Géométrie algébrique/Algebraic geometry

A Complex Hyperbolic Structure for Moduli of Cubic Surfaces

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Abstract. We show that the moduli space M of marked cubic surfaces is biholomorphic to $(B^4 - \mathcal{H})/\Gamma_0$ where B^4 is complex hyperbolic four-space, where Γ_0 is a specific group generated by complex reflections, and where $\mathcal{H}$ is the union of reflection hyperplanes for Γ_0. Thus M has a complex hyperbolic structure, i.e., an (incomplete) metric of constant holomorphic sectional curvature.

Une structure hyperbolique complexe pour les modules des surfaces cubiques

Résumé. Nous montrons que l'espace des modules M des surfaces cubiques marquées est biholomorphe à $(B^4 - \mathcal{H})/\Gamma_0$ oú B^4 est l'espace complexe hyperbolique de dimension quatre, oú Γ_0 est un groupe spécifique généré par des réflections complexes, et oú $\mathcal{H}$ est l'union de l'ensemble d'hyperplans de réflection de Γ_0. Donc M admet une structure hyperbolique complexe, c'est à dire une métrique (incomplète) de courbure holomorphe sectionnelle constante.

Version française abrégée

A une surface cubique (marquée) correspond une variété cubique de dimension trois (marquée), à savoir le revêtement de $\mathbb{P}^3$ ramifié le long de la surface. L'application des périodes f pour ces variétés de dimension trois est définie sur l'espace des modules M des cubiques marquées, et cette application f prend ses valeurs dans une quotient de la boule unitaire dans $\mathbb{C}^4$ par l'action du groupe de monodromie projective. Ce groupe Γ_0 est généré par des réflections complexes dans un ensemble d'hyperplans dont nous notons la réunion par $\mathcal{H}$. Alors nous avons le resultat suivant:

Théorème. *L'application des périodes définit une biholomorphisme*

$$f : M \longrightarrow \left(B^4 - \mathcal{H}\right)/\Gamma_0.$$

De ce théorème on obtient des résultats sur la structure métrique de M et sur son groupe fondamental:

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Corollaires. *(1) L'espace M des modules des surfaces cubiques marquées admet une structure hyperbolique complexe: une métrique (incomplète) de courbure holomorphe sectionnelle constante. (2) Le groupe fondamental de M contient un sous-groupe normale qui n'est pas de génération finie. (3) Le groupe fondamental de M n'est pas un réseau dans un groupe de Lie semisimple.*

Remarques. (1) Nos méthodes montrent aussi que la complétée métrique de $(B^4 - \mathcal{H})/\Gamma_0$ est l'orbifolde B^4/Γ_0, isomorphe à l'espace des modules des surfaces cubiques marquées stables. (2) Récemment E. Looijenga a trouvé une présentation remarkable du groupe fondamental orbifolde de l'espace des modules des cubiques lisses non-marquées.

Afin de préciser la notion de surface cubique lisse marquée, fixons un réseau L, un $\mathbb{Z}$-module libre avec une base $e_0\ ,\ldots,\ e_6$ qui est muni de la forme quadratique telle que la base soit orthogonale et telle que $(\ e_0,\ e_0\ )\ = 1$, $(\ e_k,\ e_k\ )\ = -1$ pour $k > 0$. Soit $\eta = 3e_0 - (e_1 + \cdots + e_6)$. Alors une *surface cubique marquée* est composée d'une surface cubique lisse S et d'une isométrie $\psi : L \longrightarrow H^2(S, \mathbb{Z})$ qui envoie η sur la classe d'un hyperplan. L'ensemble M de classes d'isomorphisme des surfaces cubiques marquées porte la structure d'une variété et de plus est un éspace de modules fines. Une construction de cette éspace a été donnée dans [9], oú on trouve aussi une compactification lisse C de M telle que les points de $C - M$ forment un diviseur à croisements normaux.

Pour définir le groupe Γ_0, soit $\mathcal{E}$ l'anneau des entiers d'Eisenstein $\mathbb{Z}[\omega]$ oú $\omega = (-1 + \sqrt{-3})/2$ est une troisième racine d'unité, et considérons le produit Cartesien $\mathcal{E}^5$ muni d'une forme hermitienne $h(v,\ w)\ = -v_1\bar{w}_1 + v_2\bar{w}_2 + v_3\bar{w}_3 + v_4\bar{w}_4 + v_5\bar{w}_5$. Alors $(\mathcal{E}^5, h)$ est l'unique réseau autodual sur les entiers d'Eisenstein qui est de signature $(4, 1)$. Donc $Aut(\mathcal{E}^5, h)$ est un sous-groupe discrèt du groupe unitaire $U(h)$, qui agit sur $B^4 = \{\ \ell \in \mathbb{P}^4\ :\ h|\ell < 0\ \}$. Notons que $\mathcal{E}/\sqrt{-3}\mathcal{E} \cong \mathbb{F}_3$ est un corps de trois eléments et notons aussi qu'il y a un homomorphisme naturel $Aut(\mathcal{E}^5, h) \longrightarrow Aut(\mathbb{F}_3^5, q)$ oú q est la forme quadratique obtenue par réduction de h modulo $\sqrt{-3}$. Notons par "P" la projectivisation, et définissons un groupe Γ_0 d'automorphismes de B^4 par la suite exacte

$$1 \longrightarrow \Gamma_0 \longrightarrow PAut(\mathcal{E}^5, h) \longrightarrow PAut(\mathbb{F}_3^5, q) \longrightarrow 1.$$

Ce groupe est le groupe discrèt du théorème principal. Les hyperplans de $\mathcal{H}$ sont définis par les équations $h(x, v) = 0$ pour des vecteurs v dans $\mathcal{E}^5$ avec $h(v) = 1$. Notons aussi que $PAut(\mathbb{F}_3^5, q)$ est isomorphe au groupe de Weyl du réseau E_6.

1. Main results

To a (marked) cubic surface corresponds a (marked) cubic threefold defined as the triple cover of $\mathbb{P}^3$ ramified along the surface. The period map f for these threefolds is defined on the moduli space M of marked cubic surfaces and takes its values in the quotient of the unit ball in $\mathbb{C}^4$ by the action of the projective monodromy group. This group Γ_0 is generated by complex reflections in a set of hyperplanes whose union we denote by $\mathcal{H}$. Then we have the following result:

Theorem. *The period map defines a biholomorphism*

$$f : M \longrightarrow \left(B^4 - \mathcal{H}\right)/\Gamma_0.$$

From this identification we obtain results on the metric structure and the fundamental group:

Corollaries. *(1) The moduli space of marked cubic surfaces carries a complex hyperbolic structure: an (incomplete) metric of constant holomorphic sectional curvature. (2) The fundamental group of the space of marked cubic surfaces contains a normal subgroup which is not finitely generated. (3) The fundamental group of the space of marked cubic surfaces is not a lattice in a semisimple Lie group.*

Remarks. (1) Our methods also show that the metric completion of $(B^4 - \mathcal{H})/\Gamma_0$ is the complex hyperbolic orbifold B^4/Γ_0, which is isomorphic to the moduli space of marked stable cubic surfaces. (2) Recently E. Looijenga found a remarkable presentation of the orbifold fundamental group of the moduli space of smooth unmarked cubic surfaces.

To make precise the notion of smooth marked cubic surface, fix the lattice L to be the free $\mathbb{Z}$-module with basis $e_0, \ldots, e_6$ endowed with the quadratic form for which the given basis is orthogonal and such that $(e_0, e_0) = 1$, $(e_k, e_k) = -1$ for $k > 0$. Let $\eta = 3e_0 - (e_1 + \cdots + e_6)$. Then a *marked cubic surface* consists of a smooth cubic surface S and an isometry $\psi : L \longrightarrow H^2(S, \mathbb{Z})$ which carries η to the hyperplane class. The set M of isomorphism classes of marked cubic surfaces has the structure of a variety and is a fine moduli space. A construction of it is described in [9], and a smooth compactification C is given for which the points of $C - M$ constitute a normal crossing divisor.

To define the group Γ_0, let $\mathcal{E}$ denote the ring of Eisenstein integers $\mathbb{Z}[\omega]$ where $\omega = (-1 + \sqrt{-3})/2$ is a cube root of unity, and consider the Cartesian product $\mathcal{E}^5$ endowed with the hermitian inner product $h(v, w) = -v_1\bar{w}_1 + v_2\bar{w}_2 + v_3\bar{w}_3 + v_4\bar{w}_4 + v_5\bar{w}_5$. Then $(\mathcal{E}^5, h)$ is the unique self-dual lattice over the Eisenstein integers with signature $(4, 1)$. Thus $Aut(\mathcal{E}^5, h)$ is a discrete subgroup of the unitary group $U(h)$, which acts on $B^4 = \{ \ell \in \mathbb{P}^4 \ : \ h|\ell < 0 \}$. Observe that $\mathcal{E}/\sqrt{-3}\mathcal{E} \cong \mathbb{F}_3$ is a field of three elements and that there is a natural homomorphism $Aut(\mathcal{E}^5, h) \longrightarrow Aut(\mathbb{F}_3^5, q)$ where q is the quadratic form obtained by reduction of h modulo $\sqrt{-3}$. Let "P" denote projectivization, and define a group Γ_0 of automorphisms of B^4 by

$$1 \longrightarrow \Gamma_0 \longrightarrow PAut(\mathcal{E}^5, h) \longrightarrow PAut(\mathbb{F}_3^5, q) \longrightarrow 1.$$

This is the discrete group of the main theorem. The hyperplanes of $\mathcal{H}$ are defined by the equations $h(x, v) = 0$ for vectors v in $\mathcal{E}^5$ with $h(v) = 1$. Note that $PAut(\mathbb{F}_3^5, q)$ is isomorphic to the Weyl group of the E_6 lattice.

2. Construction of a period mapping

To construct the period mapping, we examine in detail the Hodge structures for the cubic threefolds. The underlying lattice $H^3(T, \mathbb{Z})$ is ten-dimensional, carries a unimodular symplectic form Ω, and admits a Hodge decomposition of the form $H^3(T, \mathbb{C}) = H^{2,1} \oplus H^{1,2}$. Choose a generator σ for the group of automorphisms of T over $\mathbb{P}^3$, and note that it operates without fixed points on $H^3(T, \mathbb{Z})$. This action gives $H^3(T, \mathbb{Z})$ the structure of a five-dimensional module over the Eisenstein integers. It carries a hermitian form

$$h(x, y) = \frac{1}{2}(\Omega((\sigma - \sigma^{-1})x, y) + (\omega - \omega^{-1})\Omega(x, y))$$

which is unimodular and of signature $(4,1)$.

Now consider the quotient module $H^3(T,\mathbb{Z})/(1-\omega)H^3(T,\mathbb{Z})$ and observe that it can be identified isometrically with $(\mathbb{F}_3^5, q)$. We define a marking of T to be choice of such an isometry, and we claim that a marking of a cubic surface determines a marking of the corresponding threefold. Indeed, if γ is a primitive two-dimensional homology class on S then it is the boundary of a three-chain Γ on T. Since Γ and $\sigma\Gamma$ have the same boundary, the three-chain $c(\gamma) = (1-\sigma)\Gamma$ is a cycle. However, it is well-defined only up to addition of elements $(1-\sigma)\Delta$ where Δ is a three-cycle on T. Thus a homomorphism

$$c : H_2^{prim}(S,\mathbb{Z}) \longrightarrow H_3(T,\mathbb{Z})/(1-\sigma)$$

is defined. Since a marking of S can be viewed as a basis of $H_2^{prim}(S,\mathbb{Z})$, application of c to the basis elements defines a basis of $H_3(T,\mathbb{Z})/(1-\sigma)$, and this gives the required marking of the threefold.

The action of σ decomposes $H^3(T,\mathbb{C})$ into eigenspaces H^3_λ where λ varies over the primitive cube roots of unity. Because σ is holomorphic, the decomposition is compatible with the Hodge decomposition and one has

$$H^3_\omega = H^{2,1}_\omega \oplus H^{1,2}_\omega \qquad H^3_{\bar\omega} = H^{2,1}_{\bar\omega} \oplus H^{1,2}_{\bar\omega}.$$

The dimensions of the Hodge components can be found with the help of Griffiths' Poincaré residue calculus [5]. Details for this case are found in [3], section 5. One finds that

$$\dim H^{2,1}_\omega = 4, \quad \dim H^{1,2}_\omega = 1, \qquad \dim H^{2,1}_{\bar\omega} = 1, \quad \dim H^{1,2}_{\bar\omega} = 4,$$

and from the Hodge-Riemann bilinear relations one finds that h has signature $(4,1)$.

Now let ϕ be a generator of the one-dimensional space $H^{2,1}_{\bar\omega}$ and let $\gamma_1, \ldots, \gamma_5$ be a standard basis of $H^3(T,\mathbb{Z})$ considered as an $\mathcal{E}$-module. By this we mean that the γ_k are orthogonal and that $h(\gamma_1,\gamma_1) = -1$ and $h(\gamma_k,\gamma_k) = 1$ for $k > 1$. Let $v(\phi,\gamma)$ be the vector in $\mathbb{C}^5$ with components

$$v_k = \int_{\gamma_k} \phi.$$

One verifies that $h(v,v) < 0$ where now h is the hermitian form $-|v_1|^2 + |v_2|^2 + \cdots + |v_5|^2$. Thus the line generated by $v(\phi,\gamma)$ defines a point in $B^4 \subset \mathbb{P}^4$, and one checks that $v(\phi,\gamma) \notin \mathcal{H}$. By well-known constructions (the work of Griffiths), the period vector defines a holomorphic map from the universal cover of M to the ball which transforms according to the projectivized monodromy representation for marked cyclic cubic threefolds. The proof that Γ_0 is the projective monodromy group relies on the work of Libgober [6] and the first author [1]. Thus our construction yields a period map $f : M \longrightarrow (B^4 - \mathcal{H})/\Gamma_0$.

3. Properties of the period mapping

We must now show that f is bijective. For injectivity, consider once again the period vector $v(\phi,\gamma)$. The vectors γ_k can be decomposed into eigenvectors γ_k' and γ_k'' for σ, with eigenvalues ω and $\bar\omega$, respectively. Let $\hat\gamma_k'$ and $\hat\gamma_k''$ denote elements of the corresponding dual basis. Because ϕ is an eigenvector with eigenvalue $\bar{\omega}$, its integral over γ'_k vanishes, so that

$$\phi = \sum_k \hat{\gamma}''_k \int_{\gamma''_k} \phi = \sum_k \hat{\gamma}''_k \int_{\gamma_k} \phi.$$

Thus the components of $v(\phi, \gamma)$ determine ϕ as an element of $H^3_{\bar{\omega}}$. Consequently the line $\mathbb{C}v(\phi, \gamma)$ determines the complex Hodge structure $H^3_{\bar{\omega}}$. Viewing the Hodge components of $H^3_{\bar{\omega}}$ as subspaces of $H^3(T, \mathbb{C})$, we may take their conjugates to determine the complex Hodge structure H^3_{ω}. These two complex Hodge structures determine the Hodge structure on $H^3(T, \mathbb{Z})$. Thus, by the Torelli theorem of Clemens-Griffiths [4], the period vector $v(\phi, \gamma)$ determines the cubic threefold T up to isomorphism. It remains to show that T, which perforce is a cyclic cubic threefold, determines its ramification locus uniquely. This follows from the fact that the locus in question is a planar component of the Hessian surface.

To prove surjectivity we first consider a smooth compactification C of M by a normal crossing divisor D, e.g., the one given by Naruki [9], as well as the Satake compactification $\overline{B^4/\Gamma_0}$, obtained by adding fourty points, the "cusps," each corresponding to a null point of $P(\mathbb{F}_3^5, q)$. By well-known results [2] in complex variable theory, the period map has a holomorphic extension to a map $\bar{f}$ from C to the Satake compactification. Since C is compact, $\bar{f}$ is open, and $\overline{B^4/\Gamma_0}$ is connected, we conclude that $\bar{f}$ is surjective.

4. Boundary components

To pass from surjectivity of $\bar{f}$ to surjectivity of f, we must show that $\bar{f}$ maps the compactifying divisor D to the complement of $(B^4 - \mathcal{H})/\Gamma_0$ in the Satake compactification. To this end write D as a sum of irreducible components, $D = \bigcup D'_i \cup \bigcup D''_j$, where D'_i parametrizes nodal cubic surfaces via the map to the geometric invariant theory compactification of the moduli space of smooth cubics, and where in the same way the D''_j parametrize cubics with an A_2 singularity.

Now consider a one-parameter family of cubic surfaces with smooth total space acquiring a node. Its local equation near the node has the form $x^2 + y^2 + z^2 = t$ and the corresponding family of cyclic cubic threefolds has the form $x^2 + y^2 + z^2 + w^3 = t$. The local monodromy of the latter has order six, its eigenvalues are primitive sixth roots of unity, and the space of vanishing cycles is two-dimensional. (These facts are well-known and the relevant literature and arguments are summarized in [3], section 6). From [7] we conclude that coefficients of the period vector on vanishing cycles are of the form $A(t)t^{1/6} + B(t)t^{5/6}$ where A and B are holomorphic. Now the space of vanishing cycles is invariant under the action of σ and so constitutes a rank one $\mathcal{E}$-submodule. One can choose a generator δ for it so that $h(\delta, \delta) = 1$, and then one has

$$\lim_{t \to 0} \int_\delta \phi = 0.$$

Thus the limiting value of the period vector lies in the orthogonal complement of δ. In other words, $\bar{f}(D'_i)$ lies in $\mathcal{H}/\Gamma_0$, as required.

Consider next a one-parameter family of cubic surfaces with smooth total space whose central fiber acquires an A_2 singularity. Its local equation is $x^2 + y^2 + z^3 = t$ and the corresponding family of cyclic cubic threefolds has local equation $x^2 + y^2 + z^3 + w^3 = t$. In this case the local monodromy is of infinite order. After replacing t by t^3 one finds an expansion of the form $\phi(t) = A(t)(\log t)\,\hat{\gamma} +$ (terms bounded in t), where $A(0) \neq 0$ and where $\hat{\gamma}$ is an integer cohomology class which is isotropic for h. Consequently the line $\mathbb{C}\phi(t)$ converges to the isotropic line $\mathbb{C}\hat{\gamma}$ as t converges to zero, hence converges to a cusp in the Satake compactification.

6. The corollaries

Finally, we comment on the collaries. Part (a) is immediate. For part (b) let K denote the kernel of the map $\pi_1(M) \longrightarrow \Gamma_0$. Then K is isomorphic to the fundamental group of $B^4 - \mathcal{H}$ and it is easy to see that its abelianization is not finitely generated. We remark that K is not free: there are many sets of commuting elements corresponding to normal crossings of $\mathcal{H}$. For (c) we note first that for lattices in semisimple Lie group of real rank greater than one, the results of Margulis [8] imply finite generation of all normal subgroups. The rank one case can be treated separately, as was shown to us by Michael Kapovich.

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