NON-ARITHMETIC UNIFORMIZATION OF SOME REAL MODULI SPACES

DANIEL ALLCOCK, JAMES A. CARLSON, AND DOMINGO TOLEDO

Abstract. Some real moduli spaces can be presented as real hyperbolic space modulo a non-arithmetic group. The whole moduli space is made from some incommensurable arithmetic pieces, in the spirit of the construction of Gromov and Piatetski-Shapiro.

1. Introduction

The purpose of this paper is to explain how some real moduli spaces have non-arithmetic uniformizations, in the sense that they are homeomorphic to real hyperbolic space modulo the action of a non-arithmetic group. The space is assembled, in a natural way, from various pieces, each of which can be uniformized by an arithmetic group. One can check that the pieces are not all commensurable. The uniformization of the moduli space can be seen as an orbifold version of the construction of non-arithmetic groups by Gromov and Piatetski-Shapiro [5]. In other words, some real moduli spaces give very natural and concrete examples of the Gromov-Piatetski-Shapiro construction. We first found this phenomenom in the moduli space of real cubic surfaces [2]. Since some of the details there are quite technical, in this paper we outline an easier situation, namely the moduli of real polynomials in two variables, homogeneous of degree six.

The first question is why study polynomials of degree six, rather than some other degree. First of all, we want a space of complex polynomials so that their moduli space is unifomized by complex hyperbolic space. Equivalently, we want situations where the moduli space of points (possibly with weights) on the Riemann sphere is complex hyperbolic, in the sense that it is the unit ball B^n in $\mathbb{C}^n$ modulo the action of a lattice Γ of biholomorphic automorphisms of B^n. If the number of points is at least five, there are only finitely many possibilities and they are listed in [4], [6], [10]. The number of points must be at most 12, and

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if the weights are all equal, the number of points (assumed to be at least 5) must be $5, 6, 8$ or 12. In these equal weight cases the moduli space is always of the form $\Gamma\backslash B^n$ where Γ is an *arithmetic* subgroup of the group of bihilomorphic automorphisms of B^n. We chose 6 because the arithmetic group acting on the ball is the simplest, in the sense that it is the group of units of a unimodular hermitian form over a discrete subring of $\mathbb{C}$, namely the ring of Eisenstein integers (integers in $\mathbb{Q}(\sqrt{-3})$). (For 5 points discreteness of the ring fails; for 8 and 12 points unimodularity fails). Also important for us is the fact that the moduli space of six points is embedded (as the discriminant) in the moduli space of cubic surfaces. The reason is that a cubic surface with an ordinary double point can be represented as the blow-up of $\mathbb{C}P^2$ at 6 points lying on a conic.

Whenever a moduli space of complex varieties of a given type is complex hyperbolic, it seems reasonable to expect that the moduli space of the real varieties of the same type is real hyperbolic. What we will see in this example is that this is indeed the case, but not in an obvious way. The moduli space of real polynomials with distinct roots consists of 4 connected components, corresponding to the 4 possible configurations of real and complex conjugate pairs of roots: all roots being real, 4 roots being real and one complex conjugate pair, etc. We will see that each component is uniformized by a totally geodesic real hyperbolic subspace of the ball. More precisely, there is an anti-holomorphic involution (briefly: anti-involution) of the ball, so that its fixed point set H^3_i, modulo its stabilizer Γ_i in Γ, parametrizes the union of this component with the adjoining polynomials with double roots. But there is no single anti-involution of the ball whose fixed point set modulo its stabilizer in Γ parametrizes the union of the four components.

What turns out to be true is that the moduli space of real polynominals with at most double roots is homeomorphic to a real hyperbolic orbifold $\Gamma^{\mathbb{R}}\backslash H^3$ for some lattice $\Gamma^{\mathbb{R}}$ acting on real hyperbolic space H^3. This orbifold contains as open sub-orbifolds the four moduli spaces of polynomials with distinct roots, and it assembles the four distinct $\Gamma_i\backslash H^3_i$ into a single hyperbolic orbifold. The group $\Gamma^{\mathbb{R}}$ is not arithmetic, while the group Γ acting on B^3, as well as the groups Γ_i acting on H^3, are arithmetic. One can easily check that the groups Γ_i are not all commensurable (in fact, they fall into two distinct commensurability classes), so it is in this sense that $\Gamma^{\mathbb{R}}$ is a group in the spirit of Gromov and Piatetski-Shapiro. The fact that $\Gamma^{\mathbb{R}}$ is not arithmetic implies that there is no single anti-involution of B^3, respecting Γ, whose fixed point set modulo its stabilizer in Γ parametrizes all the polynomials. By examining the construction a bit more closely, one can see that $\Gamma^{\mathbb{R}}$ admits homomorphisms to Γ and that there are piecewise-geodesic equivariant maps of H^3 to B^3.

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2. Moduli of complex polynomials

We review briefly the fact, from [4], that the moduli space of 6 points on $\mathbb{C}P^1$ is complex hyperbolic. Let $\mathcal{P}$ denote the space of complex polynomials $f(x, y)$ in two variables, homogeneous of degree 6. Write $\mathcal{P}_{sm}$ for the subspace of polynomials with distinct roots (smooth zero set) and let $\mathcal{P}_{st}$ denote the subspace of polynomials with roots of multiplicity at most two (stable in the sense of geometric invariant theory for the action of $SL(2, \mathbb{C})$ on the projective space of $\mathcal{P}$). To state the uniformization theorem for this space, let B^3 be the unit ball in $\mathbb{C}^3$, and let $h(z) = |z_0|^2 - |z_1|^2 - |z_2|^2 - |z_3|^2$ be the standard hermitian form of signature $(1, 3)$ on $\mathbb{C}^4$. Recall that $PU(h) = PU(1, 3)$ is the the group of biholomorphic automorphisms of B^3, by identifying B^3 with the subspace of $\mathbb{C}P^3$ of positive lines for h. Let $\omega = e^{2\pi i/3}$ and let $\mathcal{E} = \mathbb{Z}[\omega]$, the ring of integers in $\mathbb{Q}(\sqrt{-3})$. Let $\Gamma = PU(h, \mathcal{E})$. For each vector $v \in \mathcal{E}^4 \subset \mathbb{C}^4$ with $h(v) = -1$, its orthogonal complement defines a hyperplane in $\mathbb{C}P^3$ that cuts B^3 in a totally geodesic hyperplane (or 2-dimensional sub-ball). We write $\mathcal{H} \subset B^3$ for the union of this countable collection of hyperplanes.

Theorem 2.1. *There is an isomorphism of analytic spaces between $\mathcal{P}_{st}/GL(2, \mathbb{C})$ and $\Gamma\backslash B^3$. The subspaces $\mathcal{P}_{sm}/GL(2, \mathbb{C})$ and $\Gamma\backslash(B^3 - \mathcal{H})$ correspond under this isomorphism. Moreover, Γ is generated by the complex reflections of order 6 in the hyperplanes of $\mathcal{H}$.*

We sketch some of the reasoning behind the proof of this theorem. For more details see [4]. The main point is that, to each polynomial $f \in \mathcal{P}_{sm}$ one can assign the Riemann surface X_f given by the equation $z^3 = f(x, y)$ (in the weighted projective plane of $(x : y : z)$ where x, y have weight one and z has weight two). This Riemann surface X_f is a cyclic three-fold cover of $\mathbb{C}P^1$ branched over the zeros of f. The cyclic covering group is generated by $\sigma : X_f \to X_f$ defined by $\sigma(x, y, z) = (x, y, \omega z)$. One easily checks that the genus of X_f is 4, hence $H^1(X_f, \mathbb{Z})$ is a free abelian group of rank 8 with an action of the cyclic group $\mathbb{Z}/3$ generated by σ which fixes no non-zero element (since $H^1(\mathbb{C}P^1) = 0$). Thus $H^1(X_f, \mathbb{Z}) \cong \mathcal{E}^4$, a free $\mathcal{E}$-module of rank 4, and $H^1(X_f, \mathbb{C})$ splits as a direct sum of two eigenspaces H^1_ω and $H^1_{\bar\omega}$ (since there is no non-zero eigenvector for 1) for the action of σ. There is a second direct sum decomposition of $H^1(X_f, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$ into holomorphic and anti-holomorphic forms (Hodge decomposition). While the Hodge decomposition splits $H^1(X_f, \mathbb{C})$ into two spaces of the same dimension, it does not do the same to the eigenspaces. In fact it turns out that in the decompostion

$$H^1_{\bar\omega} = H^{1,0}_{\bar\omega} \oplus H^{0,1}_{\bar\omega}$$

the first summand is one-dimensional, and spanned by the differential

$$\frac{ydx - xdy}{f^{1/3}} \tag{2.1}$$

(which is a multi-valued differential on the $\mathbb{C}P^1$ of $(x : y)$ and is easily checked to define a holomorphic differential on the branched cover X_f), while the second summand is three-dimensional. Moreover, the hermitian form

$$h(\alpha, \beta) = i\sqrt{3} \int \alpha \wedge \bar\beta$$

is $\mathcal{E}$-valued and unimodular on the projection of $H^1(X_f, \mathbb{Z})$ to $H^1_{\bar\omega}$, it is positive definite on $H^{1,0}_{\bar\omega}$ and negative definite on $H^{0,1}_{\bar\omega}$. We will write $\mathcal{E}^{1,3}$ for this module with this standard hermitian form. The family of Riemann surfaces X_f assembles into a fibration over $\mathcal{P}_{sm}$, its first cohomology gives a local system of $\mathcal{E}$ modules with fiber $\mathcal{E}^{1,3}$, and we get a reprentation from $\pi_1(\mathcal{P}_{sm})$ to the group of isometries of $\mathcal{E}^{1,3}$.

If we let $\tilde{\mathcal{P}}_{sm}$ denote the covering space of $\mathcal{P}_{sm}$ corresponding to the kernel of the resulting projective monodromy representation $\pi_1(\mathcal{P}_{sm}) \to PU(1, 3, \mathcal{E})$, then there is a well-defined period map $\tilde{\mathcal{P}}_{sm} \to B^3$ obtained by assigning to a point $\tilde f \in \tilde{\mathcal{P}}_{sm}$ lying over $f \in \mathcal{P}_{sm}$ the subspace $H^{1,0}(X_f)_{\bar\omega}$. Passing to the covering space is needed in order to trivialize the local system of projective spaces of the $H^1_{\bar\omega}$, and hence to have a fixed space in which all the lines $H^{1,0}(X_f)_{\bar\omega}$ lie. Since these are positive lines for h, this map takes values in B^3, hence we get a map $\tilde{\mathcal{P}}_{sm} \to B^3$, which by general principles is holomorphic. Explicity, this is the map given by the periods of the hypergeometric integral associated to (2.1).

This map is not surjective, in fact it misses the subspace $\mathcal{H} \subset B^3$. But the monodromy around a generic polynomial in the discriminant, namely a polynomial with a single double root, is a complex reflection of order 6 about a component hyperplane of $\mathcal{H}$, see [4]. This can also be seen as the monodromy of the singularity $z^3 = x^2$, as the Riemann surface X_f becomes singular, which can be seen to be of order 6 by the general considerations of [8]. What this eventually means is that the period map extends to a complex manifold $\tilde{\mathcal{P}}_{st}$ which is a branched cover of $\mathcal{P}_{st}$, which is branched cyclically of order 6 over the generic point of the discriminant (and with group $(\mathbb{Z}/6)^k$, $k = 1, 2, 3$ over the part of the discriminant with k double roots). The resulting map $\tilde{\mathcal{P}}_{st} \to B^3$ is constant on $GL(2, \mathbb{C})$-orbits and gives a Γ-equivariant isomorphism $\tilde{\mathcal{P}}_{st}/GL(2, \mathbb{C}) \to B^3$ and the resulting isomorphisms stated in the theorem. Finally, we remark that the group Γ is actually the full group $PU(1, 3, \mathcal{E})$, see, say, Lemma 7.12 of [1] for a proof.

3. Anti-involutions corresponding to real polynomials

Let us see what happens over the real numbers. Let us write $\mathcal{P}^{\mathbb{R}}$, $\mathcal{P}^{\mathbb{R}}_{sm}$, $\mathcal{P}^{\mathbb{R}}_{st}$ for the spaces of polynomials f with real coefficients in the corresponding spaces of complex polynomials defined in the last section. It is clear that the space $\mathcal{P}^{\mathbb{R}}_{sm}$ has 4 connected components, which we denote

$$\mathcal{P}^{\mathbb{R}}_{sm} = \mathcal{P}^{\mathbb{R}}_0 \cup \mathcal{P}^{\mathbb{R}}_1 \cup \mathcal{P}^{\mathbb{R}}_2 \cup \mathcal{P}^{\mathbb{R}}_3,$$

so that the space $\mathcal{P}^{\mathbb{R}}_i$ consists of real polynomials with distinct roots and i complex conjugate pairs of complex roots.

If $f \in \mathcal{P}^{\mathbb{R}}_i$, then the anti-involution $\kappa(x, y, z) = (\bar{x}, \bar{y}, \bar{z})$ leaves X_f invariant, so it defines an anti-involution $\kappa : X_f \to X_f$ which satisfies $\kappa\sigma = \sigma^{-1}\kappa$. Thus the induced homorphism $\kappa^* : H^1(X_f, \mathbb{Z}) \to H^1(X_f, \mathbb{Z})$ is an anti-involution of the $\mathcal{E}$-module structure. The conjugacy class of this anti-involution of the $\mathcal{E}$-module $\mathcal{E}^{1,3}$ is constant as f varies in a fixed component of $\mathcal{P}^{\mathbb{R}}_i$.

Theorem 3.1. *Let $\kappa_0, \ldots, \kappa_3$ be the anti-involutions of $\mathcal{E}^{1,3}$ defined by*

$$\begin{aligned} \kappa_0(z_0, z_1, z_2, z_3) &= (\bar{z}_0, \quad \bar{z}_1, \quad \bar{z}_2, \quad \bar{z}_3) \\ \kappa_1(z_0, z_1, z_2, z_3) &= (\bar{z}_0, \quad \bar{z}_1, \quad \bar{z}_2, -\bar{z}_3) \\ \kappa_2(z_0, z_1, z_2, z_3) &= (\bar{z}_0, \quad \bar{z}_1, -\bar{z}_2, -\bar{z}_3) \\ \kappa_3(z_0, z_1, z_2, z_3) &= (\bar{z}_0, -\bar{z}_1, -\bar{z}_2, -\bar{z}_3) \end{aligned} \tag{3.1}$$

Then κ_i is a representative of the projective conjugacy class of anti-involutions induced on $\mathcal{E}^{1,3}$ by κ and $f \in \mathcal{P}^{\mathbb{R}}_i$.

Proof. Choose disjoint closed disks D_1, D_2, D_3 in $\mathbb{C}P^1$, each invariant under complex conjugation and containing the real axis as a diameter. Choose a polynomial $f \in \mathcal{P}^{\mathbb{R}}_i$, so that each disk D_j contains exactly two roots, say p_j, q_j, of f, and which lie on a diameter of D_j, thus either on the real axis or on a diameter perpendicular to the real axis. Let C

denote the closure of $\mathbb{C}P^1 - (D_1 \cup D_2 \cup D_3)$. Let $\tilde{D}_j$, $\tilde{C}$ denote their pre-images in X_f. We want to see how the $\mathcal{E}$-module $H^1(X_f, \mathbb{Z})$ decomposes in terms of this decomposition of X_f. It is easier to visualize the isomorphic $\mathcal{E}$-module $H_1(X_f, \mathbb{Z})$, by Poincaré duality.

It is easy to check that the pre-image in X_f of the boundary of each D_j is a circle, hence from the Mayer-Vietoris sequence we see that the $\mathcal{E}$-module $H_1(\tilde{D}_j, \mathbb{Z})$ is a direct summand. Moreover, it is a free $\mathcal{E}$-module of rank one. A generator for this $\mathcal{E}$-module is the following cycle: let c_j be an oriented segment from p_j to q_j, and let $\tilde{c}_j$ be an oriented segment in $\tilde{D}_j$ that lifts this segment. Then $\tilde{c}_j - \sigma\tilde{c}_j$ is a cycle in $\tilde{D}_j$ and is a generator of the $\mathcal{E}$-module $H_1(\tilde{D}_j, \mathbb{Z})$. The other generators are its images under σ: $\sigma\tilde{c}_j - \sigma^{-1}\tilde{c}_j$, $\sigma^{-1}\tilde{c}_j - \tilde{c}_j$, and the negatives of these three generators. In fact, $\tilde{D}_j$ is homeomorphic to a torus minus a disk.

Now let's look at the action of κ on these rank one modules. If both roots in D_j are real, then $\kappa(\tilde{c}_j) = \tilde{c}_j$, and using the identity $\kappa\sigma = \sigma^{-1}\kappa$, we see that $\kappa(\sigma\tilde{c}_j - \sigma^{-1}\tilde{c}_j) = -(\sigma\tilde{c}_j - \sigma^{-1}\tilde{c}_j)$. Thus there is a generator x of this $\mathcal{E}$-module so that $\kappa(x) = -x$, in other words, the action of κ on this $\mathcal{E}$-module is isomorphic to the action of $z \to -\bar{z}$ on $\mathcal{E}$.

If the two roots in D_j are interchanged by complex conjugation, then $\kappa(\tilde{c}_j) = -\tilde{c}_j$, and the same reasoning as before gives that $\kappa(\sigma\tilde{c}_j - \sigma^{-1}\tilde{c}_j) = (\sigma\tilde{c}_j - \sigma^{-1}\tilde{c}_j)$. Thus this $\mathcal{E}$ module has a generator x which is fixed by κ, hence it is isomorphic to $\mathcal{E}$ with the usual conjugation $z \to \bar{z}$.

Next, the complementary set C defined above can be visualized as the complement of the interior of one disk, say D_1, centered at infinity, with the interior of the two other disks D_2, D_3 removed. As such, it can be visualized as a disk with diameter the real axis and with small disks centered at two points p, q removed. The surface $\tilde{C}$ is a torus with three disks removed, and the image of $H_1(\tilde{C}, \mathbb{Z})$ in $H_1(X_f, \mathbb{Z})$ is a direct summand, and it is a free $\mathcal{E}$ module on a generator somewhat like $\tilde{c} - \sigma\tilde{c}$ above. It has two segments that project to a segment on the real axis, they join two of the boundary circles of $\tilde{C}$, and are completed to a cycle by adding suitable arcs of these boundary circles. From the fact that these two segments lie over the axis one can derive that κ acts on this summand as in the summands corresponding to a D_j with two real roots, namely as $z \to -\bar{z}$.

From disjointness of support considerations one sees that $\mathcal{E}$-module $H_1(X_f, \mathbb{Z})$ decomposes as the orthogonal direct sum of the four one-dimensional $\mathcal{E}$ submodules just defined. Unimodularity forces each summand to be spanned by a vector x with $h(x) = \pm 1$. It turns out (this requires more thought) that for the summands correspoding to the D_j we have $h(x) = -1$, while for the summand corresponding to C we have $h(x) = 1$. Thus these summands make the hermitian form h standard. Our derivation of the action of κ in these coordinates shows that it acts by the negative of the formulas (3.1). Since κ and $-\kappa$ have the same projective action, in other words, they act in the same way on the ball B^3, this proves the theorem.

□

From this theorem and Theorem 2.1, it is not hard to derive that each $\mathcal{P}_i^{\mathbb{R}}/GL(2,\mathbb{R})$ is isomorphic to $\Gamma_i\backslash(H_i^3 - \mathcal{H})$, where H_i^3 denotes the fixed-point set of κ_i in B^3, which is a totally geodesic real hyperbolic 3-space, and where Γ_i denotes its stabilizer in Γ. Moreover, it is not hard to check that $\Gamma_i = PO(q_i, \mathbb{Z})$ where q_i is the quadratic form obtained by restricting the hermitian form h to the $\mathbb{Z}$-sublattice of $\mathcal{E}^{1,3}$ fixed by κ_i. Since the fixed lattice of $z \to \bar{z}$ in $\mathcal{E}$ is $\mathbb{Z} \subset \mathcal{E}$, and the fixed lattice of $z \to -\bar{z}$ in $\mathcal{E}$ is $\sqrt{-3}\,\mathbb{Z} \subset \mathcal{E}$, the fixed lattice of κ_i is a direct sum of $4-i$ copies of $\mathbb{Z}$ and i copies of $\sqrt{-3}\,\mathbb{Z}$, and restricting the hermitian form we obtain quadratic forms which are diagonal with one 1, and then -1 or -3 as the remaining diagonal entries. In summary, we get

Theorem 3.2. *The space $\mathcal{P}_i^{\mathbb{R}}/GL(2,\mathbb{R})$ is isomorphic to $\Gamma_i\backslash(H_i^3 - \mathcal{H})$, where $\Gamma_i = PO(q_i, \mathbb{Z})$ and q_i is the integral quadratic form given by*

$$\begin{aligned} q_0(x_0,\ldots,x_3) &= x_0^2 - \ \ x_1^2 - \ \ x_2^2 - \ \ x_3^2 \\ q_1(x_0,\ldots,x_3) &= x_0^2 - \ \ x_1^2 - \ \ x_2^2 - 3x_3^2 \\ q_2(x_0,\ldots,x_3) &= x_0^2 - \ \ x_1^2 - 3x_2^2 - 3x_3^2 \\ q_3(x_0,\ldots,x_3) &= x_0^2 - 3x_1^2 - 3x_2^2 - 3x_3^2\,. \end{aligned}$$

Note that the groups Γ_i fall into at least two commensurability classes, since the number of variables is even and the determinants of the forms fall into two classes in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, namely the determinants of q_0 and q_2 are in the square class of 1, while the determinants of q_1 and q_3 are in the square class of 3, see Corollary 2.7 of [5]. It is not hard to show, using either invariants of quadratic forms or explicit constructions, that Γ_0 and Γ_2 are commensurable, and that Γ_1 and Γ_3 are commensurable. Thus the Γ_i fall into exactly two commensurability classes.

4. The real moduli space is hyperbolic

Let us now consider the moduli space of stable real polynomials, namely $\mathcal{P}_{st}^{\mathbb{R}}/GL(2,\mathbb{R})$. The group $GL(2,\mathbb{R})$ acts properly with finite isotropy groups on $\mathcal{P}^{\mathbb{R}}_{st}$, thus $\mathcal{P}^{\mathbb{R}}/GL(2,\mathbb{R})$ is an orbifold. We state a uniformization theorem for this orbifold.

Theorem 4.1. *There is a lattice $\Gamma^{\mathbb{R}}$ acting on H^3 and a map of topological orbifolds $\Gamma^{\mathbb{R}}\backslash H^3 \to \mathcal{P}^{\mathbb{R}}/GL(2,\mathbb{R})$ which is a homeomorphism of the underlying spaces. The orbifold $\Gamma^{\mathbb{R}}\backslash H^3$ contains the orbifolds $\Gamma_i\backslash(H^3 - \mathcal{H})$ as open sub-orbifolds, and the map restricts to the isomorphisms of Theorem 3.2 on each of these open sub-orbifolds.*

Remark. It is not claimed that the map $\Gamma^{\mathbb{R}}\backslash H^3 \to \mathcal{P}^{\mathbb{R}}/GL(2,\mathbb{R})$ is an isomorphism of orbifolds. In fact, it is not. It fails to be so on the codimension two part of the discriminant. See [3] for a discussion of the precise relation between the two orbifold structures.

Proof. We briefly sketch the proof of this theorem. Let $\tilde{\mathcal{P}}^{\mathbb{R}}_{st}$ denote the pre-image of $\mathcal{P}^{\mathbb{R}}_{st}$ in $\tilde{\mathcal{P}}_{st}$. The first step is to show that $\tilde{\mathcal{P}}^{\mathbb{R}}_{st}/GL(2,\mathbb{R})$ immerses in $B^3 = \tilde{\mathcal{P}}_{st}$ (in the sense that the period map gives a local embedding), with image the union of a countable collection of totally geodesic real hyperbolic subspaces. Let K denote the space $\tilde{\mathcal{P}}^{\mathbb{R}}_{st}/GL(2,\mathbb{R})$ with the path metric induced from this immersion. There is a map $\Gamma\backslash K \to \mathcal{P}^{\mathbb{R}}_{st}/GL(2,\mathbb{R})$ which is a homeomorphism.

The next step is to prove that the metric space $\Gamma\backslash K$ has the structure of a real hyperbolic orbifold. This is achieved by providing local orbifold charts for $\Gamma\backslash K$. Namely, for each $x \in K$, we look at its stabilizer Γ_x in Γ, take a suitable Γ_x-invariant neighborhood U_x of x, and then find an open subset $V_x \subset H^3$ and a group G_x of isometries of H^3 leaving V_x invariant, so that $\Gamma_x\backslash U_x$ is isometric to $G_x\backslash V_x$. These orbifold charts are found by case by case analysis, depending on how singular K is at x. If x is a regular point of K, then it has a Γ_x-invariant neighborhood U_x isometric to an open set in H^3, and $U_x \to \Gamma_x\backslash U_x$ provides an orbifold chart. We next look at the generic singular point x of K. Its stabilizer $\Gamma_x = \mathbb{Z}/6$ and goes over into a complex reflection of order 6 in B^3. The point x goes to a point that lies on 6 different geodesic hyperbolic 3-spaces H^3 that interesect in a common H^2, A Γ_x-invariant neighborhood U_x is metrically the union of six open sets $W_0, \ldots, W_5$ in H^3 glued together along an open set W of an H^2. A generator of $\Gamma_x = \mathbb{Z}/6$ maps W_j to W_{j+2}; its third power reflects each W_j in the common W. Thus a fundamental domain is the union of "half" of one even-numbered W_j with "half" of one odd-numbered W_k, joined along their common W. This is isometric to an open set $V_x \subset H^3$, and V_x is a manifold chart at x, in other words, our desired orbifold is smooth at this point. Non-generic singular points of K require more care, but it is easy to classify them, and give orbifold charts in each case.

It is easy to check that this orbifold is complete. Thus the uniformization theorem [9] gives the existence of a discrete group $\Gamma^{\mathbb{R}}$ acting on H^3 with $\Gamma^{\mathbb{R}}\backslash H^3 = \Gamma\backslash K$. By the construction of the charts, it contains the orbifolds $\Gamma_i\backslash(H^3 - \mathcal{H})$ as open sub-orbifolds. Finally, one needs to check (but we do not do it here) that one gets indeed a map of orbifolds.

□

5. Non-arithmeticity

One way to prove that $\Gamma^{\mathbb{R}}$ is not arithmetic is to prove that it is essentially a Coxeter group (it contains a Coxeter subgroup of index 2), derive its Coxeter diagram, and then apply Vinberg's arithmeticity criterion [11]. Equivalently, one could, from the Coxeter diagram, derive a faithful representation of $\Gamma^{\mathbb{R}}$ as matrices with coefficients in $\mathbb{Z}[\sqrt{3}]$ and then apply the arithmeticity criterion of [4]. This approach is somewhat involved. See [3] for details of how to prove that $\Gamma^{\mathbb{R}}$ has a Coxeter sub-group of index two and how to derive its Coxeter diagram.

Instead, we look at a two-dimensional subspace of $H^2 \subset H^3$ whose stabilizer in $\Gamma^{\mathbb{R}}$ is a lattice, and prove that this lattice in H^2 is not arithmetic. This immediately implies that $\Gamma^{\mathbb{R}}$ is not arithmetic. Since it is easier to visualize fundamental domains in two dimensions, it is simpler to prove non-arithmeticity of $\Gamma^{\mathbb{R}}$ this way.

To this end, let's fix a *real* point $\infty \in \mathbb{C}P^1$ and define subspaces $\mathcal{P}_\infty \subset \mathcal{P}_{st}$ and $\mathcal{P}^{\mathbb{R}}_\infty \subset \mathcal{P}^{\mathbb{R}}_{st}$ to be the subspaces of polynomials with a double root at ∞ (and of course all other roots have multiplicity at most two). Let $G \subset GL(2,\mathbb{C})$ denote the stabilizer of ∞, and let $G^{\mathbb{R}}$ denote its intersection with $GL(2,\mathbb{R})$. The moduli space $\mathcal{P}_\infty/G$ was uniformized by the unit ball B^2 by Picard in [7], in one of the papers in which he started complex hyperbolic uniformization in dimensions greater than one. (To be strictly accurate, in this paper Picard uniformizes the branched cover corresponding to *ordered* points.) Picard's uniformization is the restriction of the Deligne-Mostow uniformization we use in this paper to an appropiate subspace. Namely, pick one irreducible component of the preimage of $\mathcal{P}_\infty$ in the branched cover $\tilde{\mathcal{P}}_{st} \to \mathcal{P}_{st}$ and denote it by $\tilde{\mathcal{P}}_\infty$. Then the period map used in the proof of Theorem 2.1 maps $\tilde{\mathcal{P}}_\infty/G$ isomorphically onto a totally geodesic sub-ball $B^2 \subset B^3$, namely one of the irreducible components of the collection of hyperplanes $\mathcal{H} \subset B^3$. This hyperplane is orthogonal to a vector $v \in \mathcal{E}^{1,3}$ with $h(v) = -1$. The sublattice of $\mathcal{E}^{1,3}$ orthogonal to v is an $\mathcal{E}^{1,2}$, hence the Eisenstein uniformization that Picard finds in [7].

The space $\mathcal{P}^{\mathbb{R}}_{\infty}$ has three connected components $\mathcal{P}^{\mathbb{R}}_{\infty,i}$, for $i = 0, 1, 2$. Arguing much as in the proof of Theorem 3.2 it is not hard to see that $\mathcal{P}^{\mathbb{R}}_{\infty,i}/G^{\mathbb{R}} = \Gamma_{\infty,i}\backslash(H^2_i - \mathcal{H})$ for a totally geodesic H^2_i and where $\Gamma_{\infty,i} = PO(q_{\infty,i}, \mathbb{Z})$ and $q_{\infty,i}$ is given by

$$\begin{aligned} q_0(x_0, x_1, x_2) &= x_0^2 - \ \ x_1^2 - \ \ x_2^2 \\ q_1(x_0, x_1, x_2) &= x_0^2 - \ \ x_1^2 - 3x_2^2 \\ q_2(x_0, x_1, x_2) &= x_0^2 - 3x_1^2 - 3x_2^2 \ , \end{aligned} \tag{5.1}$$

thus the same discussion one dimension lower. Similarly, a careful look at the proof of Theorem 4.1 shows that $\mathcal{P}^{\mathbb{R}}_{\infty}/G^{\mathbb{R}} = \Gamma^{\mathbb{R}}_{\infty}\backslash H^2$ for some subgroup $\Gamma^{\mathbb{R}}_{\infty} \subset \Gamma^{\mathbb{R}}$ acting on a totally geodesic $H^2 \subset H^3$. The equality sign is interpreted as in Theorem 4.1, namely a map of orbifolds from left to right that induces a homeomorphism of the underlying spaces. This orbifold contains the $\Gamma_{\infty,i}\backslash(H^2_i - \mathcal{H})$ as open sub-orbifolds.

It is easy to use Vinberg's algorithm [12] to find fundamental domains for the $\Gamma_{\infty,i}$. One finds that all three groups are Coxeter groups. The fundamental domain of $\Gamma_{\infty,0}$ is a $(2, 4, \infty)$-triangle, the fundamental domain of $\Gamma_{\infty,1}$ is a quadrilateral with three right angles and one vertex at infinity, and that of $\Gamma_{\infty,2}$ it is a $(2, 4, 6)$-triangle. Thus $\Gamma_{\infty,2}$ is co-compact while the others are not.

One starts the computation using integral vectors to define the sides of the fundamental domains. To compare the forms (5.1), it is easiest to go to $\mathbb{Z}[\sqrt{3}]$, since over this ring the three forms can be made standard: $-x_0^2 + x_1^2 + x_2^2$. One can then present each fundamental domain in the Klein model associated to the standard form as the convex hull of points in the unit disk with coordinates in $\mathbb{Q}(\sqrt{3})$. After possibly some motions of the domains to make them fit next to each other as they do in the orbifold $\Gamma^{\mathbb{R}}_{\infty}\backslash H^2$, the result is as shown in Figure 1.

In Figure 1, P_i is the fundamental domain of $\Gamma_{\infty,i}$ and the whole quadrilateral is the fundamental domain for $\Gamma^{\mathbb{R}}_{\infty}$. The group $\Gamma^{\mathbb{R}}_{\infty}$ is not a Coxeter group, It is generated by the reflections R_A and R_B in the sides A, B of the figure, and by the translation T (parabolic transformation) that fixes the point $(-1, 0)$ at infinity and takes the side C to the side D. Since clearly $TR_A = R_BT$, the transformations T and R_A suffice to generate the group. The sides C and D lie in the discriminant, while the sides A and B do not. This quadrilateral has piecewise geodesic maps to the complex hyperbolic plane B^2, with each P_i going into a totally geodesic H^2. (From this it is possible to construct homomorphisms from $\Gamma^{\mathbb{R}}$ to $PU(1, 2, \mathcal{E})$ and piecewise-geodesic equivariant maps from H^2 to B^2. The fact that $\Gamma^{\mathbb{R}}_{\infty}$ is not arithmetic implies that no

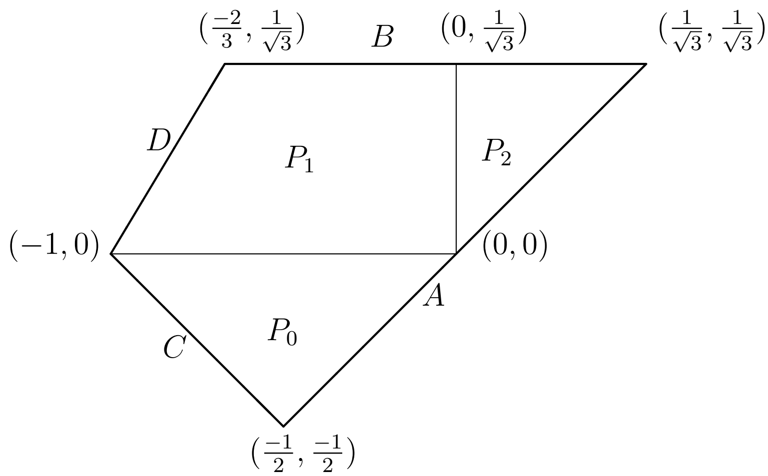

Figure 1

equivariant totally geodesic embedding is possible.) This quadrilateral has finite area, so $\Gamma_\infty^{\mathbb{R}}$ is a lattice in $PO(2,1)$.

It is easy to check that, in the coordinates x_0, x_1, x_2 that make the forms (5.1) standard, these transformations are given by the matrices

$$T = \begin{bmatrix} 3+\sqrt{3} & 2+\sqrt{3} & 1+\sqrt{3} \\ -2-\sqrt{3} & -1-\sqrt{3} & -1-\sqrt{3} \\ 1+\sqrt{3} & 1+\sqrt{3} & 1 \end{bmatrix}$$

and

$$R_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

thus the lattice $\Gamma_\infty^{\mathbb{R}} \subset PO(1,2,\mathbb{Z}[\sqrt{3}])$. We can now apply the arithmeticity criterion of Corollary 12.2.8 of [4], using for G the connected component of $PO(1,2) = SO(1,2)$, for $\mathbf{G}$ the algebraic group $\mathbf{SO}$ of the standard form above, $F = \mathbb{Q}(\sqrt{3})$, and the lattice $\Gamma_\infty^{\mathbb{R}} \cap G \subset G$. We need to check that the field generated by the traces of the matrices $\mathrm{Ad}\,\gamma$ is $\mathbb{Q}(\sqrt{3})$. It suffices to exhibit a transformation γ in the group so that $\mathrm{Tr}\,\mathrm{Ad}\,\gamma \notin \mathbb{Q}$. Let $\gamma = (TR_A)^2$ (squaring to get into the identity component G). One easily computes $\mathrm{Tr}\,\mathrm{Ad}\,\gamma = 18 + 8\sqrt{3} \notin \mathbb{Q}$. Thus there is a non-trivial Galois automorphism $\sqrt{3} \to -\sqrt{3}$, and the group of real points of the algebraic group obtained by Galois conjugating the defining equations is again the non-compact group $PO(1,2)$. This shows, by Corollary 12.2.8 of [4], that $\Gamma_\infty^{\mathbb{R}}$ is not arithmetic.

It may be instructive to look at the different pieces of this diagram. The groups associated to each P_i are actually defined over $\mathbb{Q}$, since they came from the integral quadratic forms (5.1), and one can see this from the diagram: a fundamental domain for the Galois conjugate group is obtained by taking the convex hull of the Galois conjugates of the vertices of each polygon. One sees easily that this process, applied to each P_i, yields a congruent polygon, hence the traces do not change, hence the field generated by the traces is $\mathbb{Q}$. But the same reasoning does not apply to the whole quadrilateral, and in fact the field of traces is larger than $\mathbb{Q}$.

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Department of Mathematics, University of Texas at Austin, Austin, TX 78712
E-mail address: allcock@math.utexas.edu
URL: http://www.math.utexas.edu/~allcock

Department of Mathematics, University of Utah, Salt Lake City, UT 84112; Clay Mathematics Institute, One Bow Street, Cambridge MA 02138,
E-mail address: carlson@math.utah.edu; carlson@claymath.org
URL: http://www.math.utah.edu/~carlson

Department of Mathematics, University of Utah, Salt Lake City, UT 84112
E-mail address: toledo@math.utah.edu
URL: http://www.math.utah.edu/~toledo