Hodge theory needed for the complex-moduli case, and show how the Hodge theory interacts with complex conjugation. This leads to a description (theorem 2.3) of $\mathcal{M}_0^{\mathbb{R}}$ in terms of the conjugacy classes of anti-linear involutions of $\mathcal{E}^{4,1}$. In section 3 we enumerate these classes (there are five, up to sign) and determine how they correspond to the classically-defined five components of $P\mathcal{C}_0^{\mathbb{R}}$. The main result is corollary 3.3, that $\mathcal{M}_0^{\mathbb{R}} = \coprod_{j=0}^4 P\Gamma_j^{\mathbb{R}} \backslash (H_j^4 - \mathcal{H})$, where the H_j^4 (resp. $P\Gamma_j^{\mathbb{R}}$) are the fixed-point sets (resp. centralizers in $P\Gamma$) of particular anti-linear involutions χ_0, \dots, χ_4 . In section 4 we describe the $P\Gamma_j^{\mathbb{R}}$ arithmetically (as in theorem 1.1 above) and geometrically (in terms of the Coxeter diagrams). The arithmetic description is almost immediate from the definition of the χ_j . Derivation of the diagrams relies on Vinberg's algorithm. In section 5 we determine $\mathcal{H}_j := H_j^4 \cap \mathcal{H}$, making concrete the last ingredient in our description of $\mathcal{M}_0^{\mathbb{R}}$. The main result, theorem 5.4, is that the faces of the Coxeter polyhedra corresponding to blackened nodes and triple bonds in figure 1.1 lie in \mathcal{H} . Furthermore, the $P\Gamma_j^{\mathbb{R}}$ -translates of these faces account for all of \mathcal{H}_j and form the H^3 's and H^2 's of theorem 1.1.

We have already indicated the contents of sections 6 and 7, namely the topology of $\mathcal{M}_{0,j}^{\mathbb{R}}$ and the monodromy action of $\pi_1(PC_{0,j}^{\mathbb{R}})$ on the lines of a cubic surface. Section 8 computes the hyperbolic volumes of $\mathcal{M}_s^{\mathbb{R}}$ and the $\mathcal{M}_{0,j}^{\mathbb{R}}$, using the Gauss-Bonnet theorem and a computation of the orbifold Euler characteristics of the $P\Gamma_j^{\mathbb{R}} \backslash H_j^4$. The results for the $\mathcal{M}_{0,j}^{\mathbb{R}}$ appear in table 1.2; the sum of the volumes is $37\pi^2/1080$, which is the volume of $\mathcal{M}_s^{\mathbb{R}}$. Note that the component corresponding to the simplest topology has the greatest volume, just over 40% of the total, and the component corresponding to surfaces with the most real lines has the smallest volume. Finally, as described above, sections 9, 10 and 11 treat the local description of $\mathcal{M}_s^{\mathbb{R}}$ as a hyperbolic orbifold, the global description got from the gluing process, and the nonarithmeticity of $P\Gamma^{\mathbb{R}}$.

We would like to draw the reader's attention to related work in the literature. Yoshida [29] has studied the real locus of the space of marked cubic surfaces. He obtained the hyperbolic structure on $\mathcal{M}_{0,0}^{\mathbb{R}}$ and studied it in detail. Our work completes his by studying the other $\mathcal{M}_{0,j}^{\mathbb{R}}$ and constructing the hyperbolic structures from the complex hyperbolic structure on \mathcal{M}_0 , rather than as a lucky phenomenon. The gluing and the nonarithmetic lattice $P\Gamma^{\mathbb{R}}$ are also new.

Many authors have studied real algebraic objects of various sorts in terms of Coxeter diagrams and/or the action of complex conjugation on homology. Nikulin has done extensive work on K3 surfaces, for example [22] and [23], and Kharlamov and his coauthors have studied K3 surfaces, Enriques surfaces and cubic fourfolds, for example [19], [11] and [14]. We also refer the reader to Moriceau's work on nodal quartic surfaces [21] and that of Gross and Harris on abelian varieties [17, Prop. 9.3]. There is considerable literature on moduli of n -tuples in $\mathbb{R}P^1$; see for example [26], [30], [7] and the papers they cite. We also have two expository articles [4] and [5] that develop the ideas of this paper less formally, and in the context of related but simpler moduli problems, for which the moduli space has dimension ≤ 3 . The lower dimension means that all the manipulations of Coxeter diagrams can be visualized directly. The results of this paper were announced in [6].

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2. MODULI OF SMOOTH REAL CUBIC SURFACES

The purpose of this section is to prove those results on moduli of smooth real cubic surfaces which follow easily from the general results on moduli of complex cubic surfaces that we proved in [2]. In later sections we will improve these results considerably.

j	Topology	Real Lines	Euler char.	Volume	Fraction
0	$\mathbb{R}P^2 + 3H$	27	1/1920	.00685	2.03%
1	$\mathbb{R}P^2 + 2H$	15	1/288	.04569	13.51%
2	$\mathbb{R}P^2 + H$	7	5/576	.11423	33.78%
3	$\mathbb{R}P^2$	3	1/96	.13708	40.54%
4	$\mathbb{R}P^2 \cup S^2$	3	1/384	.03427	10.14%
			37/1440	.33813	100.00%

TABLE 1.2. The hyperbolic volumes of the components of the real moduli space.

We review briefly the main constructions of [2]. Let \mathcal{C} be the space of all nonzero cubic forms in four complex variables, Δ be the discriminant locus, and \mathcal{C}_0 be the set $\mathcal{C} - \Delta$ of forms defining surfaces which are smooth (as schemes). We take $GL(4, \mathbb{C})$ to act on the left on \mathbb{C}^4 , hence on the right on \mathcal{C} , i.e., $g \in GL(4, \mathbb{C})$ carries $F \in \mathcal{C}$ to $F.g = F \circ g$. Throughout the paper, ω denotes the primitive cube root of unity $e^{2\pi i/3}$. Since the group $D = \{I, \omega I, \omega^2 I\}$ acts trivially on \mathcal{C} , we have an induced action of $G = GL(4, \mathbb{C})/D$. It is known that $GL(4, \mathbb{C})$ acts properly on \mathcal{C}_0 , so the moduli space $\mathcal{M}_0 = \mathcal{C}_0/G$ is a complex-analytic orbifold.

We write $\mathcal{C}^{\mathbb{R}}$, $\Delta^{\mathbb{R}}$ and $\mathcal{C}_0^{\mathbb{R}}$ for the subsets of the corresponding spaces whose members have real coefficients. Note that it is possible for the zero locus in $\mathbb{R}P^3$ of $F \in \Delta^{\mathbb{R}}$ to be a smooth manifold, for example it might have two complex-conjugate singularities. We write $G^{\mathbb{R}}$ for the group $GL(4, \mathbb{R})$, which is the same as the real points of G , and we write $\mathcal{M}_0^{\mathbb{R}}$ for the space $\mathcal{C}_0^{\mathbb{R}}/G^{\mathbb{R}}$. This is a real-analytic orbifold in the sense that it is locally the quotient of a real analytic manifold by a real analytic action of a finite group. (It is also a real semi-analytic space, i.e., locally defined by equalities and inequalities of real analytic functions. However, it is not a real analytic space for the same reason that $[0, \infty)$ is not. These properties are not important in this paper.) There is a natural map $\mathcal{M}_0^{\mathbb{R}} \rightarrow \mathcal{M}_0$ which is finite-to-one, surjective and generically injective, but not injective. So $\mathcal{M}_0^{\mathbb{R}}$ is not quite the same as the real locus of \mathcal{M}_0 . The purpose of this paper is to use real hyperbolic geometry to understand $\mathcal{M}_0^{\mathbb{R}}$ and its enlargement $\mathcal{M}_s^{\mathbb{R}}$, defined by replacing “smooth” in the above definitions by “stable”. See section 9 for the stable case.

We relate \mathcal{M}_0 to $\mathbb{C}H^4$ via a certain covering space of \mathcal{C}_0 , the space \mathcal{F}_0 of framed smooth cubic forms. Given $F \in \mathcal{C}_0$, let S be the surface

it defines in $\mathbb{C}P^3$ and T be the smooth threefold in $\mathbb{C}P^4$ defined by

$$(2.1) \quad Y^3 - F(X_0, \dots, X_3) = 0 .$$

Whenever we have a cubic form F in mind, we take S and T to be defined by these conventions. Let $\sigma \in \mathrm{PGL}(5, \mathbb{C})$ be given by $\sigma(X_0, \dots, X_3, Y) = (X_0, \dots, X_3, \omega Y)$. Then $H^3(T; \mathbb{Z}) \cong \mathbb{Z}^{10}$ and σ^* fixes no element of $H^3(T; \mathbb{Z})$ except 0. We may therefore regard $H^3(T; \mathbb{Z})$ as a module over the Eisenstein integers $\mathcal{E} = \mathbb{Z}[\omega]$, with $\bar{\omega}$ acting as σ^* . (In [2] we took ω to act as σ^* , but unfortunately this made the period map antiholomorphic, as discussed in the remark added in proof. This choice is in no way essential in this paper.)

We write $\Lambda(T)$ for this \mathcal{E} -module, which possesses a natural \mathcal{E} -valued Hermitian form arising from the symplectic form on H^3 . It turns out that $\Lambda(T)$ is isometric to $\Lambda = \mathcal{E}^{4,1}$, by which we mean \mathcal{E}^5 equipped with the Hermitian form

$$(2.2) \quad \langle x|y \rangle = -x_0\bar{y}_0 + x_1\bar{y}_1 + \dots + x_4\bar{y}_4 .$$

(If one takes ω to act as $(\sigma^{-1})^*$, as we are doing here, then the Hermitian form 2.3.1 of [2] should be defined by $(\Omega(\theta x, y) + \theta\Omega(x, y))/2$ in order to have signature $(4, 1)$ rather than $(1, 4)$. Here θ denotes the action of $\omega - \bar{\omega} \in \mathcal{E}$ on $\Lambda(T)$ and on \mathcal{E} and the notation is otherwise as in [2].)

A framing of F is a projective equivalence class $[i]$ of \mathcal{E} -linear isometries $i : \Lambda(T) \rightarrow \Lambda$; thus $[i] = [i']$ just if i and i' differ by multiplication by a unit of \mathcal{E} . A framed smooth cubic form is a pair $(F, [i])$ where $F \in \mathcal{C}_0$ and $[i]$ is a framing of F . We write \mathcal{F}_0 for the set of all such pairs, topologized and given a complex manifold structure as in section 3.9 of [2].

\mathcal{F}_0 is a covering space of \mathcal{C}_0 , and turns out to be connected, which is another way of saying that the projective monodromy action of $\pi_1(\mathcal{C}_0)$ on the local system of $\Lambda(T)$'s is the full projective isometry group of Λ . We describe the action of the deck group $P\mathrm{Aut} \Lambda$ by the left action

$$(2.3) \quad \gamma.(F, [i]) = (F, [\gamma \circ i]) .$$

In section 7 of [2] we computed the monodromy representation of $\pi_1(\mathcal{C}_0)$ on the local system of the $\Lambda(T)$'s and found that its image Γ is not all of $\mathrm{Aut} \Lambda$; it has index two. However, $P\Gamma = P\mathrm{Aut} \Lambda$. The slight difference between Γ and $\mathrm{Aut} \Lambda$ will play no role in this paper.

The action of G on \mathcal{C}_0 lifts to \mathcal{F}_0 as follows. If $h \in G$ then lift it to an element of $\mathrm{GL}(4, \mathbb{C})$, still denoted by h , and regard it as acting on $\mathbb{C}P^4$ by

$$h(X_0, \dots, X_3, Y) = (h(X_0, \dots, X_3), Y) .$$

If $(F, [i]) \in \mathcal{F}_0$ then h carries the points of $h^{-1}.T$ to those of T , so it defines an isometry $h^* : \Lambda(T) \rightarrow \Lambda(h^{-1}.T)$. We define

$$(2.4) \quad (F, [i]).h = (F \circ h, [i \circ (h^*)^{-1}]) .$$

This is well-defined despite the ambiguity in the lift of h to $\mathrm{GL}(4, \mathbb{C})$, because different lifts differ in their action on $\mathbb{C}P^4$ by a power of σ , which acts on $\Lambda(T)$ and $\Lambda(h^{-1}.T)$ by a scalar.

We take complex hyperbolic 4-space $\mathbb{C}H^4$ to be the set of negative lines in $\mathbb{C}^{4,1} = \Lambda \otimes_{\mathcal{E}} \mathbb{C}$. Theorem 2.20 of [2] asserts that $\mathcal{F}_0/G \cong \mathbb{C}H^4 - \mathcal{H}$, where \mathcal{H} is the union of the orthogonal complements of the norm 1 vectors of Λ . The points of \mathcal{H} correspond to cubic surfaces with nodes but no worse singularities. This isomorphism is given by the period map $g : \mathcal{F}_0 \rightarrow \mathbb{C}H^4$ of [2], defined by

$$g(F, [i]) = i_*(H_{\bar{\omega}}^{2,1}(T)) \in \mathbb{C}H^4 \subseteq P(\mathbb{C}^{4,1}) ,$$

where by $i_* : H_{\bar{\omega}}^3(T, \mathbb{C}) \rightarrow \Lambda \otimes_{\mathcal{E}} \mathbb{C}$ we mean the composition

$$(2.5) \quad H_{\bar{\omega}}^3(T; \mathbb{C}) \cong H^3(T; \mathbb{R}) = \Lambda(T) \otimes_{\mathcal{E}} \mathbb{C} \xrightarrow[i \otimes 1]{\cong} \Lambda \otimes_{\mathcal{E}} \mathbb{C} = \mathbb{C}^{4,1} .$$

Here the leftmost isomorphism is given by the eigenspace projection $H^3(T, \mathbb{R}) \rightarrow H_{\bar{\omega}}^3(T; \mathbb{C})$, the term $H^3(T; \mathbb{R})$ is a complex vector space with $\bar{\omega}$ acting as σ^* , and all the maps are isomorphisms of complex vector spaces.

Now we turn to real surfaces. We write κ for the standard complex conjugation map on \mathbb{C}^n , and also for the induced map on \mathcal{C} given by $(F.\kappa)(x) = \overline{F(\kappa.x)}$. In coordinates this amounts to replacing the coefficients of F by their complex conjugates. This convention for the action of κ on functions arranges for it to carry holomorphic functions to holomorphic functions (rather than anti-holomorphic ones). We extend this convention as follows: if α is an anti-holomorphic map of a complex variety V_1 to another V_2 , then $\overline{\alpha^*} : H^*(V_2; \mathbb{C}) \rightarrow H^*(V_1; \mathbb{C})$ is defined as the usual pullback under $V_1 \rightarrow V_2$ followed by complex conjugation in $H^*(V_1; \mathbb{C})$. This has the advantage that, if V_1 and V_2 are compact Kähler manifolds, then $\overline{\alpha^*}$ is an antilinear map that preserves the Hodge decomposition (rather than the complex linear map α^* that interchanges Hodge types). Many different complex conjugation maps appear in this paper, so we call a self-map of a complex manifold (resp. complex vector space or \mathcal{E} -module) an anti-involution if it is anti-holomorphic (resp. anti-linear) and has order 2.

The fixed-point set of κ in \mathcal{C} is $\mathcal{C}^{\mathbb{R}}$. We write $\mathcal{F}_0^{\mathbb{R}}$ for the preimage of $\mathcal{C}_0^{\mathbb{R}}$ in \mathcal{F}_0 . This is a covering space of $\mathcal{C}_0^{\mathbb{R}}$, which is disconnected because $\mathcal{C}_0^{\mathbb{R}}$ is; we will see later that $\mathcal{F}_0^{\mathbb{R}}$ has infinitely many components.

As a first step in separating these components, let \mathcal{A} denote the set of anti-involutions of Λ and let $P\mathcal{A}$ denote the set of their projective equivalence classes. If $(F, [i]) \in \mathcal{F}_0^{\mathbb{R}}$ then κ acts on $\Lambda(T)$ as an anti-involution, so $\chi = i \circ \kappa^* \circ i^{-1}$ lies in \mathcal{A} . Because of the ambiguity in the choice of representative i for $[i]$, χ is not determined by $[i]$; however, its class $[\chi]$ in $P\mathcal{A}$ is well-defined. Clearly $[\chi]$ does not change if $(F, [i])$ varies in a connected component of $\mathcal{F}_0^{\mathbb{R}}$. We will often lighten the notation by omitting the brackets. Thus we get a map $\pi_0(\mathcal{F}_0^{\mathbb{R}}) \rightarrow P\mathcal{A}$. If $C\mathcal{A}$ denotes the set of $P\Gamma$ -conjugacy classes of elements of $P\mathcal{A}$, we also get a map $\pi_0(\mathcal{C}_0^{\mathbb{R}}) \rightarrow C\mathcal{A}$. We will soon see that these two maps are surjective, and the second one is bijective.

For each $\chi \in P\mathcal{A}$, define a subspace \mathcal{F}_0^χ of $\mathcal{F}_0^{\mathbb{R}}$ by

$$(2.6) \quad \mathcal{F}_0^\chi = \{(F, [i]) \in \mathcal{F}_0^{\mathbb{R}} : [i \circ \kappa^* \circ i^{-1}] = [\chi]\}.$$

It is the subspace of $\mathcal{F}_0^{\mathbb{R}}$ where κ^* acts (projectively) as χ . The various \mathcal{F}_0^χ cover $\mathcal{F}_0^{\mathbb{R}}$, because if $(F, [i])$ lies in $\mathcal{F}_0^{\mathbb{R}}$, then it lies in \mathcal{F}_0^χ with $\chi = i \circ \kappa^* \circ i^{-1}$; we will see below that each \mathcal{F}_0^χ is nonempty. Similarly, if $\chi \in \mathcal{A}$ then we define H_χ^4 as its fixed-point set in $\mathbb{C}H^4 \subseteq P(\Lambda \otimes_{\mathbb{E}} \mathbb{C})$. The notation reflects the fact that H_χ^4 is a copy of real hyperbolic 4-space.

We need the following lemma in order to define the real period map.

Lemma 2.1. *In the notation above, $g(\mathcal{F}_0^\chi) \subset H_\chi^4$.*

Proof. For $(F, [i]) \in \mathcal{F}_0^\chi$ we must show that χ preserves $i_*(H_{\bar{\omega}}^{2,1}(T))$. We defined $\overline{\kappa^*}$ so that it is an antilinear map of $H^3(T; \mathbb{C})$ which preserves the Hodge decomposition; it also preserves each eigenspace of σ . Therefore it preserves the inclusion $H_{\bar{\omega}}^{2,1}(T) \rightarrow H_{\bar{\omega}}^3(T)$. Now, the action of κ^* on $H^3(T; \mathbb{R})$ is identified with the action of $\overline{\kappa^*}$ on $H_{\bar{\omega}}^3(T; \mathbb{C})$ under the eigenspace projection. Therefore κ^* preserves the subspace of $\Lambda(T) \otimes_{\mathbb{E}} \mathbb{C} = H^3(T; \mathbb{R})$ corresponding to $H_{\bar{\omega}}^{2,1}(T)$. The fact that χ preserves $i_*(H_{\bar{\omega}}^{2,1}(T))$ is then a formal consequence of $\chi = i \circ \kappa^* \circ i^{-1}$. \square

We define the real period map $g^{\mathbb{R}} : \mathcal{F}_0^{\mathbb{R}} \rightarrow \mathbb{C}H^4 \times P\mathcal{A}$ by

$$(2.7) \quad g^{\mathbb{R}}(F, [i]) = (g(F), [\chi = i \circ \kappa^* \circ i^{-1}]).$$

The previous lemma asserts that $g(F, [i]) \in H_\chi^4$, so $g^{\mathbb{R}}$ can be regarded as a map $\mathcal{F}_0^{\mathbb{R}} \rightarrow \coprod_{\chi \in P\mathcal{A}} H_\chi^4$. The next lemma shows that we can even regard $g^{\mathbb{R}}$ as a map $\mathcal{F}_0^{\mathbb{R}}/G^{\mathbb{R}} \rightarrow \coprod_{\chi \in P\mathcal{A}} H_\chi^4$.

Lemma 2.2. *The real period map $g^{\mathbb{R}} : \mathcal{F}_0^{\mathbb{R}} \rightarrow \coprod_{\chi \in P\mathcal{A}} H_\chi^4$ is constant on $G^{\mathbb{R}}$ -orbits.*

Proof. We must show for $(F, [i]) \in \mathcal{F}_0^{\mathbb{R}}$ and $h \in G^{\mathbb{R}}$ that $g^{\mathbb{R}}((F, [i]).h) = g^{\mathbb{R}}(F, [i])$. The essential point is that the elements of $P\mathcal{A}$ associated to $(F, [i])$ and $(F, [i]).h$ coincide. We almost know this already, because the anti-involution is constant on components of $\mathcal{F}_0^{\mathbb{R}}$. But $G^{\mathbb{R}}$ is not connected, so we need a little more. We have

$$\begin{aligned} g^{\mathbb{R}}((F, [i]).h) &= \left(g((F, [i]).h), [i \circ (h^{-1})^* \circ \kappa^* \circ h^* \circ i^{-1}] \right) \\ &= \left(g(F, [i]), [i \circ (h^{-1})^* \circ \kappa^* \circ h^* \circ i^{-1}] \right) \\ &= \left(g(F, [i]), [i \circ \kappa^* \circ i^{-1}] \right) \\ &= g^{\mathbb{R}}(F, [i]) . \end{aligned}$$

Here the first line uses the definitions (2.7) and (2.4) of $g^{\mathbb{R}}$ and the G -action, the second uses the G -invariance of the complex period map g , and the third uses the fact that κ and h commute. \square

We know that $g^{\mathbb{R}}$ cannot map $\mathcal{F}_0^{\mathbb{R}}$ onto $\coprod_{\chi \in P\mathcal{A}} H_{\chi}^4$, because $g(\mathcal{F}_0)$ contains no elements of \mathcal{H} . Therefore we define $K_0 = \coprod_{\chi \in P\mathcal{A}} (H_{\chi}^4 - \mathcal{H})$. The following is the main theorem of this section.

Theorem 2.3. *The real period map $g^{\mathbb{R}}$ defines a $P\Gamma$ -equivariant real-analytic diffeomorphism $\mathcal{F}_0^{\mathbb{R}}/G^{\mathbb{R}} \rightarrow K_0 = \coprod_{\chi \in P\mathcal{A}} (H_{\chi}^4 - \mathcal{H})$. Thus $\mathcal{F}_0^{\mathbb{R}}$ is a principal $G^{\mathbb{R}}$ -bundle over K_0 . Taking the quotient by $P\Gamma$ yields a real-analytic orbifold isomorphism*

$$\mathcal{M}_0^{\mathbb{R}} = P\Gamma \backslash \mathcal{F}_0^{\mathbb{R}} / G^{\mathbb{R}} \rightarrow P\Gamma \backslash K_0 .$$

Proof. First observe that $g^{\mathbb{R}} : \mathcal{F}_0^{\mathbb{R}} \rightarrow K_0$ is a local diffeomorphism. This follows immediately from the fact that the rank of $g^{\mathbb{R}}$ is the same as that of g , which is 4 everywhere in \mathcal{F}_0 (see [2, (9.1)]).

Next we prove surjectivity. Suppose $\chi \in P\mathcal{A}$ and $x \in H_{\chi}^4 - \mathcal{H}$. By the surjectivity of the complex period map, there exists $(F, [i]) \in \mathcal{F}_0$ with $g(F, [i]) = x$. We will exhibit an anti-involution α of T that corresponds to χ in a suitable sense. Namely, we claim there exists an anti-involution α of \mathbb{C}^4 with $F.\alpha = F$, such that $\alpha^* : \Lambda(T) \rightarrow \Lambda(T)$ coincides with $i^{-1} \circ \chi \circ i$ up to scalars. Here, we regard α as acting on \mathbb{C}^5 by

$$(2.8) \quad \alpha.(X_0, \dots, X_3, Y) = (\alpha.(X_0, \dots, X_3), \bar{Y}) ,$$

so that α preserves $T \subseteq \mathbb{C}P^4$, and α^* denotes the anti-involution of $\Lambda(T)$ given by pullback under $\alpha : T \rightarrow T$.

If such α exists then it is conjugate to κ by an element of $\mathrm{GL}(4, \mathbb{C})$ (since all real structures on a complex vector space are equivalent), so

there exists $h \in \mathrm{GL}(4, \mathbb{C})$ with $h^{-1}\alpha h = \kappa$. Then $(F, [i]).h \in \mathcal{F}_0^{\mathbb{R}}$ has image (x, χ) , because

$$(F.h).\kappa = F.h\kappa = F.\alpha h = F.h,$$

so $F.h \in \mathcal{C}_0$, and a computation like that for lemma 2.2 establishes $g^{\mathbb{R}}((F, [i]).h) = (x, \chi)$.

To construct α we will relate anti-involutions of Λ preserving x to anti-involutions of \mathbb{C}^4 preserving F . In order to do this we must enlarge the group actions (2.3) and (2.4) to include antilinear transformations. Let $\mathrm{GL}(4, \mathbb{C})'$ be the group of all linear and antilinear automorphisms of \mathbb{C}^4 , and regard it as also acting on \mathbb{C}^5 , with an element h acting by (2.4) if h is linear and by

$$(2.9) \quad h(X_0, \dots, X_1, Y) = (h(X_0, \dots, X_3), \bar{Y})$$

if h is antilinear. We let $(\mathrm{Aut} \Lambda)'$ be the group of all linear and antilinear isometries of Λ , and let \mathcal{F}'_0 be the space of all pairs $(F, [i])$ where $F \in \mathcal{C}_0$, $i : \Lambda(T) \rightarrow \Lambda$ is either a linear or antilinear isometry, and $[i]$ is its equivalence class modulo scalars. \mathcal{F}'_0 is a disjoint union of two copies of \mathcal{F}_0 . Formulas (2.3), (2.4) and (2.9) now define actions of $(\mathrm{Aut} \Lambda)'$ and $\mathrm{GL}(4, \mathbb{C})'$ on \mathcal{F}'_0 , the antilinear elements in each group exchanging the two components of \mathcal{F}'_0 . The subgroup $D = \{I, \omega I, \omega^2 I\}$ of $\mathrm{GL}(4, \mathbb{C})'$ acts trivially, inducing an action of the quotient group, which we call G' . The scalars in $(\mathrm{Aut} \Lambda)'$ also act trivially, inducing an action of the quotient group, which we call $P\Gamma'$. Each of G' and $P\Gamma'$ acts freely on \mathcal{F}'_0 , because G and $P\Gamma$ act freely on \mathcal{F}_0 .

Now we apply a general principle: let Y be a set, L a group with a free left action on Y , and R be a group with a free right action on Y , these two actions commuting. Suppose $y \in Y$ has images $\ell \in Y/R$ and $r \in L \backslash Y$. Then for every element ϕ of L preserving ℓ , there is a unique element $\hat{\phi}$ of R such that $\phi.y.\hat{\phi} = y$. Furthermore, $\phi \rightarrow \hat{\phi}$ defines an anti-isomorphism from the stabilizer L_ℓ of ℓ to the stabilizer R_r of r . (Note: the ‘‘anti-’’ comes from the fact that L acts on the left and R on the right; it has nothing to do with anti-involutions.)

We apply this with $Y = \mathcal{F}'_0$, $y = (F, [i])$, $R = G'$ and $L = P\Gamma'$. Our choice of y gives $\ell = x \in \mathbb{C}H^4 - \mathcal{H} = Y/R$ and $r = F \in \mathcal{F}_0 = L \backslash Y$. We take $\phi = \chi \in (P\Gamma')_x = L_\ell$ and obtain $\hat{\phi} \in R_r = (G')_F$ of order two, satisfying $\phi.(F, [i]).\hat{\phi}$. This amounts to the equality $[i] = [\chi \circ i \circ (\hat{\phi}^{-1})^*]$, i.e., $[\hat{\phi}^*] = [i^{-1} \circ \chi \circ i]$. Since ϕ exchanges the components of \mathcal{F}'_0 , $\hat{\phi}$ does also, so $\hat{\phi}$ is antilinear. Taking $\alpha = \hat{\phi}$ finishes the proof of surjectivity.

To prove injectivity it suffices to show that $g^{\mathbb{R}} : \mathcal{F}_0^{\chi} / G^{\mathbb{R}} \rightarrow \mathcal{F}_0 / G = \mathbb{C}H^4 - \mathcal{H}$ is injective for each $\chi \in P\mathcal{A}$. This also follows from a general principle, best expressed by regarding \mathcal{F}_0^{χ} as the fixed-point set of χ in

\mathcal{F}_0 . We have formulated an action of $P\Gamma'$ on \mathcal{F}'_0 , but we can regard it as acting on \mathcal{F}_0 by identifying \mathcal{F}_0 with $\mathcal{F}'_0/\langle\kappa\rangle$. The subgroup $P\Gamma$ acts by (2.3) as before, but an anti-linear $\gamma \in P\Gamma'$ now acts by

$$\gamma.(F, [i]) = (F.\kappa, [\gamma \circ i \circ \kappa^*]) .$$

It follows from these definitions that \mathcal{F}_0^χ is the fixed-point set of χ . The principle we invoke is: if a group G acts freely on a set X , ϕ is a transformation of X normalizing G , and Z is the centralizer of ϕ in G , then the natural map $X^\phi/Z \rightarrow X/G$ is injective. (Proof: if $h \in G$ carries $x \in X^\phi$ to $y \in X^\phi$ then so does $\phi^{-1}h\phi$, so $\phi^{-1}h\phi = h$ by freeness, so $h \in Z$.) We apply this with $X = \mathcal{F}_0$, $G = G$ and $\phi = \chi$; then $X^\phi = \mathcal{F}_0^\chi$, $Z = G^{\mathbb{R}}$, and we conclude that $\mathcal{F}_0^\chi/G^{\mathbb{R}} \rightarrow \mathcal{F}_0/G = \mathbb{C}H^4 - \mathcal{H}$ is injective. \square

In later sections we will make concrete the three elements of theorem 2.3 that are at this point just abstract. In section 3 we will make explicit the components of $P\Gamma \backslash K_0$, by giving explicit anti-involutions χ_0, \dots, χ_4 of Λ which form a set of representatives of CA . This gives $\mathcal{M}_0^{\mathbb{R}} \cong \coprod_{j=0}^4 P\Gamma_j^{\mathbb{R}} \backslash (H_j^4 - \mathcal{H})$, where H_j^4 is short for $H_{\chi_j}^4$ and $P\Gamma_j^{\mathbb{R}}$ is the subgroup of $P\Gamma$ preserving H_j^4 . Then in section 4 we will make the $P\Gamma_j^{\mathbb{R}}$ explicit by describing them arithmetically and in terms of Coxeter groups. In section 5 we will make explicit the spaces $H_j^4 - \mathcal{H}$ by describing $H_j^4 \cap \mathcal{H}$. Our main theorem for moduli of smooth real cubic surfaces, given in the introduction as theorem 1.1, is the union of theorem 2.3, corollary 3.3, and theorems 4.1 and 5.4.

3. THE FIVE FAMILIES OF REAL CUBICS

Theorem 2.3 described $\mathcal{M}_0^{\mathbb{R}}$ in terms of the H_χ^4 , where $[\chi]$ varies over the set PA of anti-involutions of $\mathbb{C}H^4$ arising from anti-involutions χ of Λ . In this section we classify the χ up to conjugacy by $\text{Aut } \Lambda$; there are exactly 10 classes, and we give a recognition principle which allows one to easily compute which class contains a given anti-involution. In fact there are only five classes up to sign, so CA has 5 elements, and there are 5 orbits of H_χ^4 's under $PA\text{Aut } \Lambda$. Unlike in the rest of the paper, in this section we will be careful to distinguish between an anti-involution χ of Λ and its scalar class $[\chi]$.

We begin by stating the classification. Using the coordinate system (2.2), we define the following five anti-involutions of Λ :

$$\begin{aligned}
 \chi_0 &: (x_0, x_1, x_2, x_3, x_4) \mapsto (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) \\
 \chi_1 &: (x_0, x_1, x_2, x_3, x_4) \mapsto (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3, -\bar{x}_4) \\
 (3.1) \quad \chi_2 &: (x_0, x_1, x_2, x_3, x_4) \mapsto (\bar{x}_0, \bar{x}_1, \bar{x}_2, -\bar{x}_3, -\bar{x}_4) \\
 \chi_3 &: (x_0, x_1, x_2, x_3, x_4) \mapsto (\bar{x}_0, \bar{x}_1, -\bar{x}_2, -\bar{x}_3, -\bar{x}_4) \\
 \chi_4 &: (x_0, x_1, x_2, x_3, x_4) \mapsto (\bar{x}_0, -\bar{x}_1, -\bar{x}_2, -\bar{x}_3, -\bar{x}_4).
 \end{aligned}$$

The subscript indicates how many of the coordinates are replaced by the negatives of their complex conjugates rather than just their conjugates. We caution the reader that the obvious analogue of the following theorem fails for some other $\mathcal{E}^{n,1}$, for example $n = 3$.

Theorem 3.1. *An anti-involution of Λ is Aut Λ -conjugate to exactly one of the $\pm\chi_j$.*

We will give an indirect proof of this theorem by combining Theorem 2.3 and classical knowledge of cubic surfaces. Our original approach was a direct algebraic proof, thereby reproving the classical fact that $\mathbb{C}_0^{\mathbb{R}}$ has five connected components. Since the proof of the classical theorem on components is quite simple (e.g., [24, §§23–24]), we omit the algebraic argument.

The key ingredient in the proof is the fact that no two of the $\pm\chi_j$ are conjugate. To see this, recall that $\theta = \omega - \bar{\omega} = \sqrt{-3}$ and consider the 5-dimensional vector space $V = \Lambda/\theta\Lambda$ over $\mathbb{F}_3 = \mathcal{E}/\theta\mathcal{E}$. V is equipped with a nondegenerate symmetric bilinear form q obtained by reducing inner products in Λ modulo θ . Any linear or anti-linear isometry of Λ reduces to a linear isometry of V , and we consider the action of $\chi = \pm\chi_j$ on V . Obviously, the dimensions of χ 's eigenspaces and the determinants of q 's restrictions to them are conjugacy invariants of χ (the determinants lie in $\mathbb{F}_3^*/(\mathbb{F}_3^*)^2 = \{\pm 1\}$). These invariants are easy to compute for χ_j , because both q and χ_j are diagonalized in our coordinate system. The result is that χ_j has negated (resp. fixed) space of dimension j (resp. $5-j$) and the restriction of q to it has determinant $+1$ (resp. -1). For $-\chi_j$, the negated and fixed spaces are reversed. Therefore $\pm\chi_0, \dots, \pm\chi_4$ all lie in distinct conjugacy classes. Now we can prove theorem 3.1 and also the following recognition principle:

Theorem 3.2. *Two anti-involutions of Λ are equivalent if and only if the restrictions of q to the two fixed spaces in V (or to the two negated spaces) are isometric.*

Proof of theorems 3.1 and 3.2. It is classical that $PC_0^{\mathbb{R}}$ has 5 connected components [24, §§23–24]. Because -1 lies in the identity component of $G^{\mathbb{R}}$, it follows that $C_0^{\mathbb{R}}$ itself has 5 components, and thence that $\mathcal{M}_0^{\mathbb{R}}$ has at most 5 components. By theorem 2.3, this number of components is at least the cardinality of CA , so $|CA| \leq 5$. If $\chi \in \mathcal{A}$, then $\chi \cdot (-\omega)^i$, $i = 0, \dots, 5$ are all of the elements of $[\chi]$, and these fall into at most two conjugacy classes (proof: conjugate by scalars). Therefore there are at most 10 classes of anti-involutions of Λ . Since we have exhibited 10 classes, they must be a complete set of representatives, justifying theorem 3.1. Since they are distinguished by their actions on V , we also have theorem 3.2. \square

Every inequality in the proof must be an equality, so each $P\Gamma_j^{\mathbb{R}} \backslash (H_j^4 - \mathcal{H})$ must be connected. This gives our first improvement on theorem 2.3. Recall that H_j^4 is the fixed-point set of χ_j in $\mathbb{C}H^4$, and $P\Gamma_j^{\mathbb{R}}$ is the stabilizer of H_j^4 in $P\Gamma$. See section 4 for more information about the $P\Gamma_j^{\mathbb{R}}$, which are nonuniform lattices in $PO(4, 1)$.

Corollary 3.3. *The set CA has cardinality 5 and is represented by χ_0, \dots, χ_4 of (3.1). We have an isomorphism $\mathcal{M}_0^{\mathbb{R}} = \coprod_{j=0}^4 P\Gamma_j^{\mathbb{R}} \backslash (H_j^4 - \mathcal{H})$ of real analytic orbifolds. For each j , $P\Gamma_j^{\mathbb{R}}$ acts transitively on the connected components of $H_j^4 - \mathcal{H}$. \square*

Now we determine the correspondence between our 5 components and the classical parameterization of the 5 types of real cubic surface. This was in terms of the topology of the real locus of S , or by the action of complex conjugation on the 27 lines of S . We will study the families in terms of the action of complex conjugation on $H^2(S)$, and relate this to its actions on $\Lambda(T)$ and the lines on S .

We write $L(S)$ for $H^2(S; \mathbb{Z})$, which is isometric to $L := \mathbb{Z}^{1,6}$ in such a way that the hyperplane class $\eta(S)$ corresponds to the norm 3 vector $\eta = (3, -1, -1, -1, -1, -1, -1)$. Here the inner product on $\mathbb{Z}^{1,6}$ is given by

$$x \cdot y = x_0y_0 - x_1y_1 - \dots - x_6y_6 .$$

The primitive cohomology $L_0(S)$ is the orthogonal complement of $\eta(S)$ and is a negative-definite copy of the E_6 root lattice. The isometries of $L(S)$ preserving $\eta(S)$ form a copy of the Weyl group $W = W(E_6) = \text{Aut}(L, \eta)$, which is generated by the reflections in the roots (norm -2 vectors) of $L_0 = \eta^\perp$. Since κ is antiholomorphic, it negates $\eta(S)$ and hence acts on $L(S)$ by the product of $-I$ and some element g of $\text{Aut}(L(S), \eta(S))$ of order 1 or 2. Therefore, to classify the possible actions of κ on $L(S)$ we will enumerate the involutions of W up to conjugacy. According to [10, p. 27] there are exactly four conjugacy

family	class of	real	non-real of		fixed	fixed	class of
	action		1st	2nd	space	space	
	on $L(S)$	lines	kind	kind	in $V(S)$	in $V(T)$	on $\Lambda(T)$
$\mathcal{C}_{0,0}^{\mathbb{R}}$	$-I$	27	0	0	$[\]$	$[++++-]$	χ_0
$\mathcal{C}_{0,1}^{\mathbb{R}}$	$-R_1$	15	0	12	$[+]$	$[++++-]$	χ_1
$\mathcal{C}_{0,2}^{\mathbb{R}}$	$-R_1R_2$	7	4	16	$[++]$	$[+++-]$	χ_2
$\mathcal{C}_{0,3}^{\mathbb{R}}$	$-R_1R_2R_3$	3	12	12	$[+++]$	$[+-]$	χ_3
$\mathcal{C}_{0,4}^{\mathbb{R}}$	$-R_1R_2R_3R_4$	3	24	0	$[++++]$	$[-]$	χ_4

TABLE 3.1. Action of complex conjugation on various objects associated to $F \in \mathcal{C}_{0,j}^{\mathbb{R}}$. The 6th and 7th columns indicate diagonalized \mathbb{F}_3 -quadratic forms with ± 1 's on the diagonal.

classes of involutions. Each class may be constructed as the product of the reflections in $1 \leq j \leq 4$ mutually orthogonal roots. To make this explicit we choose four distinct commuting reflections R_1, \dots, R_4 in W .

We write $\mathcal{C}_{0,0}^{\mathbb{R}}, \dots, \mathcal{C}_{0,4}^{\mathbb{R}}$ for the set of those $F \in \mathcal{C}_0^{\mathbb{R}}$ for which $(L(S), \eta(S), \kappa^*)$ is equivalent to $(L, \eta, -g)$ for $g = I, R_1, R_1R_2, R_1R_2R_3, R_1R_2R_3R_4$. The j in $\mathcal{C}_{0,j}^{\mathbb{R}}$ is the number of R 's involved. By the above considerations the $\mathcal{C}_{0,j}^{\mathbb{R}}$ are disjoint and cover $\mathcal{C}_0^{\mathbb{R}}$. We will write $\mathcal{M}_{0,j}^{\mathbb{R}}$ for $\mathcal{C}_{0,j}^{\mathbb{R}}/G^{\mathbb{R}}$.

Now we relate the κ -action on $L(S)$ to the configuration of lines. In the classical terminology, a line is called real if it is preserved by κ , and a non-real line is said to be of the first (resp. second) kind if it meets (resp. does not meet) its complex conjugate. The terminology becomes a little easier to remember if one thinks of a real line as being a line of the 0th kind. The lines define 27 elements of $L(S)$, which are exactly the 27 vectors of norm -1 that have inner product 1 with $\eta(S)$. Two lines meet (resp. do not meet) if the corresponding vectors have inner product 1 (resp. 0). In particular, which lines of S are real or nonreal of the first or second kind can be determined by studying the action of κ on $L(S)$. The numbers of lines of the various types depends only on the isometry class of $(L(S), \eta(S), \kappa^*)$, with the results given in the first five columns of table 3.1. This allows us to identify our five families with the classically defined ones. For example, Segre [24] names the families F_1, \dots, F_5 ; his F_{j+1} corresponds to our $\mathcal{C}_{0,j}^{\mathbb{R}}$.

Finally, we consider the action of κ on $\Lambda(T)$ and on two 5-dimensional quadratic vector spaces over \mathbb{F}_3 . These are

$$V(T) = \Lambda(T)/\theta\Lambda(T) \quad \text{and} \quad V(S) = L_0(S)/3L'_0(S),$$

where $L'_0(S)$ is the dual lattice of $L_0(S)$. The goal in the following analysis is to find the anti-involution $\chi \in \{\pm\chi_0, \dots, \pm\chi_4\}$ such that $(\Lambda(T), \kappa^*)$ is isometric to (Λ, χ) . The idea is that the action of κ on $L(S)$ determines the action on $V(S)$, hence on $V(T)$ using a construction from [2], and finally determines χ by means of our recognition principle for anti-involutions of Λ (theorem 3.2).

Lemma 3.4. *Let $F \in \mathcal{C}_0^{\mathbb{R}}$ and denote the actions of κ on $V(S)$ and $V(T)$ by $\hat{\kappa}$. Then $(V(S), \hat{\kappa})$ and $(V(T), -\hat{\kappa})$ are isomorphic as quadratic spaces equipped with isometries.*

Proof. By [2, lemma 3.14] there is a natural isometry $V(S) \rightarrow V(T)$, which we will denote by A . It is defined as follows. Given $a \in V(S)$, we choose a representative b for a in $L_0(S)$ and then take the homology class c Poincaré dual to b . Since c is an element of the primitive homology of $S \subseteq T$ and T has no primitive second homology, c bounds some 3-chain d in T . Then $e = \sigma_*(d) - \sigma_*^{-1}(d)$ defines an element of $H_3(T; \mathbb{Z})$. We let f be the element of $H^3(T; \mathbb{Z}) = \Lambda(T)$ Poincaré dual to e , and g be the element of $V(T)$ represented by f . We set $A(a) = g$. One can check that A is well-defined and an isometry.

Supposing that a, \dots, g are as above, we will work out $A(\hat{\kappa}(a))$ in terms of $A(a)$. An element of $L_0(S)$ representing $a' := \hat{\kappa}(a)$ is $b' = \kappa^*(b)$, and its Poincaré dual is $c' = \kappa_*(c)$. A 3-chain bounded by c' is $d' = \kappa_*(d)$, and then the relevant element of $H_3(T; \mathbb{Z})$ is

$$\begin{aligned} e' &= \sigma_*(d') - \sigma_*^{-1}(d') \\ &= \sigma_*\kappa_*(d) - \sigma_*^{-1}\kappa_*(d) \\ &= \kappa_*\sigma_*^{-1}(d) - \kappa_*\sigma_*(d) \\ &= -\kappa_*(\sigma_*(d) - \sigma_*^{-1}(d)) \\ &= -\kappa_*(e). \end{aligned}$$

We have used the fact that κ conjugates σ into σ^{-1} . Then the Poincaré dual of e' is $f' = -\kappa^*(f)$ and the reduction of f' modulo θ is $-\hat{\kappa}(g)$. That is, $A(\hat{\kappa}(a)) = -\hat{\kappa}(A(a))$. Therefore A is an isometry between the pairs $(V(S), \hat{\kappa})$ and $(V(T), -\hat{\kappa})$. \square

Lemma 3.5. *Suppose $F \in \mathcal{C}_{0,j}^{\mathbb{R}}$. Then the isometry classes of the fixed spaces for κ in $V(S)$ and $V(T)$ are given by the 6th and 7th columns of table 3.1, and $(\Lambda(T), \kappa^*)$ is isometric to (Λ, χ_j) as indicated in the last column.*

Proof. Since the conjugacy class of the action of κ on $L_0(S)$ is known, it is easy to compute the fixed space in $V(S)$. It is just the span of the images of the roots corresponding to R_1, \dots, R_j . This space is isometric to $[+^j]$ because the roots have norm $-2 \equiv 1 \pmod{3}$. The lemma shows that the fixed space in $V(T)$ is isometric to the negated space in $V(S)$, justifying the 7th column. The last claim follows from theorem 3.2. \square

We defined the spaces $\mathcal{C}_{0,j}^{\mathbb{R}}$ and $\mathcal{M}_{0,j}^{\mathbb{R}}$ in terms of the action of complex conjugation on the homology of the surface, and the $PI_j^{\mathbb{R}}$ in terms of the anti-involutions χ_j . The final result of this section is that the indexings by j correspond; it follows immediately from the previous lemma.

Corollary 3.6. *For each $j = 0, \dots, 4$, $\mathcal{M}_{0,j}^{\mathbb{R}} = PI_j^{\mathbb{R}} \setminus (H_j^4 - \mathcal{H})$.* \square

4. THE STABILIZERS OF THE H^4 'S

In this section we continue to make theorem 2.3 more explicit; we know that $\mathcal{M}_0^{\mathbb{R}} = \coprod_{j=0}^4 PI_j^{\mathbb{R}} \setminus (H_j^4 - \mathcal{H})$, and now we will describe the $PI_j^{\mathbb{R}}$. We give two descriptions, one arithmetic and one in the language of Coxeter groups. The arithmetic description is easy:

Theorem 4.1. *$PI_j^{\mathbb{R}} \cong \text{PO}(\Psi_j)$, where Ψ_j is the quadratic form on \mathbb{Z}^5 given by*

$$\begin{aligned}\Psi_0(y_0, \dots, y_4) &= -y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 \\ \Psi_1(y_0, \dots, y_4) &= -y_0^2 + y_1^2 + y_2^2 + y_3^2 + 3y_4^2 \\ \Psi_2(y_0, \dots, y_4) &= -y_0^2 + y_1^2 + y_2^2 + 3y_3^2 + 3y_4^2 \\ \Psi_3(y_0, \dots, y_4) &= -y_0^2 + y_1^2 + 3y_2^2 + 3y_3^2 + 3y_4^2 \\ \Psi_4(y_0, \dots, y_4) &= -y_0^2 + 3y_1^2 + 3y_2^2 + 3y_3^2 + 3y_4^2.\end{aligned}$$

The mnemonic is that j of the coefficients of Ψ_j are 3 rather than 1. To prove the theorem, write $\Lambda_j := \Lambda^{\chi_j}$ for the \mathbb{Z} -lattice of χ_j -invariant vectors in Λ , so $\Lambda_j = \mathbb{Z}^{5-j} \oplus \theta\mathbb{Z}^j \subseteq \mathcal{E}^5$. The theorem now follows from this lemma:

Lemma 4.2. *For each j , every isometry of the \mathbb{Z} -lattice Λ_j is induced by an isometry of Λ .*

Proof. One can check that the \mathbb{Z} -lattice $L = \Lambda_j \cap \theta\Lambda$ can be described in terms of Λ_j alone as $L = 3(\Lambda_j)'$, where the prime denotes the dual lattice. Therefore every isometry of Λ_j preserves the \mathcal{E} -span of Λ_j and $\frac{1}{\theta}L$, which in each case is exactly Λ . \square

Now we describe the $P\Gamma_j^{\mathbb{R}}$ more geometrically; this is interesting in its own right, and also necessary for when we allow our cubic surfaces to have singularities (section 9). Our description relies on the good fortune that the subgroup W_j generated by reflections has index 1 or 2 in each case. The W_j are Coxeter groups, described in figure 1.1 using an extension of the usual conventions for Coxeter diagrams.

Namely, the mirrors (fixed-point sets) of the reflections in W_j chop H_j^4 into components, which W_j permutes freely and transitively. The closure of any one of these components is called a Weyl chamber; we fix one and call it C_j . Then W_j is generated by the reflections across the facets of C_j , and C_j is a fundamental domain in the strong sense that any point of H_j^4 is W_j -equivalent to a unique point of C_j . We describe W_j by drawing its Coxeter diagram: its vertices (“nodes”) correspond to the facets of C_j , which are joined by edges (“bonds”) that are decorated according to how facets meet each other, using the following scheme:

$$\begin{aligned}
 & \text{no bond } \circ \quad \quad \quad \circ \iff \text{ they meet orthogonally;} \\
 & \text{a single bond } \circ \text{---} \circ \iff \text{ their interior angle is } \pi/3; \\
 (4.1) \quad & \text{a double bond } \circ \text{====} \circ \iff \text{ their interior angle is } \pi/4; \\
 & \text{a triple bond } \circ \text{=====} \circ \iff \text{ their interior angle is } \pi/6; \\
 & \text{a strong bond } \circ \text{———} \circ \iff \text{ they are parallel;} \\
 & \text{a weak bond } \circ \text{-----} \circ \iff \text{ they are ultraparallel.}
 \end{aligned}$$

Parallel walls are those that do not meet in hyperbolic space but do meet at the sphere at infinity. Ultraparallel walls are those that do not meet even at infinity.

Note that the diagram for W_j admits a symmetry for $j = 1$ or 2 ; this represents an isometry of C_j . We now state the main theorem of this section.

Theorem 4.3. *$P\Gamma_j^{\mathbb{R}}$ is the semidirect product of its reflection subgroup W_j , given in figure 1.1, by the group of diagram automorphisms, which is $\mathbb{Z}/2$ if $j = 1$ or 2 and trivial otherwise.*

The rest of the section is devoted to the proof. For the most part the argument is uniform in j , so we will write H for $H_j^4 = H_{\chi_j}^4$, W for W_j , χ for χ_j and C for C_j . We will write Λ^χ for $\Lambda_j = \Lambda^{\chi_j}$. We call $r \in \Lambda^\chi$ a root of Λ^χ if either (i) $r^2 = 1$ or 2 , or (ii) $r^2 = 3$ or 6 and $\theta|r$ in Λ . This is not a standard definition of ‘root’, but it is the natural one in this context, as the next lemma shows. Norm 3 and 6 roots are really just norm 1 and 2 roots of Λ in disguise; they occur when χ negates rather than preserves a norm 1 or 2 vector in Λ . Also, using the ideas of the proof of lemma 4.2 lets one replace the condition $r/\theta \in \Lambda$ by

the equivalent condition that $r \in 3(\Lambda^x)'$, where the prime indicates the dual \mathbb{Z} -lattice. The advantage is that this refers only to Λ^x rather than to its embedding in Λ .

Lemma 4.4. *The hyperbolic reflections in W are exactly the reflections in the roots of Λ^x .*

Proof. Since $\text{Aut } \Lambda$ contains the reflections in norm 1 and 2 vectors of Λ , it is clear that biflection in any root of Λ^x lies in W . To prove the converse, observe that a real hyperbolic reflection of H can only be induced by a biflection, i.e., a complex reflection of order 2 of $\mathbb{C}H^4$. It is not hard to see that a biflection in $P\Gamma$ arises from a biflection of Λ ; for a formal proof see [1, lemma 8.1]. The only reflections of Λ are those in lattice vectors r of norm ± 1 and ± 2 (see [1, lemma 8.2]). If $r^2 < 0$ then the “reflection” in r fixes only a single point of $\mathbb{C}H^4$, so it doesn’t act as a hyperbolic reflection. So we may suppose $r^2 = 1$ or 2 . Since biflection in r commutes with χ , $\chi(r)$ is proportional to r , so that χ preserves the \mathcal{E} -span of r . It is easy to see that an anti-involution of a free \mathcal{E} -module of rank one either fixes or negates some generator of this module. Thus χ preserves either a unit multiple of r or a unit multiple of θr . That is, the biflection in r coincides with the biflection in some root of Λ^x . \square

Given some roots r_1, \dots, r_n of Λ^x whose inner products are nonpositive, their polyhedron is defined to be a particular one of the regions bounded by the hyperplanes r_i^\perp , namely the image in H of

$$\{ v \in \Lambda^x \otimes \mathbb{R} \mid v^2 < 0 \text{ and } v \cdot r_i \leq 0 \text{ for } i = 1, \dots, n \} .$$

A set of simple roots for W is a set of roots of Λ^x whose pairwise inner products are nonpositive and whose polyhedron is a Weyl chamber C . Vinberg’s algorithm [27] seeks a set of simple roots for W . The algorithm applies to any hyperbolic Coxeter group, but we will discuss it only in our specific situation. In fact we will only treat the case $j = 3$ in detail, and comment briefly on the others.

First one chooses a vector k (the “controlling vector”) representing a point p of H . We choose $k = (1, 0, 0, 0, 0)$, which conveniently lies in all the Λ^{x_j} . Second, one considers the finite subgroup $V \subseteq W$ generated by the reflections in W that fix p . By lemma 4.4 these are the reflections in the roots of Λ^x that are orthogonal to k . In the case $j = 3$, we identify \mathbb{Z}^5 with Λ^x so that $(a, b, c, d, e) \in \mathbb{Z}^5$ corresponds to $(a, b, c\theta, d\theta, e\theta) \in \Lambda$. The quadratic form is then Ψ_3 from theorem 4.1. We will use these \mathbb{Z}^5 coordinates throughout the calculations, converting the results to elements of Λ only at the very end. The roots orthogonal to the controlling vector are $(0, \pm 1, 0, 0, 0)$ of

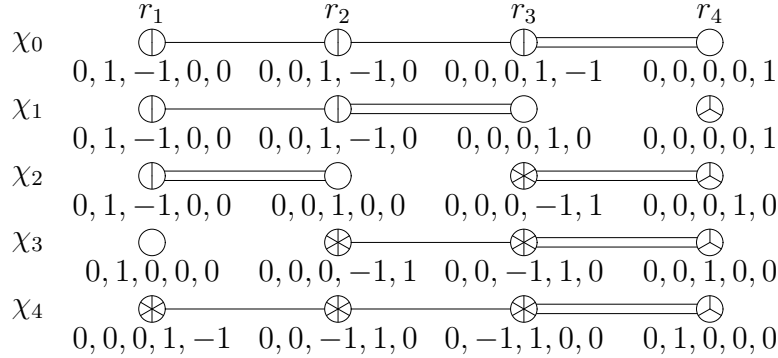


FIGURE 4.1. For each $\chi = \chi_j$ we give simple roots in Λ^χ for the stabilizer $V \subseteq W$ of $p \in H$. In each case we number the roots r_1, \dots, r_4 from left to right. A node indicated by \bigcirc (resp. $\bigoplus, \bigoplus, \bigotimes$) represents a root of norm 1 (resp. 2, 3, 6). The mnemonic is that the norm of the root is the number of white regions in the symbol. Nodes are joined according to the conventions of (4.1). The meanings of the coordinates are explained in the text.

norm 1, $(0, 0, \pm 1, 0, 0)$, $(0, 0, 0, \pm 1, 0)$ and $(0, 0, 0, 0, \pm 1)$ of norm 3, and $\pm(0, 0, 1, -1, 0)$, $\pm(0, 0, 1, 0, -1)$ and $\pm(0, 0, 0, 1, -1)$ of norm 6. Now, V was defined as the group generated by reflections in these roots, and one recognizes it as a Coxeter group of type A_1B_3 . A set of simple roots appears in figure 4.1, together with analogous data for the other χ_j . The norms of the roots are indicated by using different symbols for the nodes; this will be useful in section 5 for describing $H_j^4 \cap \mathcal{H}$. *Caution:* the meaning of a 5-tuple of integers depends on j , even though j is not visible in the notation. Given an element of \mathbb{Z}^5 representing a vector in Λ^{χ_j} , multiply the last j coordinates by θ to obtain the components of that vector with respect to the standard basis of $\Lambda = \mathcal{E}^{4,1}$.

Next, one orders the mirrors of W that miss p according to their “priority”, where the priority is a decreasing function of the distance to p . We define the priority of a mirror r^\perp associated to a root $r \in \Lambda^\chi$ to be $-2(k \cdot r)^2/r^2$. This is always an integer, so it is easy to enumerate the possible priorities, bearing in mind that if $r^2 = 3$ or 6 then $3|k \cdot r$, since r being a root requires $r \in 3(\Lambda^\chi)'$. It turns out that only the first 10 possible priorities are required for the calculations; these are given in the calculations below. The iterative step in Vinberg’s algorithm is to consider all roots of a given priority p , and suppose that previous batches have enumerated all simple roots of higher priority. Batch 0

has already been defined. We discard those roots of priority p that have positive inner product with some simple root of a previous batch. Those that remain are simple roots and form the current batch. If the polyhedron P defined by our newly-enlarged set of simple roots has finite volume then the algorithm terminates. Otherwise, we proceed to the next batch. The finite-volume condition can be checked using a criterion of Vinberg [28, p. 22] on the Coxeter diagram of P , which is computed from the inner products among the simple roots. There is no guarantee that the algorithm will terminate, but if it does then the roots obtained (the union of all the batches) form a set of simple roots for W .

Here are the details of the calculation for $j = 3$, the lengthiest of the cases. For a root $r = (a, b, c, d, e)$ to be a simple root, it must satisfy $r \cdot r_i \leq 0$ for $i = 1, \dots, 4$, which amounts to the conditions $b \leq 0$ and $e \leq d \leq c \leq 0$.

Batch 1: r has norm 2 and priority -1 . Then $r = (1, b, c, d, e)$ with $b^2 + 3(c^2 + d^2 + e^2) = 3$. This implies that one of c, d and e must be ± 1 and all the others including b must vanish. Imposing the condition $c \leq d \leq e \leq 0$ implies $r = (1, 0, 0, 0, -1)$. This is the only root of this batch, and we name it r_5 . By inspecting the Coxeter diagram for r_1, \dots, r_5 one checks that their polyhedron has infinite volume, so we continue.

Batch 2: r has norm 1 and priority -2 . Then $r = (1, b, c, d, e)$ with $a^2 + 3(c^2 + d^2 + e^2) = 2$. There are no solutions for b, \dots, e , so there are no roots in this batch.

Batch 3: r has norm 6 and priority -3 . Then $r = (3, b, c, d, e)$ with $b^2 + 3(c^2 + d^2 + e^2) = 15$. Therefore $r = (3, -3, 0, -1, -1)$ or $(3, 0, 0, -1, -2)$. If one of these failed to lie in $3(\Lambda^x)'$ then we would discard it because r would not be a root. They both lie in $3(\Lambda^x)'$, but we discard the second one anyway because it has positive inner product with r_5 . The other root is the only simple root in this batch, and we call it r_6 . As in batch 1, the polyhedron defined by r_1, \dots, r_6 has infinite volume, so we continue.

Batch 4: r has norm 2 and priority -4 . This forces $r = (2, 0, 0, -1, -1)$ but then $r \cdot r_5 > 0$. So this batch is empty.

Batch 5: r has norm 3 and priority -6 . This forces $r = (3, -3, 0, 0, -1)$ or $(3, 0, 0, 0, -2)$. Again one checks that these lie in $3(\Lambda^x)'$. But in the first case we have $r \cdot r_6 > 0$ and in the second we have $r \cdot r_5 > 0$. So this batch is empty.

Batch 6: r has norm 1 and priority -8 . There are no solutions for b, \dots, e , so this batch is empty.

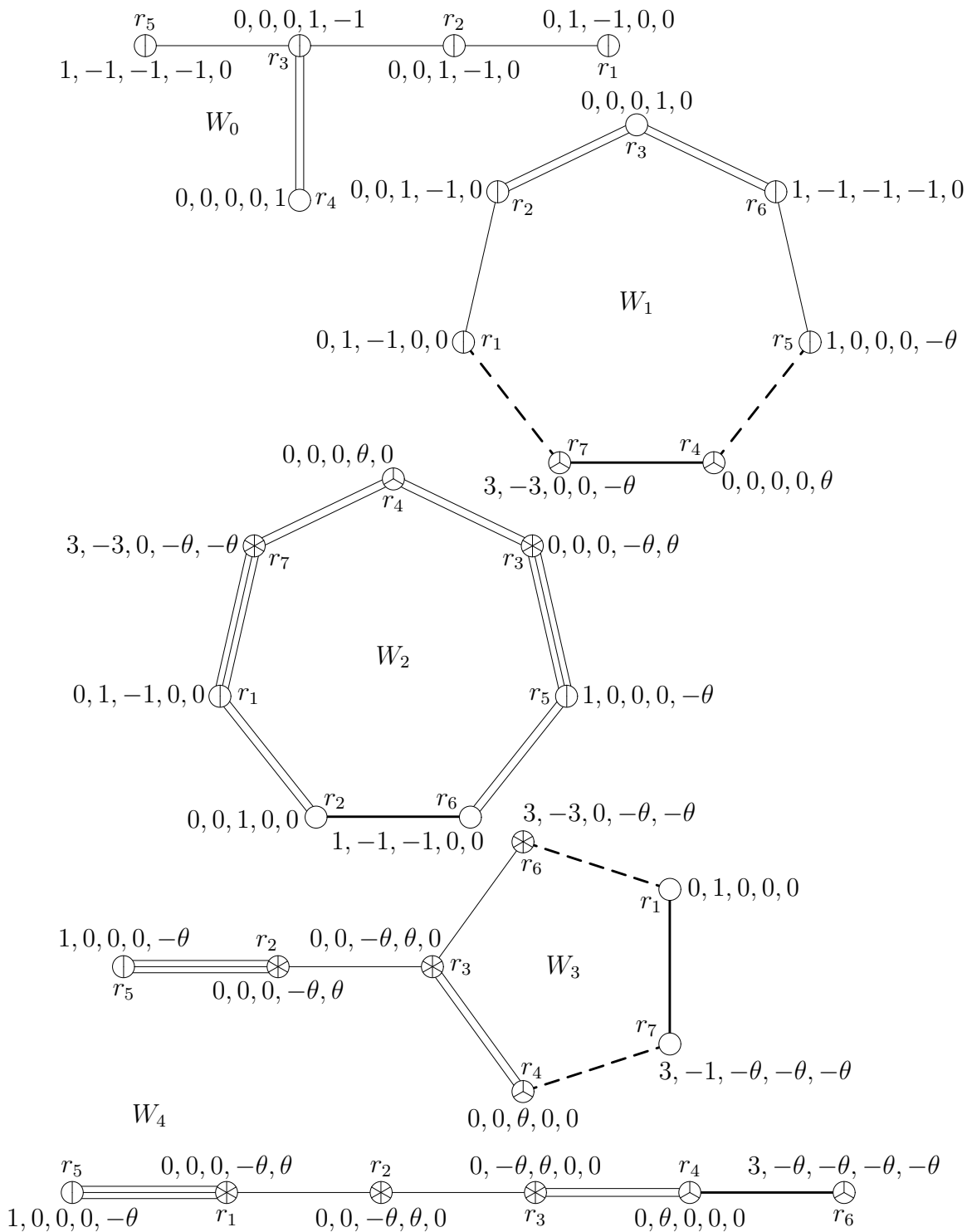


FIGURE 4.2. Simple roots for the W_j .

Batch 7: r has norm 2 and priority -9 . There are no solutions for b, \dots, e , so this batch is empty.

Batch 8: r has norm 6 and priority -12 . Then r is one of $(6, 0, -1, -2, -3)$, $(6, -3, -1, -1, -3)$ or $(6, -6, 0, -1, -1)$, which all do lie in $3(\Lambda^\chi)'$. In the first two cases we have $r \cdot r_5 > 0$ and in the last we have $r \cdot r_6 > 0$. So this batch is empty.

Batch 9: r has norm 2 and priority -16 . Then r is $(4, 0, -1, -1, -2)$ or $(4, -3, -1, -1, -1)$. In the first case we have $r \cdot r_5 > 0$ and in the second we have $r \cdot r_6 > 0$. So this batch is empty.

Batch 10: r has norm 1 and priority -18 . Then $r = (3, -2, 0, -1, -1)$ or $(3, -1, -1, -1, -1)$. The first of these has positive inner product with r_6 , and the second is the only root of this batch; we name it r_7 . One checks that the polyhedron defined by r_1, \dots, r_7 has finite volume, so the algorithm terminates and these roots are a set of simple roots for $W = W_3$.

Converting r_1, \dots, r_7 to our standard coordinates on Λ amounts to multiplying their last three components by θ , and the resulting simple roots and Coxeter diagram appear in figure 4.2. This data also appears for the other cases, which are so similar to this one that we give only the batches in which the various simple roots r_5, \dots appear.

Case $\chi = \chi_0$: r_5 appears in batch 1.

Case $\chi = \chi_1$: r_5 and r_6 appear in batch 1 and r_7 in batch 5.

Case $\chi = \chi_2$: r_5, r_6 and r_7 appear in batches 1, 2 and 3.

Case $\chi = \chi_4$: r_5 and r_6 appear in batches 1 and 5.

Now we can finish the proof of theorem 4.3, which describes $P\Gamma_j^{\mathbb{R}}$ as the semidirect product of W_j by its group of diagram automorphisms. W_j is obviously a normal subgroup of $P\Gamma_j^{\mathbb{R}}$. It follows that $P\Gamma_j^{\mathbb{R}}$ is the semidirect product of W_j by the subgroup of $P\Gamma_j^{\mathbb{R}}$ that carries C_j into itself. In cases $j = 0, 3$ and 4 , C_j has no symmetry, so $P\Gamma_j^{\mathbb{R}} = W_j$ as claimed. In the remaining cases all we have to do is check is that the nontrivial diagram automorphism γ lies in $P\Gamma_j^{\mathbb{R}}$. In each case, the simple roots span Λ_j , and γ preserves the norms and inner products of the simple roots. So $\gamma \in P\Gamma_j^{\mathbb{R}}$, by lemma 4.2.

5. THE DISCRIMINANT IN THE REAL MODULI SPACE

Theorem 2.3 identifies the moduli space $\mathcal{F}_0^{\mathbb{R}}/G^{\mathbb{R}}$ of smooth framed real cubics with the incomplete hyperbolic manifold K_0 , which is the disjoint union of the $H_\chi^4 - \mathcal{H}$. Here χ varies over the (projectivized) anti-involutions of Λ and \mathcal{H} is the locus in $\mathbb{C}H^4$ representing the singular cubic surfaces. For a concrete understanding of K_0 we need to understand how \mathcal{H} meets the various H^4 's. As explained in [2], \mathcal{H} is

the union of the orthogonal complements r^\perp of the norm 1 vectors r of Λ . Therefore we call these roots “discriminant roots” and their mirrors r^\perp “discriminant mirrors”.

If χ is an anti-involution of Λ , then one way H_χ^4 can meet r^\perp is if $\chi(r) = \pm r$; then $H_\chi^4 \cap r^\perp$ is a copy of H^3 . But a more complicated intersection can occur; to describe it we need the idea of a G_2 root system in Λ^\times . As in section 4, a root of Λ^\times means a norm 1 or 2 vector of Λ^\times , or a norm 3 or 6 vector of Λ^\times that is divisible in Λ by θ . By a G_2 root system in Λ^\times we mean a set of six roots of norm 2 and six roots of norm 6, all lying in a two-dimensional sublattice of Λ^\times . Such a set of vectors automatically forms a copy of what is commonly known as the G_2 root system. The reason these root systems are important is that each G_2 root system R in Λ^\times determines a copy of \mathcal{E}^2 in Λ , and hence two discriminant mirrors. The \mathcal{E}^2 is just $\Lambda \cap (\langle R \rangle \otimes_{\mathbb{Z}} \mathbb{C})$; to see this, introduce coordinates on the complex span of R , in which R consists of the permutations of $(1, -1, 0)$ and $\pm(2, -1, -1)$ in

$$\mathbb{C}^2 = \{(x, y, z) \in \mathbb{C}^3 : x + y + z = 0\},$$

with the usual metric. Since Λ contains $\frac{1}{\theta}$ times the norm 6 roots, it also contains

$$(5.1) \quad \begin{aligned} r_1 &= \frac{1}{\theta}(2, -1, -1) + \omega(1, -1, 0) = -\frac{1}{\theta}(\omega, \bar{\omega}, 1) \quad \text{and} \\ r_2 &= -\frac{1}{\theta}(2, -1, -1) + \bar{\omega}(1, -1, 0) = \frac{1}{\theta}(\bar{\omega}, \omega, 1). \end{aligned}$$

These have norm 1 and are orthogonal, so they span a copy of \mathcal{E}^2 . Observe also that χ exchanges the r_i , and that each of the discriminant mirrors meets H_χ^4 in the same H^2 , namely R^\perp .

The following lemma asserts that these are the only ways that H_χ^4 can meet \mathcal{H} . In terms of cubic surfaces, the first possibility parametrizes surfaces with a single real node, while the second parametrizes surfaces with a complex conjugate pair of nodes.

Lemma 5.1. *Suppose χ is an anti-involution of Λ and M is a discriminant mirror with $M \cap H_\chi^4 \neq \emptyset$. Then either*

- (i) $M \cap H_\chi^4$ is a copy of H^3 , in which case $M = r^\perp$ for a root r of Λ^\times of norm 1 or 3, or
- (ii) $M \cap H_\chi^4$ is a copy of H^2 , in which case $M = R^\perp$ for a G_2 root system R in Λ^\times .

Conversely, if r is a root of norm 1 or 3 in Λ^\times (resp. R is a G_2 root system in Λ^\times), then $H_\chi^4 \cap r^\perp$ (resp. $H_\chi^4 \cap R^\perp$) is the intersection of H_χ^4 with some discriminant mirror.

Proof. As a discriminant mirror, $M = r^\perp$ for some norm 1 vector r of Λ . Since $M \cap H_\chi^4 \neq \emptyset$, M contains points fixed by χ , so that $\chi(M)$ meets M , which is to say that r^\perp meets $\chi(r)^\perp$. By [2, lemma 7.28], either $r^\perp = \chi(r)^\perp$ or $r \perp \chi(r)$. In the first case, χ preserves the \mathcal{E} -span of r . As in the proof of lemma 4.4, Λ^\times contains a unit multiple of r or θr . Then conclusion (i) applies. In the second case, $r_1 = r$ and $r_2 = \chi(r)$ span a copy of \mathcal{E}^2 and Λ^\times contains the norm 2 roots $\alpha r_1 + \bar{\alpha} r_2$ and norm 6 roots $\alpha \theta r_1 + \bar{\alpha} \bar{\theta} r_2$, where α varies over the units of \mathcal{E} . These form a G_2 root system R in Λ^\times , and it is easy to see that

$$M \cap H_\chi^4 = M \cap \chi(M) \cap H_\chi^4 = R^\perp \cap H_\chi^4$$

is a copy of H^2 . Therefore conclusion (ii) applies.

The converse is easy: if r is a root of Λ^\times of norm 1 or 3 then we take the discriminant mirror to be r^\perp , and if R is a G_2 root system in Λ^\times then we take M to be either r_1^\perp or r_2^\perp for r_1 and r_2 as in (5.1). \square

Corollary 5.2. *For $j = 0, \dots, 4$, $H_j^4 \cap \mathcal{H}$ is the union of the orthogonal complements of the discriminant roots of Λ_j and the G_2 root systems in Λ_j . \square*

For our applications we need to re-state this result in terms of the Coxeter diagrams.

Lemma 5.3. *If $x \in C_j$ then $x \in \mathcal{H}$ if and only if either*

- (i) *x lies in r^\perp for r a simple root of W_j of norm 1 or 3, or*
- (ii) *x lies in $r^\perp \cap s^\perp$, where r and s are simple roots of W_j of norms 2 and 6, whose mirrors meet at angle $\pi/6$.*

Proof. If $x \in r^\perp$ for r a simple root of norm 1 then $x \in \mathcal{H}$ because \mathcal{H} is defined as the union of the orthogonal complements of the norm 1 vectors of Λ . If $x \in r^\perp$ for r a simple root of norm 3, then r/θ is a norm one vector of Λ and we have the same conclusion. If r and s are two roots as in (ii), then the biflections in them generate a dihedral group of order 12, and the images of r and s under this group form a G_2 root system R in Λ_j . Then $x \in \mathcal{H}$ by lemma 5.1.

To prove the converse, suppose $x \in C_j \cap \mathcal{H}$. By lemma 5.1, either $x \in r^\perp$ for a root r of Λ_j of norm 1 or 3, or else $x \in R^\perp$ for a G_2 root system R in Λ_j . We treat only the second case because the first is similar but simpler. We choose a set $\{r, s\}$ of simple roots for R , which necessarily have norms 2 and 6 and whose mirrors necessarily meet at angle $\pi/6$. Then r^\perp and s^\perp are two of the walls for some Weyl chamber C' of W_j . This uses the fact that no two distinct mirrors of W_j can meet, yet make an angle less than $\pi/6$. (If there were such a pair of mirrors then there would be such a pair among the simple roots

of W_j .) We apply the element of W_j carrying C' to C_j ; since C_j is a fundamental domain for W_j in the strong sense, this transformation fixes x . Then the images of r and s are simple roots of W_j and the facets of C_j they define both contain x . \square

We remark that all the triple bonds in figure 4.2 come from G_2 root systems, so the condition on the norms of r and s in part (ii) of the lemma may be dropped. This leads to our final description of the moduli space of smooth real cubic surfaces:

Theorem 5.4. *The moduli space $\mathcal{M}_0^{\mathbb{R}}$ falls into five components $\mathcal{M}_{0,j}^{\mathbb{R}}$, $j = 0, \dots, 4$. As a real analytic orbifold, $\mathcal{M}_{0,j}^{\mathbb{R}}$ is isomorphic to an open suborbifold of $PI_j^{\mathbb{R}} \setminus H_j^4$, namely the open subset obtained by deleting the images in $PI_j^{\mathbb{R}} \setminus H_4^j$ of the faces of C_j corresponding to the blackened nodes \bullet and triple bonds $\circ \equiv \equiv \equiv \circ$ of figure 1.1. \square*

The two kinds of walls of the C_j play such different roles that we will use the following language. In light of the theorem, a wall corresponding to a blackened node in figure 1.1 will be called a discriminant wall. The other walls will be called Eckardt walls, because the corresponding cubic surfaces are exactly those that have Eckardt points. (An Eckardt point is a point through which three lines of the surface pass. This property will play no role in this paper; we use it only as a convenient name for these walls.)

6. TOPOLOGY OF THE MODULI SPACE OF SMOOTH SURFACES

This section and the next two are applications of the theory developed so far. The theoretical development continues in section 9.

The description of $\mathcal{M}_0^{\mathbb{R}}$ in theorem 5.4 is so explicit that many facts about real cubic surfaces and their moduli can be read off the diagrams. In this section we give presentations of the orbifold fundamental groups $\pi_1^{\text{orb}}(\mathcal{M}_{0,j}^{\mathbb{R}})$ of the components of $\mathcal{M}_0^{\mathbb{R}}$ and prove that the $\mathcal{M}_{0,j}^{\mathbb{R}}$ have contractible (orbifold) universal covers.

Theorem 6.1. *The orbifold fundamental groups of the components of $\mathcal{M}_0^{\mathbb{R}}$ are:*

$$\begin{aligned} \pi_1^{\text{orb}}(\mathcal{M}_{0,0}^{\mathbb{R}}) &\cong S_5 \\ \pi_1^{\text{orb}}(\mathcal{M}_{0,1}^{\mathbb{R}}) &\cong (S_3 \times S_3) \rtimes \mathbb{Z}/2 \\ \pi_1^{\text{orb}}(\mathcal{M}_{0,2}^{\mathbb{R}}) &\cong (D_\infty \times D_\infty) \rtimes \mathbb{Z}/2 \\ \pi_1^{\text{orb}}(\mathcal{M}_{0,3}^{\mathbb{R}}) &\cong \pi_1^{\text{orb}}(\mathcal{M}_{0,4}^{\mathbb{R}}) \cong \circ \overset{\infty}{\text{---}} \circ \text{---} \circ \text{---} \circ \end{aligned}$$

where the $\mathbb{Z}/2$ in each semidirect product exchanges the displayed factors of the normal subgroup.

Here S_n is the symmetric group, D_∞ is the infinite dihedral group, and the last group is a Coxeter group with the given diagram. We have labeled the leftmost bond “ ∞ ”, indicating the absence of a relation between two generators, rather than a strong or weak bond, because we are describing the fundamental group as an abstract group, not as a concrete reflection group. We remark that $\pi_1^{\text{orb}}(\mathcal{M}_{0,2}^{\mathbb{R}})$ is isomorphic to the Coxeter group of the Euclidean $(2, 4, 4)$ triangle.

Proof of theorem 6.1. The general theory of Coxeter groups (see for example [18]) allows us to write down a presentation for W_j . The standard generators for W_j are the reflections across the facets of C_j . Two of these reflections ρ and ρ' satisfy $(\rho\rho')^n = 1$ for $n = 2$ (resp. 3, 4, or 6) if the corresponding nodes are joined by no bond (resp. a single bond, double bond, or triple bond). These relations and the relations that the generators are involutions suffice to define W_j .

We get a presentation of $\pi_1^{\text{orb}}(W_j \setminus (H_j^4 - \mathcal{H}))$ from the presentation of W_j by omitting some of the generators and relations. Since the generators of W_j correspond to the walls of C_j , and removing \mathcal{H} from C_j removes the discriminant walls, we leave out those generators. Since removing these walls also removes all the codimension two faces which are their intersections with other walls, we also leave out all the relations involving the omitted generators. Finally, we leave out the relations coming from triple bonds, because removing \mathcal{H} from C_j removes the codimension two faces corresponding to these bonds. For $j = 0, 4$ or 5 , $PT_j^{\mathbb{R}} = W_j$ and we can read off $\pi_1^{\text{orb}}(\mathcal{M}_{0,j}^{\mathbb{R}})$ from the diagram, with the results given in the statement of the theorem. For $j = 1$ or 2 the same computation shows that $\pi_1^{\text{orb}}(W_j \setminus (H_j^4 - \mathcal{H}))$ is $S_3 \times S_3$ or $D_\infty \times D_\infty$. To describe $\pi_1^{\text{orb}}(\mathcal{M}_{0,j}^{\mathbb{R}})$ one must take the semidirect product by the diagram automorphism. This action can also be read from figure 1.1. \square

Theorem 6.2. *The $\mathcal{M}_{0,j}^{\mathbb{R}}$ are aspherical orbifolds, in the sense that their orbifold universal covers are contractible manifolds.*

Proof. We write D_j for the component of $H_j^4 - \mathcal{H}$ containing $C_j - \mathcal{H}$, and think of $\mathcal{M}_{0,j}^{\mathbb{R}}$ as

$$(\text{the stabilizer of } D_j \text{ in } PT_j^{\mathbb{R}}) \setminus D_j .$$

Since D_j is an orbifold cover of $\mathcal{M}_{0,j}^{\mathbb{R}}$, it suffices to show that D_j is aspherical. One way to understand D_j is as the union of the translates of $C_j - \mathcal{H}$ under the group T_j generated by the reflections across the

Eckardt walls of C_j . (Recall that the Eckardt walls are those corresponding to white nodes in figure 1.1, while the black ones represent discriminant walls.) Alternately, T_j is the stabilizer of D_j in W_j . Now we look at the D_j individually. T_0 is the finite group S_5 , and the four Eckardt walls are the walls containing a vertex P of C_0 . (Vertices in H^n of an n -dimensional Coxeter polyhedron correspond bijectively to n -node subdiagrams of the Coxeter diagram which generate finite Coxeter groups.) Therefore D_0 is the interior of a finite-volume polyhedron centered at P , so D_0 is even contractible. The same argument works for $j = 1$, with $S_3 \times S_3$ in place of S_5 . The case $j = 2$ is more complicated, even though T_2 is still finite—it is the direct product of two copies of the Coxeter group G_2 —and the Eckardt walls are still the walls meeting at a vertex P of C_2 . The complication is that the fixed-point set of each G_2 factor lies in \mathcal{H} . The result is that D_2 is the interior of a finite-volume polyhedron centered at P , minus its intersection with two mutually orthogonal H^2 's that meet transversely at P . Therefore D_2 is homeomorphic to a product of two punctured open disks, so it is aspherical.

Now we will treat $j = 3$; the case $j = 4$ is just the same. What is new is that T_3 is infinite. However, one of the discriminant walls (the lower of the rightmost two in figure 1.1) is orthogonal to all of the Eckardt walls. Therefore T_3 preserves the hyperplane H containing this discriminant wall. Furthermore, T_3 is the Coxeter group

(6.1)

which is a nonuniform lattice in $\text{PO}(3, 1)$, acting on H in the natural way. In particular, H is a component of the boundary of D_3 , and every T_3 -translate of C_3 has one of its facets lying in H . Finally, H is orthogonal to the codimension two face of C_3 associated to the triple bond in figure 1.1, and therefore orthogonal to all of its T_3 -translates. We summarize: D_3 is the interior of an infinite-volume convex polyhedron in H^4 , minus the union of a family of H^2 's, each orthogonal to the distinguished facet H . Therefore D_3 is homeomorphic to the product of an open interval with $H - Z$, where Z is the intersection of H with the union of these H^2 's.

$H - Z$ can be understood in terms of T_3 's action on it. A fundamental domain for T_3 is a simplex with shape described in (6.1), and the edge corresponding to the triple bond lies in Z . Indeed, Z is the union of the T_3 -translates of this edge. Direct visualization in H^3 shows that Z is the union of countably many disjoint geodesics. Therefore $H^3 - Z$ is aspherical, for example by stratified Morse theory; see Theorem 10.8 of [15]. \square

7. RELATION WITH THE WORK OF SEGRE

Classical knowledge about the topology of each connected component of the space of real smooth cubic forms was restricted to the monodromy representation of the fundamental group of each component in the Weyl group of E_6 , the group of symmetries of the configuration of lines (real and complex) of the surface. The monodromy groups M_0, \dots, M_4 , one for each component of $PC_0^{\mathbb{R}}$, were computed by Segre in [24]. Our methods readily compute the fundamental groups of the components, which are almost the same as the groups $\pi_1^{\text{orb}}(\mathcal{M}_{0,j}^{\mathbb{R}})$ computed in the last section, and give algorithmic computations of the monodromy representations. In particular we show that four of Segre's computations are correct, and correct an error in his remaining computation.

Lemma 7.1. *For each $j = 0, \dots, 4$, there is an exact sequence*

$$(7.1) \quad 1 \rightarrow \mathbb{Z}/2 \rightarrow \pi_1(\mathcal{C}_{0,j}^{\mathbb{R}}) \rightarrow \pi_1^{\text{orb}}(\mathcal{M}_{0,j}^{\mathbb{R}}) \rightarrow \mathbb{Z}/2 \rightarrow 1.$$

Here, the image of the middle map is the orientation-preserving subgroup of $\pi_1^{\text{orb}}(\mathcal{M}_{0,j}^{\mathbb{R}})$ and the kernel is $\pi_1(G^{\mathbb{R}})$. \square

The proof has two ingredients. One is the exact homotopy sequence of the fibration $G^{\mathbb{R}} \rightarrow \mathcal{F}_j \rightarrow D_j$, where D_j is a component of $H_j^4 - \mathcal{H}$, as in the proof of theorem 6.2, and \mathcal{F}_j is its $g^{\mathbb{R}}$ -preimage in $\mathcal{F}_0^{\mathbb{R}}$. The other ingredient is the interaction of this sequence with the T_j -action on \mathcal{F}_j and D_j , where T_j is the stabilizer of D_j , again as in the proof of theorem 6.2. We omit the details. We also remark that it would be more classical to consider $\pi_1(PC_{0,j}^{\mathbb{R}})$ instead. This would change the $\mathbb{Z}/2 = \pi_1(G^{\mathbb{R}})$ on the left into $(\mathbb{Z}/2)^2 = \pi_1(PG^{\mathbb{R}})$, but not affect our other considerations.

The monodromy representation $\pi_1(\mathcal{C}_{0,j}^{\mathbb{R}}) \rightarrow W$, where W denotes the Weyl group of E_6 , is now quite easy to compute. Recall that W is the same as $PO(V)$, where $V = V(S) = V(T)$ is the \mathbb{F}_3 -vector space defined defined in section 3 (note the that quadratic forms on $V(S)$ and $V(T)$ differ by sign, but this does not change the orthogonal group). Since the representation $\pi_1^{\text{orb}}(\mathcal{M}_0) \rightarrow W$ is just reduction modulo θ of the monodromy representation $\pi_1^{\text{orb}}(\mathcal{M}_0) \rightarrow P\Gamma$, the following procedure computes the representations $\pi_1^{\text{orb}}(\mathcal{M}_{0,j}^{\mathbb{R}}) \rightarrow W$. Take reflections in the norm 2 and 6 roots from the diagrams in Figure 4.2 and reduce their actions modulo θ . This can be encoded in diagrams as follows: Whenever the root vector is primitive, reduce it modulo θ and then take the corresponding reflection in the \mathbb{F}_3 -vector space. If the root vector is not primitive, first divide it by θ to get a primitive vector, and then

proceed as before. To compute the representations of $\pi_1(\mathcal{C}_{0,j})$, restrict to the orientation-preserving subgroup of $\pi_1^{\text{orb}}(\mathcal{M}_{0,j}^{\mathbb{R}})$. This proves the following theorem.

Theorem 7.2. *Let M_j denote the image of the monodromy representation $\pi_1(\mathcal{C}_{0,j}) \rightarrow W(E_6)$. Then*

$$\begin{aligned} M_0 &= A_5 \\ M_1 &= S_3 \times S_3 \\ M_2 &= (\mathbb{Z}/2)^3 \rtimes \mathbb{Z}/2 \\ M_3 &= M_4 = S_4 \end{aligned}$$

In M_2 , $\mathbb{Z}/2$ acts on $(\mathbb{Z}/2)^3$ by $(a, b, c) \rightarrow (c, a + b + c, a)$.

Caution. Note that $\pi_1^{\text{orb}}(\mathcal{M}_{0,1}^{\mathbb{R}}) \cong S_3 \times S_3 : \mathbb{Z}/2$ has two subgroups isomorphic to $S_3 \times S_3$. The one which is the image of $\pi_1(\mathcal{C}_{0,1}^{\mathbb{R}})$, here manifesting as M_1 , is not the obvious one but the other one.

Remark. In the two cases where $\pi_1(\mathcal{C}_{0,j}^{\mathbb{R}})$ is finite, namely $j = 0$ or 1 , its representation in W is almost faithful. The kernel is precisely the central $\mathbb{Z}/2 = \pi_1(G^{\mathbb{R}})$.

The groups we denote M_0, \dots, M_4 are the ones denoted $\Gamma_1, \dots, \Gamma_5$ by Segre in §34 of [24], and computed in §35 to §54. Segre not only computes each group as an abstract group, but gives a very detailed description of how it acts on various configurations of lines and tritangent planes on a surface in the appropriate component. This more detailed information can also be obtained from the diagrams of figure 4.2 by reduction modulo θ and the dictionary between the geometry of the \mathbb{F}_3 -vector space V and the configuration of 27 lines provided on p. 26 of [10]. We look at this more detailed information only in the case of $M_2 = \Gamma_3$ because this is the only case in which our answer for the abstract group disagrees with Segre’s answer. He gives it as $\mathbb{Z}/2 \times \mathbb{Z}/2$ at the end of §46 (page 72), while we get a non-abelian group of order 16.

In fact, a careful reading of §46 of [24] shows that he discusses actions of subgroups of M_2 on two distinct sets, but does not seem to discuss the whole group. It is not clear how he reaches his conclusion that $M_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.

We now sketch how to get the description for the action of this group on the lines and compare our statements with Segre’s. Let (V, q) denote the quadratic \mathbb{F}_3 -vector space \mathbb{F}_3^5 with $q = -x_0^2 + x_1^2 + \dots + x_4^2$, which is the $\text{mod}(\theta)$ reduction of (Λ, h) . The tritangent planes of S correspond to the “plus points” of $P(V)$, see p. 26 of [10]. With our choice of q , these are the lines $\langle v \rangle$ in V with $q(v) = -1$; see p. xii of [10]. A line of S corresponds to a “base”, which, with our choice of q , means

a collection of five mutually orthogonal lines in V , each spanned by a vector v with $q(v) = -1$. The fact that a base contains five plus-points corresponds to the fact that each line on S is contained in 5 tritangent planes to S . One checks that each plus-point is contained in exactly 3 bases, corresponding to the fact that each tritangent plane contains 3 lines of S .

The anti-involution χ_2 that defines the component $\mathcal{C}_{0,2}$ acts on V by

$$\chi_2(x_0, \dots, x_4) = (x_0, x_1, x_2, -x_3, -x_4).$$

The real tritangent planes correspond to the lines in V spanned by vectors v with $q(v) = -1$ and $\chi_2(v) = \pm v$. These are the lines spanned by the vectors

$$\begin{aligned} v_0 &= (0, 1, 1, 0, 0) = a_0 = \alpha \\ v_1 &= (1, 0, 0, 0, 0) = r_5 = a_{22} = \delta \\ v_2 &= (0, 1, -1, 0, 0) = r_1 = a_{21} = \delta_1 \\ v_3 &= (0, 0, 0, 1, -1) = r_3 = a_{11} = \beta \\ v_4 &= (0, 0, 0, 1, 1) = r_7 = a_{12} = \gamma, \end{aligned}$$

where r_1, r_3, r_5, r_7 is the notation of figure 4.2, the notation a_0, a_{ij} will be used shortly, and the notation $\alpha, \delta, \delta_1, \beta, \gamma$ is the one used by Segre. Note that v_1, v_2, v_3, v_4 are the $\text{mod}(\theta)$ reductions of primitive vectors proportional to r_5, r_1, r_3, r_7 respectively. Thus a surface in the component $\mathcal{C}_{0,2}$ has 5 real tritangent planes, all containing the line $\{v_0, \dots, v_4\}$. This is the hyperbolic line r of the second kind of §31(iii) of [24].

One can write down the bases corresponding to the lines contained in each real tritangent plane, and check that the two other lines contained in each of v_0, v_1, v_2 are real, while the ones in each of v_3, v_4 are complex conjugate. In the notation a_0, a_{ij} for v_0, \dots, v_4 , let A_{ij} denote the reflection in the root vector a_{ij} and let S denote the diagram automorphism, which by definition satisfies $S(a_{i2}) = a_{i1}$, and is easily checked to satisfy $S(a_0) = a_0$. Then $M_2 \subset PO(V, q)$ is a subgroup of index two of the group M'_2 generated by the involution S and the commuting reflections A_{ij} , where we have the further relations $SA_{11} = A_{12}S$ and $SA_{21} = A_{22}S$; M_2 is the subgroup of M'_2 where the number of A_{ij} is even. Let l denote the line common to the 5 real tritangent planes, let l_0, l'_0 denote the remaining lines in a_0 , let l_{ij}, l'_{ij} denote the remaining lines in a_{ij} , and choose the labeling so that S takes l_{i2} to l_{i1} . Then a mechanical verification shows that A_{ij} interchanges l_0 and l'_0 , it preserves l_{ij} (and hence l'_{ij}) and for $(m, n) \neq (i, j)$, A_{ij} interchanges l_{mn} and l'_{mn} . Finally, one checks that S preserves l_0 and l'_0 . This gives the

action of M'_2 on this configuration of lines, hence, by restriction, that of M_2 . In summary, we have the following theorem:

Theorem 7.3. *The group M_2 leaves the lines l, l_0, l'_0 invariant and permutes the lines l_{ij}, l'_{ij} . It consists of all even permutations of this collection of eight lines that (i) preserve each pair $\{l_{11}, l'_{11}\}, \{l_{12}, l'_{12}\}$ or exchange these pairs, and (ii) preserve each pair $\{l_{21}, l'_{21}\}, \{l_{22}, l'_{22}\}$ or exchange these pairs. \square*

We remark that the subgroup $\langle A_{11}A_{12}, A_{21}A_{22}, S \rangle \cong (\mathbb{Z}/2)^3$ acts on the set of l_{1j}, l'_{1j} of complex conjugate lines in the manner described by Segre in the body of §46 of [24], and acts on the set of real lines l_{2j}, l'_{2j} in the manner described in the final statement of §46. This $(\mathbb{Z}/2)^3$ is the subgroup of M_2 that permutes evenly each of the sets of real and complex lines, but M_2 contains even permutations of the whole set of lines that restrict to odd permutations of each of these two subsets, for instance $A_{11}A_{22}$, a possibility that seems to be implicitly excluded in [24].

8. VOLUMES

In this section we compute the volume of each $P\Gamma_j^{\mathbb{R}} \backslash H_j^4$ by computing its orbifold Euler characteristic and using the general relation

$$\text{vol}(M) = \frac{\text{vol}(S^n)}{\chi(S^n)} |\chi(M)| = \frac{2^n \pi^{n/2} (n/2)!}{n!} |\chi(M)|$$

for a hyperbolic orbifold M with $n = \dim M$ even. For the Euler characteristic, consider the subgroup W_j generated by reflections, and its fundamental polyhedron C_j , described by the its Coxeter diagram in figure 1.1. W_j has index δ in $P\Gamma_j^{\mathbb{R}}$ with $\delta = 1$ or 2 . The latter case occurs when the diagram has an automorphism of order two. Consider therefore the orbifold $W_j \backslash H_j^4$. Since

$$\delta \cdot \chi(P\Gamma_j^{\mathbb{R}} \backslash H_j^4) = \chi(W_j \backslash H_j^4)$$

it suffices to compute the right-hand side. To this end, consider a face F of C_j and its stabilizer $P\Gamma_j^{\mathbb{R}}(F)$ in $P\Gamma_j^{\mathbb{R}}$. If Φ stands for the set of proper faces of C_j , then

$$\chi(W_j \backslash H_j^4) = 1 + \sum_{F \in \Phi} \frac{(-1)^{\dim F} \chi(F)}{|P\Gamma_j^{\mathbb{R}}(F)|} = 1 + \sum_{F \in \Phi} \frac{(-1)^{\dim F}}{|P\Gamma_j^{\mathbb{R}}(F)|}.$$

Let Δ be a Coxeter diagram, let $\Sigma(\Delta)$ be the set of nonempty sub-diagrams describing finite Coxeter groups, and for E in Σ , let $|E|$ be the number of its nodes and $W(E)$ be the associated Coxeter group.

The face of C_j corresponding to E has codimension $|E|$ in an even-dimensional space, so the previous relation can be written as

$$\chi(W_j \setminus H_j^4) = 1 + \sum_{E \in \Sigma} \frac{(-1)^{|E|}}{|W(E)|}.$$

For the Coxeter diagrams that occur in this paper, the enumeration of subdiagrams is lengthy but easy. We did the computations by hand and then checked them with a computer. Consider, for instance, the case of W_0 . Every proper subdiagram except the one got by omitting the rightmost node describes a finite Coxeter group; for example, the other four-node subdiagrams (which describe vertices of C_0) have types B_4 , $A_1^2 \times A_2$, $A_1 \times B_3$ and A_4 . The resulting contribution to the Euler characteristic is

$$(-1)^4 \left(\frac{1}{2^4 \cdot 4!} + \frac{1}{2^2 \cdot 3!} + \frac{1}{2^4 \cdot 3!} + \frac{1}{5!} \right) = \frac{121}{1920}.$$

Carrying out the full enumeration and computing the orders of the corresponding Weyl groups, one finds that

$$\chi(P\Gamma_0^{\mathbb{R}} \setminus H^4) = 1 - \frac{5}{2} + \frac{17}{8} - \frac{11}{16} + \frac{121}{1920} = \frac{1}{1920}.$$

This gives the first entry in table 1.2. The other calculations are similar.

9. MODULI OF STABLE REAL CUBIC SURFACES

The goal of this section is to understand the moduli space $\mathcal{M}_s^{\mathbb{R}}$ of stable real cubic surfaces as a quotient of H^4 . To define this space we begin with the space \mathcal{C}_s of cubic forms defining surfaces in $\mathbb{C}P^3$ which are stable in the sense of geometric invariant theory. It is classical that a cubic surface is stable if and only if its singularities are all ordinary double points (nodes), and that the most nodes any cubic surface can have is four. Since G acts properly on the open set $\mathcal{C}_s \subseteq \mathcal{C}$, the quotient $\mathcal{M}_s = \mathcal{C}_s/G$ is a complex analytic space and a complex analytic orbifold. The main result of [2] is the construction of an isomorphism $g : \mathcal{M}_s \rightarrow P\Gamma \setminus \mathbb{C}H^4$ in the category of analytic spaces. This is not an orbifold isomorphism, but in [2, §3.19] we described how to modify the orbifold structure on $P\Gamma \setminus \mathbb{C}H^4$ so that g becomes one. This issue, which is absent in the smooth case, causes considerable complication in the treatment of real stable cubics.

Mimicking the construction in the smooth case, we take $\mathcal{C}_s^{\mathbb{R}} = \mathcal{C}_s \cap \mathcal{C}^{\mathbb{R}}$ and define the moduli space $\mathcal{M}_s^{\mathbb{R}}$ as the real analytic orbifold $\mathcal{C}_s^{\mathbb{R}}/G^{\mathbb{R}}$. In this section we will find a real-hyperbolic orbifold structure on $\mathcal{M}_s^{\mathbb{R}}$ by identifying it with $P\Gamma^{\mathbb{R}} \setminus H^4$ for a suitable lattice $P\Gamma^{\mathbb{R}}$ in $\mathrm{PO}(4, 1)$. It will be obvious that this structure agrees with the moduli-space

orbifold structure on $\mathcal{M}_0^{\mathbb{R}}$. (Less obvious is that the two structures differ on $\mathcal{M}_s^{\mathbb{R}} - \mathcal{M}_0^{\mathbb{R}}$. But they do define the same topological orbifold structure, except along the locus of real surfaces having a conjugate pair of nodes. There, even the topological orbifold structures differ. We omit further discussion of this issue.)

It is possible (and perhaps easier) to skip the theory below and “construct” $P\Gamma^{\mathbb{R}}$ by gluing together the 5 orbifolds $P\Gamma_j^{\mathbb{R}} \setminus H_j^4$ along their discriminant walls. There is an essentially unique way to do this that makes sense (the one in section 10), and one “obtains” the orbifold $P\Gamma^{\mathbb{R}} \setminus H^4$. This is what we did at first, and it led to the problem of how this construction relates to $\mathcal{M}_s^{\mathbb{R}}$. The essential content of this section is to give an intrinsic definition of the hyperbolic structure. Then section 10 plays the role of computing a known-to-exist orbifold structure, rather than constructing it.

In [2] we related \mathcal{M}_s and $\mathbb{C}H^4$ via the space \mathcal{F}_s of framed stable cubic forms, which is the Fox completion (or normalization) of \mathcal{F}_0 over \mathcal{C}_s . \mathcal{F}_s is a branched cover, with ramification over Δ which can be described explicitly (see below). The actions of $P\Gamma$ and G on \mathcal{F}_0 extend to \mathcal{F}_s , by [2, §3.9].

We define $\mathcal{F}_s^{\mathbb{R}}$ as the preimage of $\mathcal{C}_s^{\mathbb{R}}$ in \mathcal{F}_s . We will see that it is not a manifold, because of the ramification, but is a union of embedded submanifolds. We define K to be $\mathcal{F}_s^{\mathbb{R}}/G^{\mathbb{R}}$, which is not a manifold either. At this point it is merely a topological space; below, we will equip it with a metric structure. Essentially by definition, $\mathcal{M}_s^{\mathbb{R}}$ coincides with $P\Gamma \setminus K$. If K were a manifold then this would define an orbifold structure on $\mathcal{M}_s^{\mathbb{R}}$. But it is not, so we must take a different approach. First we will give a local description of $\mathcal{F}_s^{\mathbb{R}} \subseteq \mathcal{F}_s$, and then show that $g : \mathcal{F}_s \rightarrow \mathbb{C}H^4$ induces a local embedding $K \rightarrow \mathbb{C}H^4$. This makes K into a metric space, using the path metric obtained by pulling back the metric on $\mathbb{C}H^4$. Finally, we will study the action of $P\Gamma$ on K to deduce that $P\Gamma \setminus K$, as a metric space, is locally modeled on quotients of H^4 by finite groups. Such a metric space has a unique hyperbolic orbifold structure. The completeness of this structure on $\mathcal{M}_s^{\mathbb{R}}$ then follows from the completeness of $P\Gamma \setminus \mathbb{C}H^4$, and orbifold uniformization then implies the existence of a discrete group $P\Gamma^{\mathbb{R}}$ acting on H^4 with $\mathcal{M}_s^{\mathbb{R}} \cong P\Gamma^{\mathbb{R}} \setminus H^4$. See section 10 for a concrete description of $P\Gamma^{\mathbb{R}}$ and section 11 for a proof that it is not arithmetic.

We begin with a local description of \mathcal{F}_s . Let $f \in \mathcal{F}_s$ lie over $F \in \mathcal{C}_s$, let k be the number of nodes of the cubic surface S , and let $x = g(f) \in \mathbb{C}H^4$. By [2, §3.10 and §3.17], there exist coordinates (t_1, \dots, t_4) identifying $\mathbb{C}H^4$ with the unit ball B^4 , and x with the origin, such that (1) the components of \mathcal{H} passing through x are the hyperplanes

$t_1 = 0, \dots, t_k = 0$; (2) the stabilizer $P\Gamma_f$ of f is $(\mathbb{Z}/6)^k$, acting on $\mathbb{C}H^4$ by multiplying t_1, \dots, t_k by sixth roots of unity; (3) the pullbacks of the globally-defined coordinate functions t_1, \dots, t_4 to \mathcal{F}_s may be extended to a system of local coordinates (t_1, \dots, t_{20}) around f ; (4) the functions $u_1 = t_1^6, \dots, u_k = t_k^6, u_{k+1} = t_{k+1}, \dots, u_{20} = t_{20}$ give local coordinates around $F \in \mathcal{C}_s$; and (5) the discriminant $\Delta \subseteq \mathcal{C}_s$ near F is the union of the hypersurfaces $u_1 = 0, \dots, u_k = 0$. A consequence of (3) is that the period map $g : \mathcal{F}_s \rightarrow \mathbb{C}H^4$ is given near f by forgetting t_5, \dots, t_{20} .

Now suppose $f \in \mathcal{F}_s^{\mathbb{R}}$, and that $2a$ of the nodes are non-real and b are real. Because the components of Δ at $F \in \mathcal{C}_s$ correspond to the nodes of S , complex conjugation κ permutes them in the same way as it permutes the nodes. Therefore we may assume that it acts by

$$(9.1) \quad u_i \mapsto \begin{cases} \bar{u}_{i+1} & \text{for } i \text{ odd and } i \leq 2a \\ \bar{u}_{i-1} & \text{for } i \text{ even and } i \leq 2a \\ \bar{u}_i & \text{for } i > 2a. \end{cases}$$

In these local coordinates, $\mathcal{C}_s^{\mathbb{R}}$ is the fixed-point set of κ . To describe $\mathcal{F}_s^{\mathbb{R}}$ near f , we simply compute the preimage of $\mathcal{C}_s^{\mathbb{R}}$. The most important cases are first, a single real node ($a = 0, b = 1$), and second, a single pair of conjugate nodes ($a = 1, b = 0$).

In the case of a single real node, $\mathcal{F}_s^{\mathbb{R}}$ near f is modeled on a neighborhood of the origin in

$$(9.2) \quad \{(t_1, \dots, t_{20}) \in \mathbb{C}^{20} : t_1^6, t_2, \dots, t_{20} \in \mathbb{R}\}.$$

That is, a neighborhood of f is modeled on six copies of \mathbb{R}^{20} , glued together along a common \mathbb{R}^{19} .

In the case of two complex conjugate nodes, $\mathcal{F}_s^{\mathbb{R}}$ near f is modeled on a neighborhood of the origin in

$$(9.3) \quad \{(t_1, \dots, t_{20}) \in \mathbb{C}^{20} : t_2^6 = \bar{t}_1^6 \text{ and } t_3, \dots, t_{20} \in \mathbb{R}\}.$$

That is, on the union of six copies of \mathbb{R}^{20} , glued together along a common \mathbb{R}^{18} . The \mathbb{R}^{18} is given by $t_1 = t_2 = 0$ and maps diffeomorphically to $\Delta \cap \mathcal{C}_s^{\mathbb{R}}$, and each component of the complement is a six-fold cover of the part of $\mathcal{C}_s^{\mathbb{R}} - \Delta$ near F .

Recall from the proof of theorem 2.3 that $P\Gamma'$ is the group of all isometries of Λ which are either linear or anti-linear, modulo the scalars. At the end of that proof, we defined an action of $P\Gamma'$ on \mathcal{F}_0 , such that the anti-involutions $\chi \in P\mathcal{A} \subseteq P\Gamma'$ are the maps $\mathcal{F}_0 \rightarrow \mathcal{F}_0$ that have order 2 and cover $\kappa : \mathcal{C}_0 \rightarrow \mathcal{C}_0$. The $P\Gamma'$ -action on \mathcal{F}_0 extends to one on \mathcal{F}_s . One can choose a small neighborhood U of f in \mathcal{F}_s which is preserved by every $\chi \in P\mathcal{A}$ carrying f to itself, and write down these

χ in terms of our local coordinates. For example, in the one-real-node case there are 6 lifts of κ , given by

$$(t_1, \dots, t_{20}) \mapsto (\bar{t}_1 \zeta^i, \bar{t}_2, \dots, \bar{t}_{20}),$$

where $\zeta = e^{\pi i/3}$, and in the conjugate-pair case there are also 6 lifts, given by

$$(t_1, \dots, t_{20}) \mapsto (\bar{t}_2 \zeta^i, \bar{t}_1 \zeta^i, \bar{t}_3, \dots, \bar{t}_{20}).$$

We define \mathcal{F}_s^χ to be the fixed-point set of $\chi \in P\mathcal{A}$. As the fixed-point set of a real-analytic involution, \mathcal{F}_s^χ is a real-analytic manifold. Recall that $g : \mathcal{F}_s \rightarrow \mathbb{C}H^4$ is the complex period map. Lemma 9.2 below is the extension of the diffeomorphism $\mathcal{F}_0^\chi/G^{\mathbb{R}} \cong H_\chi^4 - \mathcal{H}$ of theorem 2.3 to $\mathcal{F}_s^\chi/G^{\mathbb{R}} \cong H_\chi^4$; to prove it we need the following general principle.

Lemma 9.1. *Let G be a Lie group acting properly and with finite stabilizers on a smooth manifold X , let F be a finite group of diffeomorphisms of X normalizing G , let X^F be its fixed-point set, and let G^F be its centralizer in G . Then the natural map $X^F/G^F \rightarrow X/G$ is proper.*

Proof. We write π and π^F for the maps $X \rightarrow X/G$ and $X^F \rightarrow X^F/G^F$, and f for the natural map $X^F/G^F \rightarrow X/G$. We prove the theorem under the additional hypothesis that F and G meet trivially; this is all we need and the proof in the general case is similar. This hypothesis implies that the group L generated by G and F is $G \rtimes F$. Begin by choosing a complete L -invariant Riemannian metric on X .

To prove f proper it suffices to exhibit for any G -orbit $\mathcal{O} \subseteq X$ a G -invariant neighborhood $V \subseteq X$ with $f^{-1}(\pi(V))$ precompact. Since G has finite index in L , $\mathcal{O}.L \subseteq X$ is the union of finitely many G -orbits. Using properness and Riemannian geometry one finds $\varepsilon > 0$ such that (1) distinct G -orbits in $\mathcal{O}.L$ lie at distance $> \varepsilon$, and (2) any point of X at distance $< \varepsilon$ from \mathcal{O} has a unique nearest point in \mathcal{O} . We take V to be the open $\varepsilon/2$ -neighborhood of \mathcal{O} .

To show that $f^{-1}(\pi(V))$ is precompact we will exhibit a compact set $K \subseteq X^F$ with $\pi^F(K)$ containing $f^{-1}(\pi(V))$. We claim that there are finitely many G^F -orbits in $\mathcal{O} \cap X^F$, so we can choose orbit representatives $\tilde{x}_1, \dots, \tilde{x}_n$. If $\mathcal{O} \cap X^F$ is empty then this is trivial. If $\mathcal{O} \cap X^F$ is nonempty, say containing \tilde{x} , then the G^F -orbits in $\mathcal{O} \cap X^F$ are in bijection with the conjugacy classes of splittings of

$$1 \rightarrow G_{\tilde{x}} \rightarrow G_{\tilde{x}} \rtimes F \rightarrow F \rightarrow 1,$$

where $G_{\tilde{x}}$ is the G -stabilizer of \tilde{x} . Since $G_{\tilde{x}}$ is finite, there are finitely many splittings, hence finitely many orbits. We take K to be the

union of the closed $\varepsilon/2$ -balls around $\tilde{x}_1, \dots, \tilde{x}_n$, intersected with X^F . (In particular, K is empty if $\mathcal{O} \cap X^F$ is.)

K is obviously compact, so all that remains is to prove $f^{-1}(\pi(V)) \subseteq \pi^F(K)$. If $f^{-1}(\pi(V))$ is empty then we are done. Otherwise, suppose $y \in f^{-1}(\pi(V)) \subseteq X^F/G^F$ and let $\tilde{y} \in X^F$ lie over it. Now, \tilde{y} is F -invariant and F permutes the G -orbits in $\mathcal{O}.L$. Since \tilde{y} lies within $\varepsilon/2$ of \mathcal{O} , it lies at distance $> \varepsilon/2$ of every other G -orbit in $\mathcal{O}.L$, so F preserves \mathcal{O} . Therefore F preserves the unique point \tilde{x} of \mathcal{O} closest to \tilde{y} , so $\tilde{x} \in X^F$. We choose $g \in G^F$ with $\tilde{x}.g$ equal to one of the \tilde{x}_i . Then $\tilde{y}.g$ lies within $\varepsilon/2$ of $\tilde{x}.g = \tilde{x}_i$, hence lies in K , and $\pi^F(\tilde{y}.g) = y$, proving $f^{-1}(\pi(V)) \subseteq \pi^F(K)$. \square

Lemma 9.2. *For every $\chi \in PA$, $g : \mathcal{F}_s^\chi/G^{\mathbb{R}} \rightarrow H_\chi^4$ is an isomorphism of real-analytic manifolds.*

Proof. It is a local diffeomorphism because its rank is everywhere 4, just as in theorem 2.3. Injectivity also follows from the argument used there. (This uses the freeness of G 's action on \mathcal{F}_s , not just on \mathcal{F}_0 ; see [2, Lemma 3.14].) To see surjectivity, we apply the previous lemma with $G = G$, $X = \mathcal{F}_s$, $F = \{1, \chi\}$, $X^F = \mathcal{F}_s^\chi$ and $G^F = G^{\mathbb{R}}$. Therefore the map $\mathcal{F}_s^\chi/G^{\mathbb{R}} \rightarrow \mathcal{F}_s/G = \mathbb{C}H^4$ is proper, so its image is closed. Theorem 2.3 tells us that the image contains the open dense subset $g(\mathcal{F}_0^\chi) = H_\chi^4 - \mathcal{H}$, so the map is surjective. \square

Lemma 9.3. *Suppose $f \in \mathcal{F}_s^{\mathbb{R}}$, and let $\alpha_1, \dots, \alpha_\ell$ be the elements of PA such that $f \in \mathcal{F}_s^{\alpha_i}$. Then the map*

$$(9.4) \quad (\cup_{i=1}^{\ell} \mathcal{F}_s^{\alpha_i})/G^{\mathbb{R}} \rightarrow \cup_{i=1}^{\ell} H_{\alpha_i}^4$$

induced by g is a homeomorphism.

The left side of (9.4) contains a neighborhood of the image of f in K , so the lemma implies that $g : K \rightarrow \mathbb{C}H^4$ is a local embedding. We will write K_f for the right side of (9.4).

Proof. We first claim that for all i and j ,

$$g : \mathcal{F}_s^{\alpha_i} \cap \mathcal{F}_s^{\alpha_j} \rightarrow H_{\alpha_i}^4 \cap H_{\alpha_j}^4$$

is surjective. To see this, let C be the component of $\mathcal{F}_s^{\alpha_i} \cap \mathcal{F}_s^{\alpha_j}$ containing f . This is a component of the fixed-point set of the finite group generated by α_i and α_j . In particular, it is a smooth manifold whose tangent spaces are all totally real. Since C is connected, its dimension everywhere is its dimension at f , which by our local coordinates is $16 + \dim_{\mathbb{R}} H_{\alpha_i}^4 \cap H_{\alpha_j}^4$. Since the tangent spaces are totally real and the kernel of the derivative of the period map has complex dimension 16,

the (real) rank of $g|_C$ equals $\dim_{\mathbb{R}} H_{\alpha_i}^4 \cap H_{\alpha_j}^4$ everywhere. Therefore $g(C)$ is open. It is also closed, since g induces a diffeomorphism from each $\mathcal{F}_s^{\alpha_i}/G^{\mathbb{R}}$ to $H_{\alpha_i}^4$ for each i . This proves surjectivity, since $H_{\alpha_i}^4 \cap H_{\alpha_j}^4$ is connected.

Now we prove the lemma itself; the map (9.4) is surjective and proper because $\mathcal{F}_s^{\alpha_i}/G^{\mathbb{R}} \rightarrow H_{\alpha_i}^4$ is surjective and proper for each i . To prove injectivity, suppose $a_i \in \mathcal{F}_s^{\alpha_i}/G^{\mathbb{R}}$ for $i = 1, 2$ have the same image in $\mathbb{C}H^4$. Then their common image lies in $H_{\alpha_1}^4 \cap H_{\alpha_2}^4$, so by the claim above there exists $b \in (\mathcal{F}_s^{\alpha_i} \cap \mathcal{F}_s^{\alpha_j})/G^{\mathbb{R}}$ with the same image. Since each $\mathcal{F}_s^{\alpha_i}/G^{\mathbb{R}} \rightarrow H_{\alpha_i}^4$ is injective, each a_i coincides with b , so $a_1 = a_2$. \square

At this point we know that g locally embeds $K = \mathcal{F}_s^{\mathbb{R}}/G^{\mathbb{R}}$ into $\mathbb{C}H^4$, and even have an identification of small open sets in K with open sets in unions of copies of H^4 in $\mathbb{C}H^4$. The induced path-metric on K is the largest metric which preserves the lengths of paths; under it, K is piecewise isometric to H^4 . K is not a manifold—it may be described locally by suppressing the coordinates t_5, \dots, t_{20} from our local description of $\mathcal{F}_s^{\mathbb{R}}$. Nevertheless, corollary 9.5 below shows us that the path metric on $\mathcal{M}_s^{\mathbb{R}} = P\Gamma \backslash K$ is locally isometric to quotients of H^4 by finite groups. This is the key idea of this section.

To prepare for this, let f and the α_i be as in lemma 9.3. Let A_f be the subgroup of $P\Gamma$ fixing the image of f in $\mathcal{F}_s^{\mathbb{R}}/G^{\mathbb{R}}$. Then the natural map

$$(9.5) \quad A_f \backslash K_f \cong A_f \backslash (\cup_{i=1}^{\ell} \mathcal{F}_s^{\alpha_i})/G^{\mathbb{R}} \rightarrow P\Gamma \backslash \mathcal{F}_s^{\mathbb{R}}/G^{\mathbb{R}} = \mathcal{M}_s^{\mathbb{R}}$$

is a homeomorphism in a neighborhood of the image of f in $A_f \backslash K_f$. So, to describe the metric near the image of f in $\mathcal{M}_s^{\mathbb{R}}$, it suffices to describe it near the image of f in $A_f \backslash K_f$. To get such a description we consider $B_f \backslash K_f$, where $B_f \subseteq A_f$ is the subgroup of $P\Gamma$ generated by the hexaflections associated to the *real* nodes of S . $P\Gamma \backslash K$, $A_f \backslash K_f$ and $B_f \backslash K_f$ are all metric spaces, equipped with their natural path metrics. The last of these spaces can be understood in coordinates. For the all-real-nodes case, only the first sentence of the next lemma is needed, and this case should allow the reader to understand the rest of this section.

Lemma 9.4. *If S has only real nodes, then $B_f \backslash K_f$ is isometric to H^4 . If S has a single pair of conjugate nodes, and possibly also some real nodes, then $B_f \backslash K_f$ is isometric to the union of six copies of H^4 identified along a common H^2 . If S has two pairs of conjugate nodes then $B_f \backslash K_f = K_f$ is the union of 36 copies of H^4 , any two of which meet along an H^2 or at a point. In each case, A_f acts transitively*

on the indicated H^4 's. If H is any one of them, and $(A_f/B_f)_H$ its stabilizer, then the natural map

$$(A_f/B_f)_H \backslash H \rightarrow (A_f/B_f) \backslash (B_f \backslash K_f) = A_f \backslash K_f$$

is an isometry.

Corollary 9.5. *Every point of $PG \backslash K$ has a neighborhood isometric to the quotient of an open set in H^4 by a finite group of isometries.* \square

Proof of lemma 9.4. We take $x = g(f)$ as before and refer to the coordinates t_1, \dots, t_4 that identify $\mathbb{C}H^4$ with B^4 . Recall that the cubic surface S has $2a$ non-real and b real nodes, with $k = 2a + b$. $PG_f \subseteq A_f$ acts on $\mathbb{C}H^4$ by multiplying t_1, \dots, t_k by 6th roots of unity, and B_f acts by multiplying $t_{2a+1}, \dots, t_{2a+b}$ by 6th roots of unity. K_f may be described in the manner used to obtain (9.2) and (9.3), with t_5, \dots, t_{20} omitted. With concrete descriptions of K_f and B_f in hand, one can work out $B_f \backslash K_f$. Here are the results for the various cases.

First suppose S has only real nodes. Then

$$K_f = \{(t_1, \dots, t_4) \in B^4 : t_1^6, \dots, t_k^6, t_{k+1}, \dots, t_4 \in \mathbb{R}\}.$$

Each of the 2^k subsets

$$K_{f, \varepsilon_1, \dots, \varepsilon_k} = \{(t_1, \dots, t_4) \in B^4 : i^{\varepsilon_1} t_1, \dots, i^{\varepsilon_k} t_k \in [0, \infty) \\ \text{and } t_{k+1}, \dots, t_4 \in \mathbb{R}\},$$

indexed by $\varepsilon_1, \dots, \varepsilon_k \in \{0, 1\}$, is isometric to the closed region in H^4 bounded by k mutually orthogonal hyperplanes. Their union U is a fundamental domain for B_f in the sense that it maps homeomorphically and piecewise-isometrically onto $B_f \backslash K_f$. Under its path metric, U is isometric to H^4 , say by the following map, defined separately on each $K_{f, \varepsilon_1, \dots, \varepsilon_k}$ by

$$(t_1, \dots, t_k) \mapsto (-i^{\varepsilon_1} t_1, \dots, -i^{\varepsilon_k} t_k, t_{k+1}, \dots, t_4).$$

This identifies $B_f \backslash K_f$ with the standard H^4 in $\mathbb{C}H^4$.

If S has a single pair of non-real nodes and no real nodes, then B_f is trivial and $B_f \backslash K_f = K_f$. The α_i are the 6 maps

$$\alpha_i : (t_1, \dots, t_4) \mapsto (\bar{t}_2 \zeta^i, \bar{t}_1 \zeta^i, \bar{t}_3, \bar{t}_4)$$

with $i \in \mathbb{Z}/6$, whose fixed-point sets are

$$H_{\alpha_i}^4 = \{(t_1, \dots, t_4) \in B^4 : t_2 = \bar{t}_1 \zeta^i \text{ and } t_3, t_4 \in \mathbb{R}\}.$$

It is obvious that any two of these H^4 's meet along the $H^2 \subseteq B^4$ described by $t_1 = t_2 = 0$ and $t_3, t_4 \in \mathbb{R}$.

If S has two pairs of non-real nodes (hence no real nodes at all) then the argument is essentially the same. The difference is that there are now 36 anti-involutions

$$\alpha_{m,n} : (t_1, \dots, t_4) \mapsto (\bar{t}_2\zeta^m, \bar{t}_1\zeta^m, \bar{t}_4\zeta^n, \bar{t}_3\zeta^n),$$

where $m, n \in \mathbb{Z}/6$, with fixed-point sets

$$H_{\alpha_{m,n}}^4 = \{(t_1, \dots, t_4) \in B^4 : t_2 = \bar{t}_1\zeta^m, t_4 = \bar{t}_3\zeta^n\}.$$

If $(m', n') \neq (m, n)$ then $H_{\alpha_{m,n}}^4$ meets $H_{\alpha_{m',n'}}^4$ in an H^2 if $m = m'$ or $n = n'$, and otherwise only at the origin.

If S has a pair of non-real nodes and also a single real node then the argument is a mix of the cases above. $B_f \cong \mathbb{Z}/6$ acts by multiplying t_3 by powers of ζ , and there are 36 anti-involutions, namely

$$\alpha_{m,n} : (t_1, \dots, t_4) \mapsto (\bar{t}_2\zeta^m, \bar{t}_1\zeta^m, \bar{t}_3\zeta^n, \bar{t}_4).$$

We have

$$K_f = \{(t_1, \dots, t_4) \in B^4 : t_2^6 = \bar{t}_1^6, t_3^6 \in \mathbb{R}, t_4 \in \mathbb{R}\}.$$

The union U of the subsets with t_3 or it_3 in $[0, \infty)$ is a fundamental domain for B_f ; applying the identity map to the first subset and $t_3 \mapsto -it_3$ to the second identifies U with

$$\{(t_1, \dots, t_4) \in B^4 : t_2^6 = \bar{t}_1^6, t_3, t_4 \in \mathbb{R}\}.$$

That is, $B_f \setminus K_f$ is what K_f was in the case of no real nodes, as claimed. If S has two non-real and two real nodes then the argument is only notationally more complicated.

The remaining claims are trivial unless there are non-real nodes. In every case, the transitivity of A_f on the H^4 's in $B_f \setminus K_f$ is easy to see because A_f contains transformations multiplying t_1, \dots, t_{2a} by powers of ζ . If H is one of the H^4 's and $J = (A_f/B_f)_H$ is its stabilizer, then it remains to prove that $J \setminus H \rightarrow A_f \setminus K_f$ is an isometry. Surjectivity follows from the transitivity of A_f on the H^4 's. It is obviously a piecewise isometry, so all we must prove is injectivity. That is, if two points of H are equivalent under A_f/B_f , then they are equivalent under J . To prove this it suffices to show that for all $y \in B_f \setminus K_f$, the stabilizer of y in A_f/B_f acts transitively on the H^4 's in $B_f \setminus K_f$ containing y . This is easy, using the stabilizer of y in $P\Gamma_f/B_f \cong (\mathbb{Z}/6)^{2a}$. \square

We have equipped $\mathcal{M}_s^{\mathbb{R}}$ with a path metric which is locally isometric to quotients of H^4 by finite groups. It is easy to see that if X is such a metric space then there is a unique real-hyperbolic orbifold structure on X whose path metric is the given one. (The essential point is that if V and V' are open subsets of H^4 and Γ and Γ' are finite groups of

isometries of H^4 preserving V and V' respectively, with V/Γ isometric to V'/Γ' , then there is an isometry of H^4 carrying V to V' and Γ to Γ' .) Therefore $\mathcal{M}_s^{\mathbb{R}}$ is a real hyperbolic orbifold.

For completeness, we give explicit orbifold charts. Take f as before, and H one of the H^4 's comprising $B_f \setminus K_f$. Recall that $(A_f/B_f)_H$ is its stabilizer in A_f/B_f . The orbifold chart is the composition

$$\begin{aligned} H &\rightarrow (A_f/B_f)_H \backslash H \\ &\cong (A_f/B_f) \backslash (B_f \setminus K_f) \\ &= A_f \setminus K_f \\ &\cong A_f \backslash (\cup \mathcal{F}_s^{\alpha_i}) / G^{\mathbb{R}} \\ &\rightarrow P\Gamma \backslash \mathcal{F}_s^{\mathbb{R}} / G^{\mathbb{R}} = \mathcal{M}_s^{\mathbb{R}}. \end{aligned}$$

(Rather, it is the restriction of this composition to a suitable open subset of H .) The homeomorphism of the second line is part of lemma 9.4, and that of the fourth is lemma 9.3. The map in the last line is a homeomorphism in a neighborhood U of the image of f in $A_f \backslash (\cup \mathcal{F}_s^{\alpha_i}) / G^{\mathbb{R}}$. We take the domain of the orbifold chart to be the subset of H which is the preimage of U .

Theorem 9.6. *With the orbifold structure given above, $\mathcal{M}_s^{\mathbb{R}}$ is a complete real hyperbolic orbifold of finite volume, and there is a properly discontinuous group $P\Gamma^{\mathbb{R}}$ of motions of H^4 such that $\mathcal{M}_s^{\mathbb{R}}$ and $P\Gamma^{\mathbb{R}} \backslash H^4$ are isomorphic hyperbolic orbifolds.*

Proof. To prove $\mathcal{M}_s^{\mathbb{R}}$ complete, consider $K = \mathcal{F}_s^{\mathbb{R}} / G^{\mathbb{R}}$. We know that g maps K to $\mathbb{C}H^4$; this is proper because any compact set in $\mathbb{C}H^4$ meets only finitely many H_{χ}^4 , $\chi \in PA$, and g carries each $\mathcal{F}_s^{\chi} / G^{\mathbb{R}}$ homeomorphically to H_{χ}^4 . Since $K \rightarrow \mathbb{C}H^4$ is proper and $P\Gamma \backslash \mathbb{C}H^4$ is complete, so is $P\Gamma \backslash K$.

The uniformization theorem for complete hyperbolic orbifolds implies the existence of $P\Gamma^{\mathbb{R}}$ with the stated properties. See Proposition 13.3.2 of [25] or Chapter III G of [8] for discussion and proofs of this theorem. The volume of $\mathcal{M}_s^{\mathbb{R}}$ is the sum of the volumes of the $P\Gamma_j^{\mathbb{R}} \backslash H_j^4$. Since these have finite volume, so does $\mathcal{M}_s^{\mathbb{R}}$. \square

10. A FUNDAMENTAL DOMAIN FOR $P\Gamma^{\mathbb{R}}$

In the previous section we equipped the moduli space $\mathcal{M}_s^{\mathbb{R}}$ of stable real cubic surfaces with a complete hyperbolic orbifold structure, so $\mathcal{M}_s^{\mathbb{R}} \cong P\Gamma^{\mathbb{R}} \backslash H^4$ for some discrete group $P\Gamma^{\mathbb{R}}$. In this section we construct a fundamental domain and the associated generators for $P\Gamma^{\mathbb{R}}$. Besides its intrinsic interest, this allows us to prove in section 11 that

$P\Gamma^{\mathbb{R}}$ is nonarithmetic. Throughout this section, when we refer to $\mathcal{M}_s^{\mathbb{R}}$ as an orbifold, we refer to the hyperbolic structure.

We first explain how the orbifold universal cover $H \cong H^4$ of $\mathcal{M}_s^{\mathbb{R}}$ is tiled by copies of the polyhedra C_j of section 4. Consider the set of points in the orbifold $\mathcal{M}_0^{\mathbb{R}} \subseteq \mathcal{M}_s^{\mathbb{R}}$ whose local group contains no reflections, and its preimage under the orbifold covering map $H \rightarrow \mathcal{M}_s^{\mathbb{R}}$. Because the restriction of the hyperbolic structure of $\mathcal{M}_s^{\mathbb{R}}$ to $\mathcal{M}_0^{\mathbb{R}}$ is the (incomplete) structure described in section 2, each component of the preimage is a copy of the interior of one of the C_j . We call the closure of such a component a chamber of type j . It is clear that the union of the chambers is H and that their interiors are disjoint, so that they tile H .

Recall that we call a wall of a chamber a discriminant wall if it lies over the discriminant, and an Eckardt wall otherwise. By theorem 5.4, it is a discriminant wall if and only if it corresponds to a blackened node of C_j in figure 1.1. Because the orbifold structure on $\mathcal{M}_s^{\mathbb{R}}$ restricts to that on $\mathcal{M}_0^{\mathbb{R}}$, every point of an Eckardt wall is fixed by some reflection of $P\Gamma^{\mathbb{R}}$. Therefore $P\Gamma^{\mathbb{R}}$ contains the reflections across the Eckardt walls of the chambers. The same argument shows that if a chamber has type 1 or 2, so that it has a diagram automorphism, then some element of $P\Gamma^{\mathbb{R}}$ carries it to itself by this automorphism.

We have seen that across any Eckardt wall of a chamber lies another chamber of the same type, in fact the mirror image of the first. Now we describe how the chambers meet across the discriminant walls. This is most easily understood by considering the 5 specific chambers $C_j \subseteq H_j^4$ given in section 4, regarding all the H_j^4 's as lying in $\mathbb{C}H^4$. Using the labeling of figure 4.2, we refer to the k th simple root of C_j as r_{jk} and to the corresponding wall of C_j as C_{jk} . The following lemma leads to complete information about how chambers meet across discriminant walls.

Lemma 10.1. *As subsets of $\mathbb{C}H^4$, we have $C_{04} = C_{14}$, $C_{13} = C_{24}$, $C_{22} = C_{34}$ and $C_{31} = C_{44}$. There is an element of $P\Gamma$ carrying C_{37} isometrically to C_{46} .*

Proof. The first assertion is just a calculation; it is even easy if organized along the lines of the following treatment of the first equality. It is obvious that $r_{04}^{\perp} \subseteq H_0^4$ and $r_{14}^{\perp} \subseteq H_1^4$ coincide, since $r_{14} = \theta \cdot r_{04}$. Simple roots describing C_{04} may be obtained by projecting the simple roots of C_0 into r_{04}^{\perp} , which amounts to zeroing out the last coordinate. Simple roots describing C_{14} may be obtained by listing the walls of C_1 meeting C_{14} , namely C_{11} , C_{12} , C_{13} and C_{16} , and projecting the corresponding roots into r_{14}^{\perp} , which again amounts to zeroing out the last

coordinate. The two 4-tuples of vectors so obtained coincide, so they define the same polyhedron in $H_0^4 \cap H_1^4 \cong H^3$.

Now we prove the second claim. Since only two discriminant walls remain unmatched, we expect C_{37} to coincide with some $P\Gamma$ -translate of C_{46} . One can argue that this must happen, but it is easier to just find a suitable element γ of $P\Gamma$. It should take θr_{37} to r_{46} ; it should also carry r_{35}, r_{32}, r_{33} and r_{36} to r_{45}, r_{41}, r_{42} and r_{43} in the order stated. These conditions determine γ , which turns out to be

$$\gamma = \begin{pmatrix} 10 + 6\omega & 4 + 2\omega & 1 - 4\omega & 1 - 4\omega & 1 - 4\omega \\ 2 - 2\omega & 1 & -2 - 2\omega & -2 - 2\omega & -2 - 2\omega \\ 1 - 4\omega & -2\omega & -2 - 2\omega & -3 - 2\omega & -3 - 2\omega \\ 1 - 4\omega & -2\omega & -3 - 2\omega & -2 - 2\omega & -3 - 2\omega \\ 1 - 4\omega & -2\omega & -3 - 2\omega & -3 - 2\omega & -2 - 2\omega \end{pmatrix},$$

where we regard vectors as column vectors and γ acts on the left. Since γ has entries in \mathcal{E} and satisfies

$$\gamma^T \cdot \text{diag}[-1, 1, 1, 1, 1] \cdot \bar{\gamma} = \text{diag}[-1, 1, 1, 1, 1],$$

it lies in $P\Gamma$. By construction, it carries C_{37} to C_{46} . \square

The lemma completes our picture of how the chambers meet along walls, as follows. Suppose P (resp. P') is a chamber of type 0 (resp. 1), with walls named P_1, \dots, P_5 (resp. P'_1, \dots, P'_7) according to an isometric identification of P with C_0 (resp. P' with C_1). The lemma implies that P_4 and P'_4 are identified under the map $H \rightarrow \mathcal{M}_s^{\mathbb{R}}$, so there must be an element of $P\Gamma^{\mathbb{R}}$ carrying P_4 to P'_4 . This implies that P_4 is a wall not only of P but also of another chamber, of type 1. Applying this argument to the other cases of the lemma implies that every discriminant wall of a chamber is also a discriminant wall of another chamber, of known type.

Now we construct what will turn out to be a fundamental domain for a subgroup $\frac{1}{2}P\Gamma^{\mathbb{R}}$ of index 2 in $P\Gamma^{\mathbb{R}}$. We choose a chamber P_0 of type 0 and write P_{0k} for its walls corresponding to the C_{0k} under the unique isometry $P_0 \cong C_0$. Across its discriminant wall P_{04} lies a chamber P_1 of type 1; write P_{1k} for its walls corresponding to C_{1k} under the unique isometry $P_1 \cong C_1$ that identifies $P_{04} \subseteq P_1$ with C_{14} . In particular, $P_{04} = P_{14}$. P_1 shares its discriminant wall P_{17} with the image P'_0 of P_0 under the diagram automorphism of P_1 ; we label the walls of P'_0 by P'_{0k} just as we did for P_0 . We write P_2 for the chamber of type 2 on the other side of P_{13} . There are two isometries $P_2 \cong C_2$, both of which identify $P_{13} \subseteq P_2$ with C_{24} , so we must work a little harder to fix our labeling of the walls of P_2 . We choose the identification of P_2 with C_2 that identifies $P_{13} \cap P_{11} \subseteq P_2$ with $C_{24} \cap C_{21}$, and label the walls P_{2k}

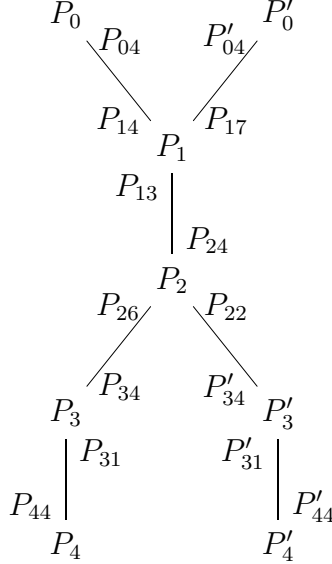


FIGURE 10.1. Assembly of the polyhedron Q from 8 chambers.

of P_2 accordingly. Now, P_2 has three discriminant walls: it shares P_{24} with P_1 , and across P_{22} and P_{26} lie chambers of type 3. We write P_3 for the one across P_{22} and P'_3 for the one across P_{26} ; these chambers are exchanged by the diagram automorphism. Label the walls of P_3 by P_{3k} according to the unique isometry $P_3 \cong C_3$, and similarly for P'_3 . Finally, across P_{31} lies a chamber P_4 of type 4, whose walls we name P_{4k} according to the isometry $P_4 \cong C_4$. Similarly, P'_3 shares P'_{31} with a chamber P'_4 which the diagram automorphism exchanges with P_4 . We label the walls of P'_4 accordingly. Let Q be the union of all eight chambers $P_0, P'_0, P_1, P_2, P_3, P'_3, P_4$ and P'_4 . The construction of Q is summarized in figure 10.1.

We remark that the diagram automorphisms of P_1 and P_2 coincide, in the sense that they are the same isometry of H , which we will call S ; this isometry preserves Q . Throughout this section, “the diagram automorphism” refers to S .

Lemma 10.2. Q is a Coxeter polyhedron.

Proof. As a set, the boundary of Q is the union of the Eckardt walls of the P_j and P'_j , together with P_{37}, P_{46}, P'_{37} and P'_{46} . Suppose W is an Eckardt wall of one of the P_j or P'_j and H_W^3 is the hyperplane in H that it spans. Then Q lies entirely in one of the closed half-spaces bounded by H_W^3 , because $P\Gamma^{\mathbb{R}}$ contains the reflection across H_W^3 , while no point in the interior of Q can be stabilized by a reflection of $P\Gamma^{\mathbb{R}}$. We call $H_W^3 \cap Q$ an Eckardt wall of Q . Two Eckardt walls of Q that

meet make interior angle π/n for some integer n , for otherwise some point in the interior of Q would be stabilized by a reflection.

Now we claim that for $W = P_{37}, P_{46}, P'_{37}$ or P'_{46} , the wall of Q containing W coincides with W , and its only meetings with other walls of Q are orthogonal intersections with Eckardt walls. We verify this for $W = P_{37}$; the key point is that P_{37} is orthogonal to all the walls of P_3 that it meets, namely P_{35}, P_{32}, P_{33} and P_{36} , and all these walls are Eckardt walls of P_3 . By the above, we know that Q lies in the region bounded by the H^3 's containing P_{35}, P_{32}, P_{33} and P_{36} , so the only walls of Q which could meet W are these walls (or rather their extensions to walls of Q). More precisely, there is a neighborhood of P_{37} in H whose intersection with Q coincides with its intersection with P_3 . All our claims follow from this. The same argument applies to P_{46} , and for the remaining two walls we appeal to symmetry. \square

Since Q is a Coxeter polyhedron, it may be described as the image in H^4 of the set of vectors having $x \cdot s \leq 0$ where s varies over a set of simple roots for Q . There is one simple root for each wall of Q , so we may find simple roots for Q by taking all the simple roots for the P_j and P'_j , and discarding the ones associated to the walls along which the P_j and P'_j meet. We will also discard duplicates, which occur when walls of two different P_j or P'_j lie in the same wall of Q .

Therefore we will need to know simple roots for all the P_j and P'_j . We identify H with $H_1^4 \subseteq \mathbb{C}H^4$, such that P_0 is C_0 . Then P_1 is the image of $C_1 \subseteq H_1^4 \subseteq \mathbb{C}H^4$ under the map

$$T_1 : (x_0, \dots, x_4) \mapsto (x_0, x_1, x_2, x_3, ix_4),$$

which is an isometry of $\mathbb{C}H^4$ but not an element of PT . This uses the fact that C_{04} and C_{14} coincide as subsets of $\mathbb{C}H^4$, and T_1 carries r_{14} to a negative multiple of r_{04} . Similarly, using the intersections of P_1 with P_2, P_2 with P_3 , and P_3 with P_4 described in lemma 10.1, we find

$$\begin{aligned} P_2 &= T_2(C_2) \text{ where } T_2 : (x_0, \dots, x_4) \mapsto (x_0, x_1, x_2, ix_3, ix_4), \\ P_3 &= T_3(C_3) \text{ where } T_3 : (x_0, \dots, x_4) \mapsto (x_0, x_1, ix_2, ix_3, ix_4), \text{ and} \\ P_4 &= T_4(C_4) \text{ where } T_4 : (x_0, \dots, x_4) \mapsto (x_0, ix_1, ix_2, ix_3, ix_4). \end{aligned}$$

For uniformity of notation we define T_0 to be the identity map. In all cases we have $P_{jk} = T_j(C_{jk})$; we selected our labelings of the walls of the P_j so that this would hold. We write s_{jk} for $T_j(r_{jk})$, yielding simple roots for the P_j . Given r_{jk} from figure 4.2, s_{jk} is got by replacing θ by $-\sqrt{3}$ wherever it appears. Since simple roots for P_1 are now known, the

root	coordinates	root [⊥]	contains
s_A	$(3, -1, \sqrt{3}, \sqrt{3}, \sqrt{3})$	P_{37}	
s_B	$(\sqrt{3}, 1, 1, 1, 1)$	P_{46}	
s_C	$(1, -1, -1, -1, 0)$	$P_{05}, P_{16}, P'_{02}, P'_{33}, P'_{42}$	
s_D	$(\sqrt{3}, -\sqrt{3}, 0, 1, 1)$	$P_{27}, P_{36}, P'_{03}, P'_{32}, P'_{41}$	
s_E	$(0, 1, -1, 0, 0)$	$P_{01}, P_{11}, P_{21}, P_{43}, P'_{35}, P'_{45}$	
s'_E	$(1, 0, 0, 0, \sqrt{3})$	$P_{15}, P_{25}, P_{35}, P_{45}, P'_{01}, P'_{43}$	
s'_D	$(0, 0, 0, 1, -1)$	$P_{03}, P_{23}, P_{32}, P_{41}, P'_{36}$	
s'_C	$(0, 0, 1, -1, 0)$	$P_{02}, P_{12}, P_{33}, P_{42}, P'_{05}$	
s'_B	$(3 + 2\sqrt{3}, -2 - \sqrt{3}, -2 - \sqrt{3}, 1, 2 + \sqrt{3})$	P'_{46}	
s'_A	$(4 + \sqrt{3}, -2 - \sqrt{3}, -2 - \sqrt{3}, \sqrt{3}, \sqrt{3})$	P'_{37}	

TABLE 10.1. Simple roots for the polyhedron Q .

matrix for the diagram automorphism S can be worked out, yielding

$$(10.1) \quad S = \begin{pmatrix} 3 & 2 & 1 & 0 & -\sqrt{3} \\ -2 & -1 & -1 & 0 & \sqrt{3} \\ -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \sqrt{3} & \sqrt{3} & 0 & 0 & -1 \end{pmatrix}.$$

Since P'_0, P'_3 and P'_4 are the images of P_0, P_3 and P_4 under S , they are described by simple roots $s'_{jk} = S \cdot s_{jk}$. We now have explicit simple roots for all eight chambers comprising Q .

To obtain simple roots for Q , we take all the s_{jk} and s'_{jk} and discard those involved in the gluing of figure 10.1, namely $s_{04}, s'_{04}, s_{14}, s_{17}, s_{13}, s_{24}, s_{26}, s_{22}, s_{34}, s'_{34}, s_{31}, s'_{31}, s_{44}$ and s'_{44} . This leaves us with 36 simple roots. There is a great deal of duplication, for example s_{01} and s_{43} are positive scalar multiples of each other. After eliminating duplicates, only 10 remain, given in table 10.1. We will indicate the walls of Q by $A, \dots, E, E', \dots, A'$ and corresponding simple roots by $s_A, \dots, s_E, s'_E, \dots, s'_A$. We have scaled them so that s_A, s_B, s'_B and s'_A have norm 1 and the rest have norm 2. In the table we also indicate which P_{jk} and P'_{jk} lie in each wall of Q . The diagram automorphism acts by exchanging primed and unprimed letters. With simple roots in hand, one can work out Q 's dihedral angles, yielding figure 10.2 as the Coxeter diagram of Q .

We already know that $P\Gamma^{\mathbb{R}}$ contains the reflections across C, D, E, E', D' and C' . By lemma 10.1, P_{37} and P_{46} are identified in $\mathcal{M}_s^{\mathbb{R}}$, so there exists an element τ of $P\Gamma^{\mathbb{R}}$ carrying $A = P_{37}$ to $B = P_{46}$. This transformation must carry P_3 to the type 3 chamber on the other side

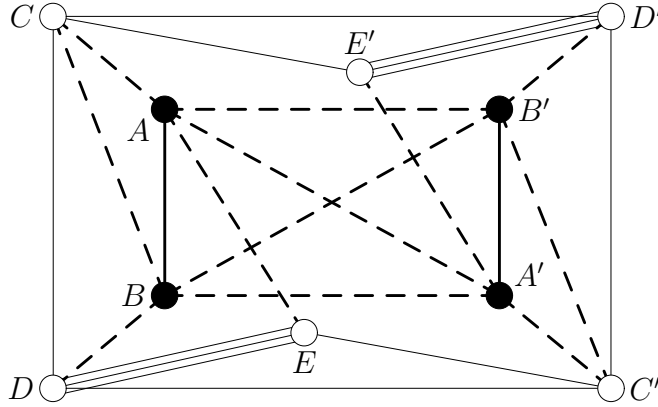


FIGURE 10.2. The polyhedron Q .

of P_{46} from P_4 , and so it carries s_A to $-s_B$. By considering how the walls of Q meet A and B , one sees that τ must fix each of s'_E , s'_D and s'_C , and carry s_D to s_E . This determines τ uniquely:

$$(10.2) \quad \tau = \begin{pmatrix} 7 + 3\sqrt{3} & 3 + \sqrt{3} & -3 - 2\sqrt{3} & -3 - 2\sqrt{3} & -3 - 2\sqrt{3} \\ 3 + \sqrt{3} & 1 & -1 - \sqrt{3} & -1 - \sqrt{3} & -1 - \sqrt{3} \\ 3 + 2\sqrt{3} & 1 + \sqrt{3} & -1 - \sqrt{3} & -2 - \sqrt{3} & -2 - \sqrt{3} \\ 3 + 2\sqrt{3} & 1 + \sqrt{3} & -2 - \sqrt{3} & -1 - \sqrt{3} & -2 - \sqrt{3} \\ 3 + 2\sqrt{3} & 1 + \sqrt{3} & -2 - \sqrt{3} & -2 - \sqrt{3} & -1 - \sqrt{3} \end{pmatrix}.$$

Of course, $P\Gamma^{\mathbb{R}}$ also contains $\tau' = S\tau S$, which carries A' to B' . We define $\frac{1}{2}P\Gamma^{\mathbb{R}}$ to be the subgroup of $P\Gamma^{\mathbb{R}}$ generated by τ , τ' and the reflections in C , D , E , E' , D' and C' .

Lemma 10.3. *Q is a fundamental domain for $\frac{1}{2}P\Gamma^{\mathbb{R}}$. More precisely, the $\frac{1}{2}P\Gamma^{\mathbb{R}}$ -images of Q cover $H \cong H^4$, and the only identifications among points of Q under $Q \rightarrow \frac{1}{2}P\Gamma^{\mathbb{R}} \backslash H$ are that A (resp. A') is identified with B (resp. B') by the action of τ (resp. τ').*

Proof. All our claims follow from Poincaré’s polyhedron theorem, as formulated in [20, sec. IV.H]. There are 7 conditions to verify. The key points are that any two Eckardt walls that intersect make an angle of the form $\pi/(\text{an integer})$, and that the 4 discriminant walls are disjoint from each other and orthogonal to the Eckardt walls that they meet. These properties dispose of Maskit’s conditions (i)–(vi). Condition (vii) is that Q modulo the identifications induced by τ and τ' is metrically complete. This follows because we already know that $H^4/P\Gamma^{\mathbb{R}}$ is complete, and $\frac{1}{2}P\Gamma^{\mathbb{R}} \subseteq P\Gamma^{\mathbb{R}}$. \square

The main theorem of this section is now an easy consequence:

Theorem 10.4. $P\Gamma^{\mathbb{R}} = \left(\frac{1}{2}P\Gamma^{\mathbb{R}}\right) \rtimes \mathbb{Z}/2$, the $\mathbb{Z}/2$ being the diagram automorphism S .

Proof. By the lemma, S does not lie in $\frac{1}{2}P\Gamma^{\mathbb{R}}$. Since S normalizes $\frac{1}{2}P\Gamma^{\mathbb{R}}$, we have

$$P\Gamma^{\mathbb{R}} = \langle \frac{1}{2}P\Gamma^{\mathbb{R}}, S \rangle = \frac{1}{2}P\Gamma^{\mathbb{R}} \rtimes \langle S \rangle.$$

Since this larger group lies in $P\Gamma^{\mathbb{R}}$ and has the same covolume as $P\Gamma^{\mathbb{R}}$, it equals $P\Gamma^{\mathbb{R}}$. \square

Remark. Poincaré’s polyhedron theorem readily gives a presentation for $P\Gamma^{\mathbb{R}}$: there are generators C, C', D, D', E, E' (the reflections in the Eckardt walls of Q), τ, τ' (the maps identifying A with B , respectively A' with B'), and S (the diagram automorphism) with the following relations. (1) The subgroup generated by C, D, E, C', D', E' has the Coxeter presentation indicated in the diagram. (2) S is an involution and conjugation by S interchanges all the primed and unprimed generators. (3) τ commutes with C', D', E' while $\tau D = E\tau$. (4) The relations obtained from (3) by interchanging the primed and unprimed letters. A presentation for $\frac{1}{2}P\Gamma^{\mathbb{R}}$ is obtained by deleting the generator S and all the relations involving it.

We have now established theorem 1.2, except for the nonarithmeticity and the fact that $\mathcal{M}_0^{\mathbb{R}} \subseteq \mathcal{M}_s^{\mathbb{R}}$ corresponds to $P\Gamma^{\mathbb{R}} \backslash (H^4 - \mathcal{H}')$ where \mathcal{H}' is a union of H^2 ’s and H^3 ’s. We will now address \mathcal{H}' ; see the next section for the nonarithmeticity. The part of Q that lies over the discriminant in $\mathcal{M}_s^{\mathbb{R}}$ consists of (1) the walls A, B, A' and B' , (2) the faces corresponding to triple bonds in figure 10.2, and (3) the walls of the P_j and P'_j along which we glued the 8 chambers to obtain Q . We will refer to a wall of case (3) as an ‘interior wall’. Setting \mathcal{H}' to be the preimage of $\mathcal{M}_s^{\mathbb{R}} - \mathcal{M}_0^{\mathbb{R}}$ in H^4 , we see that \mathcal{H}' is the union of the $\frac{1}{2}P\Gamma^{\mathbb{R}}$ -translates of these three parts of Q . The wall A is orthogonal to all the walls of Q that it meets, all of which are Eckardt walls, so it’s easy to see that the H^3 containing A is covered by the $\frac{1}{2}P\Gamma^{\mathbb{R}}$ -translates of A . The same argument applies with B, A' or B' in place of A , and also applies in case (2), yielding H^2 ’s.

The essential facts for treating case (3) are the following. If I is an interior wall, then every wall w of Q with which I has 2-dimensional intersection is an Eckardt wall of Q , and is either orthogonal to I or makes angle $\pi/4$ with it. In the orthogonal case, it is obvious that \mathcal{H}' contains the image of I under reflection across w . In the $\pi/4$ case, one can check that there is another interior wall I' with $I' \cap w = I \cap w$, $\angle(w, I') = \pi/4$ and $I \perp I'$. Then the image of I' under reflection across

w lies in the same H^3 as I does. Repeating this process, we see that the H^3 containing I is tiled by $\frac{1}{2}P\Gamma^{\mathbb{R}}$ -translates of interior walls of Q . It follows that \mathcal{H} is a union of H^2 's and H^3 's.

We remark that the H^3 tiled by translates of interior walls can be viewed as a 3-dimensional analogue of our gluing process, describing moduli of real 6-tuples in $\mathbb{C}P^1$; see [4] and [5] for details. In particular, its stabilizer in $P\Gamma^{\mathbb{R}}$ is the nonarithmetic group discussed there. Also, see [5] for the 2-dimensional analogue.

11. NONARITHMETICITY

This section is devoted to proving the following result:

Theorem 11.1. *$P\Gamma^{\mathbb{R}}$ is a nonarithmetic lattice in $PO(4, 1)$.*

Our main tool is Corollary 12.2.8 of [12]. We recall the context: G is an adjoint connected absolutely simple non compact real Lie group, \mathbf{G} is an adjoint connected simple algebraic group over \mathbb{R} so that G is the identity component of $\mathbf{G}(\mathbb{R})$ and Γ is a lattice in G . Let $E = \mathbb{Q}[\text{Tr Ad } \Gamma]$, the field generated over \mathbb{Q} by $\{\text{Tr Ad } \gamma : \gamma \in \Gamma\}$. Assume that there is a totally real number field F and a form \mathbf{G}_F of \mathbf{G} over F so that a subgroup of finite index of Γ is contained in $\mathbf{G}(\mathcal{O}_F)$, where \mathcal{O}_F is the ring of integers in F . It follows that $E \subset F$. With this context in mind, the statement of Corollary 12.2.8 of [12] is

Theorem 11.2. *A lattice $\Gamma \subset G$ is arithmetic in G if and only if for each embedding σ of F in \mathbb{R} , not inducing the identity embedding of E in \mathbb{R} , the real group $\mathbf{G}_F \otimes_{F, \sigma} \mathbb{R}$ is compact.*

To apply this, we take G to be the connected component of $SO(4, 1)$ and Γ to be the subgroup of $P\Gamma^{\mathbb{R}}$ that acts on H^4 by orientation-preserving isometries. Note that $\text{Isom } H^4 = \text{PO}(4, 1) = \text{SO}(4, 1)$, so that Γ is indeed a subgroup of G . Note that $SO(4, 1) = PO(4, 1)$, so that we may regard $P\Gamma^{\mathbb{R}}$ as a subgroup of G . We take \mathbf{G} to be the special orthogonal group of the form $\text{diag}\{-1, 1, 1, 1, 1\}$ and $F = \mathbb{Q}(\sqrt{3})$. Note that \mathbf{G} is defined over \mathbb{Q} , hence over F , and that $\Gamma \subset \mathbf{G}(\mathcal{O}_F)$. To see the last statement, it is clear that all the the matrices of the generating reflections of $P\Gamma^{\mathbb{R}} \subseteq \text{SO}(4, 1)$ have entries in F , also from formulas (10.2) and (10.1) that the matrices of S and τ have entries in F .

Next we show that $E = \mathbb{Q}(\sqrt{3}) = F$. It is clear that E is either \mathbb{Q} or $\mathbb{Q}(\sqrt{3})$. To prove that $E = \mathbb{Q}(\sqrt{3})$ it suffices to exhibit a single $\gamma \in \Gamma$ with $\text{Tr Ad } \gamma \notin \mathbb{Q}$. Almost any γ will do; we take $\gamma = (R_C R_{D'} R_{E'})^2$, where the R 's are the reflections in the corresponding simple roots from

table 10.1. One can compute a matrix for γ and its square and compute their traces, yielding $\text{Tr}(\gamma) = 13 + 6\sqrt{3}$ and $\text{Tr}(\gamma^2) = 209 + 120\sqrt{3}$. Since the adjoint representation of $O(4, 1)$ is the exterior square of the standard one, we can use the formula

$$\text{Tr Ad}(\gamma) = \frac{1}{2} ((\text{Tr}(\gamma))^2 - \text{Tr}(\gamma^2)) = 34 + 18\sqrt{3} \notin \mathbb{Q}.$$

This proves that $E = \mathbb{Q}(\sqrt{3})$.

Finally, if σ denotes the non-identity embedding of F in \mathbb{R} , then since $E = F$ it does not induce the identity embedding of E , and since the form $\text{diag}\{-1, 1, 1, 1, 1\}$ defining \mathbf{G} is fixed by σ , the group $\mathbf{G}_F \otimes_{F, \sigma} \mathbb{R}$ is again the non-compact group $SO(4, 1)$. Thus Γ is not arithmetic.

In the introduction we indicated that while our gluing construction is philosophically that of Gromov and Piatetski-Shapiro, their results do not directly apply in our situation. There are two reasons for this. They consider two hyperbolic n -manifolds M_1 and M_2 , with ∂M_1 and ∂M_2 totally geodesic and isometric to each other, and they glue them together to get a hyperbolic manifold M . If $\pi_1(M_1), \pi_1(M_2) \subseteq \pi_1(M)$ lie in noncommensurable arithmetic lattices and are Zariski-dense in $PO(n, 1)$, then M is nonarithmetic. The first and more minor obstruction to applying this is that we are gluing orbifolds with boundary and corners, not just boundary. The second obstruction is that none of the images in $P\Gamma^{\mathbb{R}}$ of the fundamental groups $\pi_1^{\text{orb}}(\mathcal{M}_{0,j}^{\mathbb{R}})$ of the pieces are Zariski-dense in $PO(4, 1)$. These images are the groups T_j from section 6; three are finite and two are lattices in $PO(3, 1)$.

Remarks. (1) We wonder whether the unimodular lattice L over $\mathbb{Z}[\sqrt{3}]$ plays some deeper geometric or arithmetic role. For example, $P\Gamma^{\mathbb{R}}$ maps to $W(E_6) \cong PO(5, \mathbb{F}_3)$ by reduction of L modulo $\sqrt{3}$. On each component of the smooth moduli space, the action on this \mathbb{F}_3 vector space is the same as the action on V from section 3. But it is not clear what this really means. (2) The group generated by reflections in the facets of Q , while being quite different from $P\Gamma^{\mathbb{R}}$, also preserves L and is also nonarithmetic.

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