Mixed Hodge Structures and compactifications of Siegel's space

(Preliminary Report)

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1 Introduction

Let $\mathscr{H}_g$ be the Siegel upper half-space of order g, and let $\Gamma = \mathrm{Sp}(g, \mathbf{Z})$ be the Siegel modular group. The quotient $\mathscr{H}_g/\Gamma$ is a normal analytic variety, which we shall call *Siegel's space.* It admits certain natural compactifications; in particular, the minimal compactification of Satake [12] and the various toroidal compactifications of Mumford [10] (see also [1], [2], [13], [14]). The aim of this report is to describe a compactification which is natural from the point of view of mixed Hodge theory: The boundary components will be distinguished quotients of classifying spaces for mixed Hodge structures. This compactification is also good from the standpoint of curve degenerations: the limit point at the boundary detects the extension-theoretic part (cf. [3]) of the limiting mixed Hodge structure, and this in turn carries geometric information about the central fiber of the degeneration.

Although the compactification presented here for Siegel's space turns out to agree with Mumford's smooth compactification for arithmetic quotients of Hermitian symmetric spaces, we have, nevertheless, included a direct construction in the hope that this will clarify some of the subtler geometric aspects of the toroidal techniques, while at the same time lay a foundation for the study of Hodge-theoretic compactifications for classifying spaces of Hodge structures of higher weight. To this end, the emphasis of this report is on descriptions and motivations, rather than on the technical details. These will be supplied in a subsequent paper where we shall also discuss some important aspects of the compactification which are not treated here: analytic structure and extension of period mappings, among others.

The first two sections are of an introductory nature: after recalling some basic facts about degenerations of Hodge structure, Schmid's nilpotent orbit

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theorem and the limiting mixed Hodge structure, we analyze an example to show the geometric significance of the extension data and of the information given by the Satake compactification. In §4, we give a detailed description of the boundary components and pre-components, while in §5, we show that the latter are a C^∞-product of a Satake boundary component and a nilpotent group. Finally in section 6 we sketch the construction of the compactification, showing the simplicial data on which the compactification depends and indicating how the topology is defined.

2 Nilpotent orbits

In this section we explain how an abstract degeneration of Hodge structures in the Siegel space defines a nilpotent orbit of mixed Hodge structures with a distinguished semigroup (integral cone) of polarizations. We view these orbits as the natural "limit objects" with which one should try to compactify the Siegel modular space. As indicated in the next section, this is reasonable on geometric as well as formal grounds. In fact, we will show that an irreducible nodal rational curve Y is determined up to a group of order two by the nilpotent orbit associated to a degeneration with Y as central fiber.

We begin by establishing the basic notation. Let

$$\phi = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

be the standard symplectic form on $H_{\mathbf{Z}} = \mathbf{Z}^{2g}$, let Γ be the group of integral symplectic matrices, and let $\mathscr{H}_g$ be the Siegel upper half-space. Let Δ, Δ^* and $\mathscr{H}$ denote the unit complex disk, punctured disk, and complex upper half plane, respectively, and let

$$t : z \longrightarrow \exp(2\pi \mathrm{i} z)$$

represent the universal covering projection from $\mathscr{H}$ to Δ^*. Set $P = \Delta^n$, $P^* = \Delta^{*n}$, and $\overline{P} = \mathscr{H}^n$. Then a *local degeneration* is given by a holomorphic map

$$f : P^* \longrightarrow \mathscr{H}_g/\Gamma,$$

which is required to lift along the natural projections to

$$\tilde{f} : \tilde{P} \longrightarrow \mathscr{H}_g$$

Let $(\gamma, x) \mapsto \gamma x$ denote the action of $\pi_1(P^*)$ on $\tilde{P}$ and observe that $\tilde{f}(\gamma x)$ is another lifting of $\tilde{f}$. Consequently there is an element $\mu(\gamma) \in \Gamma$ such that

$$\tilde{f}(\gamma x) = \mu(\gamma)\tilde{f}(x).$$

This defines the *monodromy representation*

$$\mu : \pi_1(P^*) \longrightarrow \Gamma.$$

According to the monodromy theorem, the eigenvalues of $\mu(\gamma)$ are roots of unity. Consequently there is a minimal unramified cover $P^* \to P^*$ such that each $\mu(\gamma)$ for the induced representation is *unipotent*: its eigenvalues are all equal to unity.

For unipotent degenerations the standard power series defines the logarithm of a typical monodromy transformation T:

$$N = \log T = \sum \frac{(-1)^{j+1}}{j}(T-I)^j.$$

In the Siegel case, the monodromy theorem implies that $(T-I)^2 = 0$, so that $N = T - I$ is integral and $N^2 = 0$. Another special feature of the Siegel case, as we shall see later, is that

$$NN' = 0 \tag{$*$}$$

for any two monodromy elements. Now observe that $\phi(Tx, Ty) = \phi(x, y)$ implies that

$$\phi(Nx, y) = -\phi(x, Ny),$$

so that N is in the symplectic Lie algebra. Consequently the bilinear form

$$\phi_N(x, y) = \phi(x, Ny)$$

is symmetric, and its null space is precisely the kernel of N. Asymptotic estimates from Hodge theory then show that ϕ_N is positive:

$$\phi_N \geq 0.$$

(We always mean positive semidefinite, unless otherwise stated. Proofs of the positivity are given in [8], [9], [15].)

Let us now define the canonical semigroup associated to a local degeneration. Let ξ_j be the loop in the j-th punctured disk which turns once counterclockwise around the origin, with orientation defined by the complex structure. Then

$$\gamma_{\mathbf{Z}}(P^*) = \{\xi_1^{k_1} \cdots \xi_n^{k_n} \,|\, k_j \geq 0\}$$
$$\gamma_{\mathbf{Z}}^+(P^*) = \{\xi_1^{k_1} \cdots \xi_n^{k_n} \,|\, k_j > 0\}$$

define commutative semigroups, natural with respect to the homomorphisms induced by holomorphic maps. Thus we may define sets

$$\nu_{\mathbf{Z}}(f) = \log \mu\gamma_{\mathbf{Z}}(P^*)$$
$$\nu_{\mathbf{Z}}^+(f) = \log \mu\gamma_{\mathbf{Z}}^+(P^*),$$

and according to the Campbell-Hausdorff formula,

$$\log TT' = \log T + \log T' + [\log T, \log T'] + \cdots,$$

so that

$$\log TT' = \log T + \log T'$$

in view of the relation (∗). The sets just defined are therefore closed under addition, hence give natural semigroups. We remark that $\nu_{\mathbf{Z}}(f)$ is the convex integral hull of $N_1, \ldots, N_n$ where

$$N_j = \log T_j$$
$$T_j = \mu(\xi_j).$$

Consequently $\nu_{\mathbf{Z}}(f)$ has the structure of an integral polyhedral cone, with the N_j generating its edges.

Next, we define the weight filtration associated to f: set

$$W_2(N) = H_{\mathbf{Z}}$$
$$W_1(N) = \mathrm{kernel}(N)$$
$$W_0(N) = \mathrm{image}(N)$$

for any $N \in \nu_{\mathbf{Z}}(f)$. Since N lies in the symplectic Lie algebra, this filtration is self-dual with respect to ϕ: W_0 is the ϕ-orthogonal complement of W_1, and W_1 is the nullspace of ϕ_N.

Proposition 2.1. (1) *For any two* N, $N' \in \nu^+_{\mathbf{Z}}(f)$, *we have*

$$W_l(N) = W_l(N').$$

(2) *If* $N^+ \in \nu^+_{\mathbf{Z}}(f)$ *and* $N \in \nu_{\mathbf{Z}}(f)$, *then*

$$W_1(N^+) \subseteq W_1(N)$$

and

$$W_0(N^+) \supseteq W_0(N).$$

As a consequence of the proposition, $\nu^+_{\mathbf{Z}}(f)$ defines an unambiguous weight filtration.

To give the proof, set

$$N_\lambda = \sum \lambda_j N_j$$

where λ_j is a non-negative integer.

Lemma 2.2. (1) $\mathrm{kernel}(N_\lambda) = \bigcap_{\lambda_j > 0} \mathrm{kernel}(N_j)$

(2) $\mathrm{image}(N_\lambda) = \sum_{\lambda_j > 0} \mathrm{image}(N_j)$.

Proof. Certainly the intersection of the kernels is contained in the kernel of N_λ. To prove the reverse inclusion, suppose that $N_\lambda x = 0$, and observe that

$$0 = \phi(x, N_\lambda x) = \sum_{\lambda_j > 0} \lambda_j \phi(x, N_j x).$$

But the right hand side is a sum of non-negative terms, so $\phi(x, N_j x)$ must vanish for each j. Consequently x lies in the kernel of each N_j, as desired. The second assertion now follows from the first by duality.

Corollary 2.3.

(1) $N_i N_j = 0$

(2) $\nu_{\mathbf{Z}}(f)$ *is generated by the* N_j

(3) $NN' = 0$

Proof. (1) By the lemma,

$$\text{image}(N_i + N_j) \subseteq \text{kernel}(N_i + N_j) \subseteq (N_i).$$

Therefore $N_i(N_i + N_j) = 0$ which implies $N_i N_j = 0$.

(2) The Campbell-Hausdorff formula gives, by virtue of (1), the relation

$$\log T_1^{\lambda_1} \cdots T_n^{\lambda_n} = \lambda_1 N_1 + \cdots + \lambda_n N_n.$$

(3) This follows from (1) and (2).

The proposition now follows from the lemma and the second assertion of the corollary.

Let us denote the common weight filtration defined from $\nu_{\mathbf{Z}}^+$ by $W_*(f)$, and let us define the lattice of type $(1,1)$ elements

$$L^1(f) = W_2(f)/W_1(f).$$

Then the correspondence

$$N \longmapsto \phi_N$$

maps $\nu_{\mathbf{Z}}(f)$ injectively to a semigroup of positive forms on $L^1(f)$, which we shall denote by $\sigma_{\mathbf{Z}}(f)$. It is an integral polyhedral cone and $\sigma_{\mathbf{Z}}^+(f)$ is the infinitely generated sub-cone of definite elements.

Finally, we examine the "asymptotic Hodge structure": Let $F^1(z)$ be the filtration on $H_{\mathbf{C}}$ defined by $\tilde{f}(z) \in \mathscr{H}_g$. Then Schmid's nilpotent orbit theorem asserts that

$$F^1_\infty = \lim_{\text{Im}(z_j) \to \infty} \exp(-N_z) F^1(z)$$

defines a filtration on $H_{\mathbf{C}}$ provided that the real parts of the z_j remain within suitable bounds as the limit is taken. Note, however, that the limit does depend on the parametrization: multiplication of t_j in the j-th punctured disk by $\exp(2\pi\mathrm{i}\alpha_j)$ results in multiplication of the limit filtration by $\exp \alpha_j N_j$. Consequently the object naturally defined by the degeneration is not a single filtration, but rather a nilpotent orbit of such:

$$\mathcal{N}(f) = \{(\exp N) F^1_\infty \mid N \in \nu_{\mathbf{C}}\},$$

where in general

$$\nu_\Lambda = \{\Lambda - \text{module spanned by the semigroup } \nu\}.$$

Remark 2.4. The nilpotent orbit theorem states further that the two holomorphic maps $\tilde{f}$ and

$$\tilde{\nu}(z) = \exp(N_z) F^1_\infty$$

are asymptotic in the sense that for any $\mathrm{Sp}(g, \mathbf{R})$-invariant distance on $\mathscr{H}_g$, we have

$$d(\tilde{\nu}(z), \tilde{f}(z)) = 0 \left(\sum |t_j|\right),$$

where

$$t_j = \exp(2\pi\mathrm{i} z_j).$$

The fact that arbitrary degenerations lift to maps asymptotic to nilpotent orbits is further motivation for the use of the latter as boundary points for $\mathscr{H}_g$.

We claim next that the degeneration f defines a nilpotent orbit of Mixed Hodge structures:

Proposition 2.5. *Let f be a local unipotent degeneration of Hodge structures in the Siegel space. Then the triple $(W_*(f), F^1, \Lambda)$ defines a polarized mixed Hodge structure for any F^1 in $\mathcal{N}(f)$ and any $\Lambda \in \sigma^+_{\mathbf{Z}}(f)$.*

Proof. Because $\exp(N_\alpha)$ acts by the identity on W_2/W_1 and on W_1, it suffices to show that F^1 defines a mixed Hodge structure for some $F^1 \in \mathcal{N}(f)$. To see this, observe that

$$t \longmapsto f(t_1^{\lambda_1}, \ldots, t_n^{\lambda_n})$$

defines a one-variable degeneration with monodromy transformation $\exp(N_\lambda)$. Application of the SL_2-orbit theorem to this degeneration shows that F^1_∞ defines a mixed Hodge structure, polarized with respect to $\phi_{N_\lambda} \in \sigma^+_{\mathbf{Z}}$. Since any element of $\sigma^+_{\mathbf{Z}}$ arises from a one-variable degeneration in this way, the proposition is proved.

In order to give the geometric applications, let us translate this result into the language of one-motifs. A *one-motif* is a homomorphism

$$u : L \longrightarrow J$$

of a lattice into an extension of an abelian variety by a complex multiplicative group $(\mathbb{C}^*)^r$. According to a theorem of Deligne, there is an equivalence of categories between mixed Hodge structures of level one and one-motifs (cf. [6, III, Section 10]). Thus, we know from general principles that the nilpotent orbit of mixed Hodge structures constructed above is naturally equivalent to a nilpotent orbit of one-motifs. To describe it, recall that to any mixed Hodge structure H are associated canonical lattices and generalized Jacobians, defined by

$$L^p H = (Gr^W_{2p})^{p,p}_{\mathbf{Z}}$$
$$J^p H = W_{2p-1,\mathbf{C}}/(F^p W_{2p-1} + W_{2p-1,\mathbf{Z}}).$$

The mixed Hodge structure then gives a homomorphism

$$u : L^p H \longrightarrow J^p H$$

which carries the structure of a one-motif if it has level one (cf. [3, section 2]).

In the case of degenerations in the Siegel space, the above construction applies, with $p = 1$, to each structure in the nilpotent orbit, giving a one-motif

$$u_f : L^1(f) \longrightarrow J^1(f).$$

The domain and range are independent of the structure chosen from the orbit because N_α acts by zero on both W_1 and $H_{\mathbf{Z}}/W_1$. On the other hand, it defines a non-zero homomorphism from $H_{\mathbf{Z}}/W_1$ to $W_0 \otimes \mathbf{C}$, hence defines a one-motif

$$N_\alpha : L^1(f) \longrightarrow J^1(f).$$

Observe that it takes values in the multiplicative group of the extension, namely

$$J^1 W_0 = \frac{W_{0,\mathbf{C}}}{W_{0,\mathbf{Z}}} \cong (\mathbf{C}^*)^r,$$

where $r = \operatorname{rank} W_{0,\mathbf{Z}}$. In conclusion, *a degeneration of Hodge structures in the Siegel space defines a canonical orbit of one-motifs* $\mathcal{U}(f)$ *which is homogeneous under the effective action* $u \mapsto u + N$*, where* $N \in \nu_{\mathbf{C}}$.

Remark 2.6. The complete formal object $\mathcal{U}$ defined by a degeneration consists of

(i) A family of one-motifs

$$u : L^1 \longrightarrow J^1.$$

(ii) A polyhedral cone $\sigma_{\mathbf{Z}}$ of positive integral forms on L^1.

(iii) A perfect pairing

$$\phi : L^1 \times L^0 \longrightarrow \mathbf{Z}$$

where $L^0 = W_0 H_{\mathbf{Z}}$.

The perfect pairing is that defined by cup-product; it defines an isomorphism of L^0 with the dual of L^1, hence defines an isomorphism between the space of bilinear forms on L^1 and the space of homomorphisms from L^1 to L^0. In this way σ itself defines the canonical action of ν upon the one-motifs of the nilpotent orbit.

3 Curve degenerations

3a. The polyhedral cone

Consider now a local degeneration of curves

$$f : \mathscr{X} \longrightarrow P \times P'$$

smooth above $P^* \times P'$ and with Y as central fiber. Any such degeneration is the pullback of the versal deformation of Y, up to a possible preliminary base extension. Consequently all polyhedral cones $\sigma(f)$ arise as subobjects of the "versal cone" $\sigma(Y)$. Our aim in this section is to give a direct description of the versal cone when Y is stable.

To this end, recall that Y has a canonical Mayer-Vietoris sequence,

$$0 \longrightarrow H_1(\tilde{Y}) \xrightarrow{\pi_*} H_1(Y) \xrightarrow{\partial} H_0(\tilde{\Sigma}) \xrightarrow{\delta_*} H_0(\tilde{Y}) \oplus H_0(\Sigma) \longrightarrow H_0(Y) \longrightarrow 0.$$

Here $\pi : \tilde{Y} \to Y$ is the normalization, Σ is the singular locus, $\tilde{\Sigma} = \pi^{-1}(\Sigma)$, and $\delta_* = I_* \oplus (-\pi_*)$. Define groups

$$\mathscr{V}_Y = \text{kernel}\{\pi_* : H_0(\tilde{\Sigma}) \longrightarrow H_0(\Sigma)\}$$
$$\mathscr{Z}_Y = \text{kernel}\{i_* : \mathscr{V}_Y \longrightarrow H_0(\tilde{Y})\}$$

and filter $H_1(Y)$ by

$$W_0 = H_1(Y, \mathbf{Z})$$
$$W_{-1} = \pi_* H_1(\tilde{Y}, \mathbf{Z}).$$

Then the boundary operator gives an isomorphism

$$L^0 H_1(Y) \longrightarrow \mathscr{Z}_Y.$$

Note that $\mathscr{Z}_Y$ consists of zero-cycles on $\tilde{\Sigma}$ which have a degree zero on each component of Y and on each fiber of π.

Next, define a bilinear form Λ_Y on $H_0(\tilde{\Sigma})$ such that

$$\Lambda_Y(x,x) = \frac{1}{2}$$
$$\Lambda_Y(x,x') = 0$$

for arbitrary points $x, x' \in \tilde{\Sigma}$, provided that $x \neq x'$. Set

$$\hat{x} = x' - x''$$

and observe that the $\hat{x}$ give an orthonormal basis for $\mathscr{V}_Y$. The restriction of Λ_Y to $\mathscr{Z}_Y$ gives the *distinguished polarization* on the weight zero component of H_1.

Define also the rank-one forms Λ_x by

$$\Lambda_x(\xi,\xi') = \Lambda_Y(\hat{x},\xi)\Lambda_Y(\hat{x},\xi'),$$

and let $\Lambda_i = \Lambda_{x_i}$.

Proposition 3.1. *The versal cone $\sigma(Y)$ is naturally isomorphic to the cone of positive forms on $\mathscr{Z}_Y$ generated by the Λ_i, and the distinguished polarization is given by*

$$\Lambda_Y = \sum \Lambda_i.$$

Remark. Define the subordinates of Λ to be the integral forms satisfying

$$0 \leq \Lambda' \leq \Lambda.$$

Let $\sigma(\Lambda)$ be the cone generated by the subordinates of Λ. It is evident from the proposition that Λ_x is a subordinate to Λ_Y, so that

$$\sigma(Y) \subseteq \sigma(\Lambda_Y).$$

In many, but not all cases, equality holds, giving an arithmetic characterization of $\sigma(Y)$ in terms of the polarization.

Example 1. Suppose Y is irreducible. Then $\mathscr{V}_Y = \mathscr{Z}_Y$, so that in the basis $\hat{x}_i$, Λ_Y is given by the identity matrix. The rank-one subordinates are exactly the Λ_i, which are represented by diagonal matrices with a one in the i-th position.

Example 2. Let Y be the stable model of an irreducible curve with an ordinary triple point:

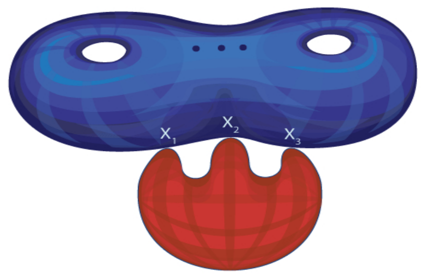

Fig. 1

Then $\hat{x}_2 - \hat{x}_1$ and $\hat{x}_3 - \hat{x}_2$ generate $\mathscr{Z}_Y$, and in that basis

$$\Lambda_Y = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

the matrix of the Dynkin diagram A_2. The forms associated to the double points are

$$\Lambda_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \Lambda_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \Lambda_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

and these are precisely the subordinates of Λ_Y.

The class of stable curves for which $\sigma(Y) = \sigma(\Lambda_Y)$ is large but not exhaustive: it includes the stable models of all irreducible curves with ordinary multiple points, but it excludes curves such as the one below:

Fig.2

In this case the distinguished polarization is twice the identity, and its four subordinates are the three matrices listed above, together with the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

On the other hand, $\sigma(Y)$ is given by the diagonal matrices with non-negative integer entries.

Proof of the proposition:

(1) *Generic degenerations.* Let Y be stable with double points $x_1, \dots, x_r$. We shall say that $f : \mathscr{X} \to \Delta^r$ is *generic* if (i) $\mathscr{X}$ is smooth (ii) there are functions u_j, v_j on $\mathscr{X}$ such that Y is given near x_j by $u_j v_j = t_j$, where t_j is a coordinate for the j-th factor of Δ^r. The cone of a generic deformation is isomorphic to the versal cone; indeed the versal fiber space is homotopy equivalent to that of a generic deformation. We may therefore confine our calculations to generic deformations.

(2) *Transport of $\sigma(Y)$ to $\mathscr{Z}(Y)$.* The cone $\sigma(Y)$ is defined on the lattice of integral $(1,1)$ cohomology,

$$L^1H^1(X_t) = H^1(X_t, \mathbf{Z})/W_1.$$

Cup-product gives a perfect pairing of L^1 with the type $(0,0)$ lattice,

$$L^1H^1 \times L^0H^1 \longrightarrow \mathbf{Z},$$

hence an isomorphism of L^1H^1 with the dual of L^0H^1, that is, with $L^0H_1(X_t)$. Because $\mathscr{X}$ is generic, there is a deformation retraction

$$k : \mathscr{X} \longrightarrow Y$$

which induces a morphism

$$k_* : H_1(X_t) \longrightarrow H_1(Y).$$

Its kernel is W_{-2}, and it induces an isomorphism on the level of L^0. The composition

$$L^1H^1(X_t) \longrightarrow L^0H_1(X_t) \xrightarrow{k_*} L^0H_1(Y) \xrightarrow{\partial} \mathscr{Z}_Y$$

yields an identification by means of which we may view $\sigma(Y)$ as naturally defined on $\mathscr{Z}_Y$.

(3) *The Lefschetz formula.* The collapsing map k can be chosen so that

$$\delta_j = k^{-1}(x_j) \cap X_t$$

is an imbedded circle, the *vanishing cycle* attached to x_j. The monodromy transformation associated with the loop δ_j is then

$$T_j(\gamma) = \gamma + \phi(\delta_j, \gamma)\delta_j,$$

and its logarithm is

$$N_j(\gamma) = \phi(\delta_j, \gamma)\phi_j.$$

Consequently W_{-2} is generated by the vanishing cycles. Since Y is obtained, up to homotopy, by attaching disks to X_t along the δ_j, this shows that W_{-2} is indeed the kernel of k_*.

(4) *Completion of the proof.* The positive form associated to N_j is

$$\phi_j(\gamma, \gamma') = \phi(\delta_j, \gamma)\phi(\delta_j, \gamma').$$

Thus it suffices to show that

$$\phi_j(\gamma, \gamma') = \Lambda_j(\delta k_* \gamma, \delta k_* \gamma').$$

Equivalently, we must show that

$$\phi(\delta_j, \gamma) = \pm\, \Lambda_Y(\hat{x}_j, \partial k_* \gamma). \qquad (*)$$

But this is clear: First, suppose that γ is a loop which meets δ_j transversely in a single point. Then $k_*\gamma$ is a cycle whose boundary is $\pm\,\hat{x}_j$, so that both sides of $(*)$ equal $\pm\,1$. Second, suppose that γ does not meet δ_j. Then $k_*\gamma$ does not meet x_j, so both sides of $(*)$ vanish. Since $H_1(Y, \mathbf{Z})$ is generated by cycles satisfying one of the above cases, the proof is complete.

Remarks 3.2. (1) For stable curves, the group $\mathscr{Z}_Y$ is naturally isomorphic to $H_1(\Gamma)$, where Γ is the dual graph of Y (a vertex for each component, an edge for each double point). Indeed, we can construct a natural exact commutative diagram in which all vertical maps are isomorphisms:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathscr{Z}_Y & \longrightarrow & \mathscr{V}_Y & \xrightarrow{i_*} & H_0(\tilde{Y}) & \longrightarrow & H_0(Y) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_1(\Gamma) & \longrightarrow & C_1(\Gamma) & \longrightarrow & C_o(\Gamma) & \longrightarrow & H^0(\Gamma) & \longrightarrow & 0 \end{array}$$

It is therefore easy to compute Λ_Y and the Λ_i from an orientation of Γ : If ξ, ξ' define simple closed curves in Γ, then

$$\Lambda_Y(\xi, \xi) = \text{ number of edges in } \xi$$
$$\Lambda_Y(\xi, \xi') = \sum \epsilon_i(\xi, \xi').$$

The symbol ϵ_i is $\pm\,1$ if the i-th edge is common to both cycles and is zero otherwise. The plus sign is taken if ξ and ξ' assign the same orientation to the i-th edge, the minus sign is taken in the contrary case. Finally, each edge defines a generator of $\sigma(Y)$ by $\Lambda_i = \epsilon_i$.

(2) We say that $\sigma(Y)$ *separates* ξ and ξ' in $\mathscr{Z}_Y$ if $\Lambda(\xi, \xi') = 0$ for all $\Lambda \in \sigma(Y)$. Zero-cycles are separated by $\sigma(Y)$ if and only if their supports are disjoint. This fundamental property of the versal cone is essential to our applications.

3b. The one-motifs

We shall give a direct interpretation of the nilpotent orbit associated to a curve degeneration. To do this, observe first that an element in $F^1H^1(X_t)$, for any mixed Hodge structure in the orbit, is represented by a logarithmic differential on $\tilde{Y}$ whose residue cycle lies in $\mathscr{Z}_Y \otimes \mathbf{C}$. Consequently there is a natural isomorphism

$$\mathrm{res} : L^1(f) \longrightarrow \mathscr{Z}_Y .$$

Next, observe that the collapsing map gives a canonical isomorphism

$$k^* : J^1(f) \longrightarrow J^1(Y),$$

where

$$J^1(Y) = \frac{H^1(Y, \mathbf{C})}{F^1H^1(Y) + H^1(Y, \mathbf{Z})}$$

is the Hodge-theoretic Jacobian. Consequently we may transpose the original nilpotent orbit of one-motifs to the $\sigma(Y)$-orbit

$$\{u_Y\} : \mathscr{Z}_Y \longrightarrow J^1(Y)$$

using the correspondence

$$u_Y : k^{*-1} \circ u_f \circ \mathrm{res}^{-1} .$$

To make sense out of the values of u_Y, we must introduce the following groups:

$$M(\xi) = \{\gamma \in H_1(Y, \mathbf{Z}) \,|\, \Lambda(\partial\gamma, \xi) = 0 \text{ for all } \Lambda \in \sigma_{\mathbf{C}}(Y)\}$$
$$\sigma_{\mathbf{C}}(\xi) = \{\text{the linear functionals } \gamma \to \Lambda(\partial\gamma, \xi),\ \Lambda \in \sigma_{\mathbf{C}}(Y)\}.$$

Because $\sigma_{\mathbf{C}}(\xi)$ annihilates the kernel of ∂, it lies in $W_0H^1(Y)$. From this one derives the exact sequence of mixed Hodge structures

$$0 \longrightarrow \sigma_{\mathbf{C}}(\xi) \longrightarrow H^1(Y, \mathbf{C}) \longrightarrow M^*(\xi) \otimes \mathbf{C} \longrightarrow 0.$$

Passing to Jacobians, we obtain

$$0 \longrightarrow J^1\sigma_{\mathbf{C}}(\xi) \longrightarrow J^1(Y) \longrightarrow J^1M^*(\xi) \longrightarrow 0,$$

where

$$J^1\sigma_{\mathbf{C}}(\xi) = \sigma_{\mathbf{C}}(\xi)/\sigma_{\mathbf{Z}}(\xi)$$

is a subgroup of the multiplicative part of the canonical extension of $J^1(Y)$. *The point of all this is that*

$$u_Y(\xi) \in J^1M^*(\xi)$$

is well-defined, independent of the motivic homomorphism chosen from the orbit.

Let us therefore calculate $u_Y(\xi)$: Choose a logarithmic differential ω_ξ whose residue is ξ and define a linear functional by integration:

$$u_\xi : \gamma \longrightarrow \int_\gamma \omega_\xi .$$

By the final remark of the previous section, cycles in $M(\xi)$ can be moved off the polar locus of ω_ξ, so that the integral converges. However, because the value of the integral depends on the homology class of γ in Y minus the support of ξ, it is well-defined only up to a sum of residues, hence up to an integer. It follows that u_ξ is well-defined in the multiplicative group $M^*_{\mathbf{C}}(\xi)/M^*_{\mathbf{Z}}(\xi)$. Furthermore, the differential in ω_ξ is defined by its residue only up to abelian differentials on Y-elements of $F^1H^1(Y)$. Therefore u_ξ determines an element of the generalized torus $J^1M^*(\xi)$ which is independent of all choices. To summarize, we have obtained the following result:

Proposition 3.3. *Let Y be a stable curve, $\{u_Y\}$ the canonical nilpotent orbit of a generic degeneration, transposed to Y. Then $u_Y(\xi)$, projects to u_ξ, the linear functional given by integration, in the canonical quotient torus associated to ξ.*

Remark 3.4. In certain cases we can interpret u_ξ as giving a set of cross-ratios on Y. To explain this, note first that the canonical extension associated to $J^1(Y)$ is isomorphic to

$$0 \longrightarrow W_0J^1(Y) \longrightarrow J^1(Y) \xrightarrow{\pi^*} J^1(\tilde{Y}) \longrightarrow 0$$

where $J^1(\tilde{Y})$ is the usual Jacobian. Therefore ξ is linearly equivalent to zero precisely when u_ξ lies in the multiplicative group W_0J^1. In this case we may write

$$u_\xi = \mathrm{d}\log f_\xi / 2\pi\mathrm{i}$$

for a suitable meromorphic function on $\tilde{Y}$. Next, let $\langle\, ,\, \rangle$ denote the canonical pairing which integration gives between $W_0H^1(Y)$ and $\mathscr{Z}_Y \cong Gr_0H_1(Y, \mathbf{Z})$, and let η be a primitive element of $\mathscr{Z}_Y$. Then the map defined on $W_0H_1(Y, \mathbf{C})$ by

$$\alpha \longmapsto \exp(2\pi\mathrm{i}\langle\alpha, \eta\rangle)$$

produce a homomorphism

$$\eta : J^1W_0 \longrightarrow \mathbf{C}^*.$$

If η is separated from ξ, we may therefore write

$$\langle u_\xi, \eta\rangle = \exp \int_{\partial^{-1}\eta} \mathrm{d}\log f_\xi,$$

where $\partial^{-1}\eta$ is any one-cycle on Y whose boundary is η. The fundamental theorem of calculus then yields

$$\langle u_\xi, \eta\rangle = \prod_p f_\xi(p)^{\nu_p(\eta)}$$

where $\nu_p(\eta)$ is the multiplicity of p in η. Moreover, the change-of-variables theorem in the calculus, coupled with the essential uniqueness of f_ξ, implies that $\langle u_\xi, \eta\rangle$ is a projective invariant of Y. Finally, if $\tilde{Y} \cong \mathbb{P}_1, \xi = b - a$, and $\eta = d - c$, then $\langle U_\xi, \eta\rangle$ is the cross-ratio of the four points a, b, c, d. In fact, let t be the unique coordinate such that $t(a) = \infty, t(b) = 0$ and $t(c) = 1$. Then $f_\xi = t$ and $\langle u_\xi, \eta\rangle = d$, which is precisely the asserted cross-ratio.

Let us end this discussion with an application to a Torelli-type problem.

Proposition 3.5. *Let Y be an irreducible rational curve with r nodes as its only singularities, and let $\mathscr{N}(Y)$ be the nilpotent orbit of a generic degeneration with Y as central fiber.*

(1) *If $r = 2$, then $\mathscr{N}(Y)$ determines Y up to isomorphism.*

(2) *If $r = 3$, then the map $Y \longmapsto \mathscr{N}(Y)$ is generically two-to-one.*

(3) *If $r \geq 4$, then the map $Y \mapsto \mathscr{N}(Y)$ is generically one-to-one.*

Proof. (1) Let $x_1, \ldots x_r$ be nodes of Y, and let $\pi^{-1}(x_i) = (a_i, b_i)$ in $\tilde{Y}$. The pairs (a_i, b_i), which depend on $2r-3$ projective moduli, determine Y. We claim first that $\mathscr{N}(Y)$ determines the cross-ratios

$$\mu_{ij} = \frac{b_j - b_i}{b_j - a_i} \cdot \frac{a_j - a_i}{a_j - b_i}$$

in a canonical way, and second that the degree of

$$\kappa : \frac{\{(a_1, b_1); \ldots; (a_r, b_r)\}}{\text{Action of } \mathbf{PGL}(1)} \longrightarrow \{(\mu_{ij}) \big| i > j\}$$

is at most two:

Lemma 3.6. *The map κ is an isomorphism if $r = 2$, is generically two-to-one onto its image if $r = 3$ and is generically one-to-one onto its image if $r > 3$.*

The proof follows jointly from the two claims.

(2) *Proof of first claim.* Recall that $\mathscr{N}(Y)$ gives a cone $\sigma(Y)$ of polarizing forms, and that its central element $\sum \Lambda_i$ is the distinguished polarization, Λ_Y. Furthermore, a Λ_Y-orthonormal basis $\{\xi_i\}$ for the weight-zero lattice may be identified with a basis $\{b_i - a_i\}$ for $\mathscr{Z}_Y$, given a suitable ordering of the nodes and the corresponding fibers of π. By the remarks preceding the proposition, we obtain

$$\mu_{ij} = \langle u_{\xi_i}, \xi_j\rangle$$

as desired.

(3) *Proof of second claim.* If $k=2$, there is only one cross-ratio, and this is certainly and invariant of the quadruple (a_1, b_1, a_2, b_2). If $k > 2$, we attempt to construct an inverse for κ as follows: First, move (a_1, b_1, a_2) to $(\infty, 0, 1)$ by the unique projective transformation which does this. Determine b_2 by the equation

$$b_2 = \mu_{21}.$$

Second, adjoin the $2r-4$ equations

$$\frac{b_k}{a_k} = \mu_{k1}$$

$$\frac{b_k - \mu_{21}}{b_k - 1} \cdot \frac{a_k - 1}{a_k - \mu_{21}} = \mu_{k2},$$

where $k \geq 3$.

Each pair, consisting of a linear and a quadratic equation determines a pair of conjugate solutions (a_j, b_j) and $(\overline{a}_j, \overline{b}_j)$ which have the indicated cross-ratios. If $r=3$, there is nothing left to do. If $r > 3$, we must consider the supplementary equations imposed by the cross-ratios μ_{ij}, where $j > 3$ and $i > 2$. As the reader may verify with a modest amount of calculation, these are not satisfied generically by any choice of solutions other than $\{(a_i, b_i)\}$. This completes the proof.

4 Polyhedral cones and boundary components

In this section we present a Hodge-theoretic construction of the rational boundary components of $\mathscr{H}_g$. As mentioned in the introduction, these will be classifying spaces for nilpotent orbits of mixed Hodge structures, or equivalently, of one-motifs.

As before, we fix a rank $2g$ lattice $H_{\mathbf{Z}}$ and an integral, nonsingular skew form ϕ. Then the Siegel space classifies ϕ-polarized Hodge structures:

$$\mathscr{H}_g = \{F^1 \in \mathrm{Grass}(g, H_{\mathbf{C}}) \,|\, \phi(F^1, F^1) = 0, \mathrm{i}\phi(F^1, \overline{F}^1) > 0\}.$$

Its compact dual is

$$\check{\mathscr{H}}_g = \{F^1 \in \mathrm{Grass}(g, H_{\mathbf{C}}) \,|\, \phi(F^1, F^1) = 0\},$$

a nonsingular subvariety of the Grassmannian, homogeneous under the complex symplectic group

$$G_{\mathbf{C}} \cong Sp(H_{\mathbf{C}}, \phi).$$

Moreover $\mathscr{H}_g$ sits inside $\check{\mathscr{H}}_g$ as an open set, homogeneous under the action of the real form

$$G = \{g \in G_{\mathbf{C}} \,|\, g(H_{\mathbf{R}}) = H_{\mathbf{R}}\}.$$

Now let

$$\mathfrak{g}_0 = \{X \in gl(H_{\mathbf{R}}) \big| \, \phi(Xu, v) + \phi(u, Xv) = 0\},$$

let

$$\{0\} = S_0 \subset S_1 \subset \cdots \subset S_g$$

be a maximal flag of rational ϕ-isotropic subspaces of $H_{\mathbf{R}}$, and define abelian subalgebras of $\mathfrak{g}_0$ by

$$\eta_i = \{N \in \mathfrak{g}_0 \big| \, \mathrm{Im} N \subseteq S_i\}.$$

For each $N \in \eta_i$, the form ϕ_N is symmetric and its null space contains the ϕ-orthogonal complement $S_i^{\perp}$ of S_i. We can therefore consider the open real cone

$$\eta_i^+ = \{N \in \eta_i \big| \, \phi_N > 0 \text{ on } H_{\mathbf{R}}/S_i^{\perp}\}$$

and its closure $\mathrm{Cl}(\eta_i^+)$.

Given linearly independent elements $N_1, \ldots, N_r$ in $\mathrm{Cl}(\eta_i^+)$, let

$$\sigma = \sigma(N_1, \ldots, N_r)$$

be the closed polyhedral cone they generate, let $\mathrm{Int}(\sigma)$ be the interior, and for a subring $\Lambda \subset \mathbf{C}$ let

$$\sigma_A = \sum \lambda_i N_i \, \big| \, \lambda_i \in \Lambda\}.$$

Because the proof of (2.1) depends only on the positivity properties of ϕ_N, we have the following:

Lemma 4.1. *For any $N \in \mathrm{Int}(\sigma)$, we have*

(i) $W_1(N) = \cap \ker N_i$

(ii) $W_0(N) = \sum \mathrm{image}\, N_i$.

Consequently the cone itself defines a weight filtration,

$$W_*(\sigma) = W_*(N),$$

where N is an interior element of σ. Moreover, there is a correspondence between nilpotent orbits and mixed Hodge structures:

Proposition 4.2. *Let $F^1 \in \check{\mathscr{H}}_g$ and let $\sigma \subset \mathrm{Cl}(\eta_i^+)$ be a polyhedral cone. Then the following are equivalent:*

(i) *The two filtrations $F^* = \{H_{\mathbf{C}} \supset F^1 \supset \{0\}\}$ and $W_*(\sigma)$ define a ϕ_N-polarized mixed Hodge structure for all $N \in \mathrm{Int}(\sigma)$.*

(ii) *For any* $N \in \mathrm{Int}(\sigma)$, *the map*

$$z \longmapsto (\exp zN)F^1$$

defines a nilpotent orbit, i.e., there exist $\alpha \in \mathbf{R}$ *such that*

$$\exp(zN) \cdot F^1 \in \mathscr{H}_g \text{ for } \mathrm{Im}(z) > \alpha.$$

Proof. Assuming (i) we can consider the generalized Hodge decomposition of $H_{\mathbf{C}}$

$$H_{\mathbf{C}} = I^{1,1} \oplus I^{1,0} \oplus I^{0,1} \oplus I^{0,0}$$

where

$$\begin{aligned} I^{1,1} &= F^1 \cap (\overline{F}^1 + W_0) \\ I^{1,0} &= F^1 \cap W_1 \\ I^{0,1} &= \overline{F}^1 \cap W_1 \\ I^{0,0} &= W_0. \end{aligned}$$

Thus $I^{1,0}$ is complex conjugate to $I^{0,1}$, $I^{0,0}$ is self-conjugate, but $I^{1,1}$ is not in general self-conjugate. Define hermitian forms

$$\begin{aligned} \psi(x,y) &= \mathrm{i}\phi(x,\overline{y}) \\ \psi_N(x,y) &= \phi(x, N\overline{y}) \end{aligned}$$

and observe that for interior elements of σ, ψ_N is positive definite on $I^{1,0}$, whereas ψ is positive definite on $I^{1,1}$ and identically zero on $I^{1,0}$.

Now given $f \in F^1$ we want to show

$$\mathrm{i}\phi\big(\exp(\mathrm{i}yN)\cdot f,\, \exp(-\mathrm{i}yN)\cdot \overline{f}\,\big) > 0$$

for y sufficiently large. A standard compactness argument will then imply (ii). But

$$\begin{aligned} &\mathrm{i}\phi\big(\exp(\mathrm{i}yN)\cdot f,\, \exp(-\mathrm{i}yN)\cdot \overline{f}\,\big) = \mathrm{i}\phi\big(f, \overline{f} - 2\mathrm{i}yN\overline{f}\,\big) \\ &= \mathrm{i}\phi(f,\overline{f}) + 2y\phi(f, N\overline{f}) \end{aligned}$$

and the last expression is clearly greater than zero for $y \gg 0$.

Conversely, let $\exp(zN)\cdot F^1$ be a nilpotent orbit. We first notice that $F^1 \cap W_0(\sigma) = \{0\}$; in fact, let $f \in F^1 \cap W_0(\sigma)$, in particular $f \in \mathrm{Im}(N)$ and therefore

$$\mathrm{i}\phi\left(\exp(zN)\cdot f, \exp(\overline{z}N)\cdot \overline{f}\,\right) \equiv 0.$$

Hence F^* induces a Hodge structure of weight zero on $Gr_0(W_*(\sigma))$. In order to prove the corresponding statement for $Gr_1(W_*(\sigma))$ it is enough to show that

$$W_1(\sigma) = W_0(\sigma) \oplus \big(W_1(\sigma) \cap F^1\big) \oplus \Big(W_1(\sigma) \cap \overline{F}^1\Big). \tag{1}$$

If $f \in F^1 \cap W_1(\sigma)$ is such that $f + \overline{f} \in W_0(\sigma)$ then

$$\phi(f, \overline{f}\,) = \phi(f, f + \overline{f}\,) = 0$$

but, clearly $\phi_N(f, \overline{f}\,)$ also vanishes and thus $\mathrm{i}\phi(\exp(zN) \cdot f, \exp(\overline{z}N) \cdot \overline{f}\,) \equiv 0$. This implies $f = 0$ and consequently the sum (1) is direct.

We now have

$$\begin{aligned} g &= \dim F^1 = \dim(F^1 \cap \mathrm{Ker} N) + \dim(N(F^1)) \\ &\leq \dim(F^1 \cap \mathrm{Ker} N) + \dim W_0(\sigma) \end{aligned}$$

from which it follows that the dimension of the right-hand side in (1) is greater than or equal to $2g - \dim W_0(\sigma) = \dim W_1(\sigma)$. This proves (1). Moreover, we also have $N(F^1) = \mathrm{Im}(N)$, which implies

$$H_{\mathbf{C}} = W_1(\sigma) + F^1 = W_1(\sigma) + \overline{F^1}$$

which is equivalent to the statement that F^* defines a Hodge structure of weight two, and pure type $(1,1)$, on $Gr_2(W_*(\sigma))$. Since the polarization statements are clear, this completes the proof of (4.2).

We are now in the position to define the notion of rational boundary components.

Definition 4.3. *Let $N_1, \ldots, N_r \in Cl(\eta_i^+)$ be rational elements, $\sigma = \sigma(N_1, \ldots, N_r)$ be the polyhedral cone they generate. The subset $B(\sigma) \subset \check{\mathscr{H}}_g$ of all $F^* \in \check{\mathscr{H}}_g$ which, together with $W_*(\sigma)$, define a polarized mixed Hodge structure will be called the pre-boundary component associated to σ. We also define the boundary component associated to σ as the quotient*

$$\mathbf{B}(\sigma) = B(\sigma)/\exp(\sigma_{\mathbf{C}}).$$

We note that (4.2) means that $B(\sigma) = \exp(\sigma_{\mathbf{C}}) \cdot \mathscr{H}_g$ and is therefore an open subset of $\check{\mathscr{H}}_g$ which contains $\mathscr{H}_g$. We shall sometimes identify $\mathbf{B}(\{0\})$ with $\mathscr{H}_g$.

Proposition 4.4. *Let σ_1 and σ_2 be rational polyhedral cones and assume that $\sigma_1 < \sigma_2$ (i.e. σ_1 is a face of σ_2). Then*

(i) *$B(\sigma_1) \subset B(\sigma_2)$ and*

(ii) *There exists a projection $p : \mathbf{B}(\sigma_1) \to \mathbf{B}(\sigma_2)$ such that the diagram*

$$\begin{array}{ccc} B(\sigma_1) & \hookrightarrow & B(\sigma_2) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathbf{B}(\sigma_1) & \xrightarrow{\;p\;} & \mathbf{B}(\sigma_2) \end{array}$$

commutes, where π_i, $i = 1, 2$, are the natural projections.

Proof. The only statement that needs to be checked is the surjectivity of p. Let $F^1 \in B(\sigma_2)$ and $N \in \operatorname{Int}(\sigma_2)$ then for some $y \in \mathbf{R}$, $F^1_y = \exp(iyN).F^1 \in B(\sigma_1)$ and $\pi_2(F^1) = \pi_2(F^1_y)$.

Recall [12] that the Satake rational boundary components of $\mathscr{H}_g$ are in one to one correspondence with the rational totally isotropic subspaces of $H_{\mathbf{C}}$. In fact, given such a subspace U, the corresponding Satake boundary component is given by

$$\mathbf{B}_S(U) \cong \mathscr{H}(U^\perp/U, \tilde{\phi})$$

where as before $U^\perp$ denotes the ϕ-annihilator of U and $\tilde{\phi}$ the non-degenerate skew-symmetric form induced by ϕ on $U^\perp/U$.

Given a polyhedral cone $\sigma \subset \operatorname{Cl}(\eta_i^+)$, defined over $\mathbf{Q}$, then $W_0(\sigma)$ is a totally isotropic subspace and $W_1(\sigma) = W_0(\sigma)^\perp$. Moreover, the map $\hat{\zeta} = \hat{\zeta}_\sigma$

$$\hat{\zeta} : B(\sigma) \longrightarrow \mathbf{B}_S(W_0(\sigma)) \tag{2}$$

which assigns to each $F^* \in B(\sigma)$, the polarized Hodge structure of weight one defined by F^* in $Gr_1(W_*(\sigma))$, is onto and smooth relative to the natural differentiable structures. Note also that since $\exp(\sigma_{\mathbf{C}})$ acts trivially on the first graded quotient $Gr_1(W_*(\sigma))$, the map $\hat{\zeta}$ factors through a map

$$\zeta : \mathbf{B}(\sigma) \longrightarrow \mathbf{B}_S(W_0(\sigma)). \tag{3}$$

If σ_1 and σ_2 are rational polyhedral cones and σ_1 is a face of σ_2 (denoted by $\sigma_1 < \sigma_2$), then $W_0(\sigma_1) \subset W_0(\sigma_2)$ and $W_1(\sigma_1) \supset W_1(\sigma_2)$. Hence, $W_0(\sigma_2)$ defines a $\tilde{\phi}$-totally isotropic rational subspace $\tilde{W}_0(\sigma_2)$ in $Gr_1(W_*(\sigma_1))$. Moreover $Gr_1(W_*(\sigma_2))$ may be identified with the quotient space $\tilde{W}_0^\perp(\sigma_2)/(\tilde{W}_0(\sigma_2))$. Hence the Siegel upper half space $\mathscr{H}(Gr_1(W_*(\sigma_2)), \tilde{\phi})$ may be thought of as a Satake boundary component in $\mathscr{H}(Gr_1(W_*(\sigma_1)), \tilde{\phi})$. Also, the following diagram commutes

$$\begin{array}{ccc}
\mathbf{B}(\sigma_1) & \xrightarrow{\;p\;} & \mathbf{B}(\sigma_2) \\
\downarrow{\scriptstyle \zeta_1} & & \downarrow{\scriptstyle \zeta_2} \\
\mathbf{B}_S(W_0(\sigma_1)) & \xrightarrow{\;\omega\;} & \mathbf{B}_S(W_0(\sigma_2))
\end{array}$$

where ω is the projection from a Siegel upper half space to one of its Satake boundary components [12].

Proposition 4.5. *The fibration* $\hat{\zeta} : B(\sigma) \to \mathbf{B}_S(W_0(\sigma))$ *is* C^∞*-trivial. The fiber is isomorphic to the subset of* $M = \operatorname{Grass}(\omega_1 - \omega_0, H_{\mathbf{R}}) \times \operatorname{Grass}(2g - \omega_1, H_{\mathbf{C}})$, $\omega_i = \dim_{\mathbf{C}} W_i(\sigma)$, *given by*

$$A = \left\{ \begin{array}{r} (U_1, U_2) \in M : W_1(\sigma)_{\mathbf{R}} = W_0(\sigma)_{\mathbf{R}} \oplus U_1,\ H_{\mathbf{C}} = W_1(\sigma) \oplus U_2 \\ \phi(U_1, U_2) = \phi(U_1, \overline{U}_2) = \phi(U_2, U_2) = 0 \end{array} \right\}.$$

Proof. Given $F^* \in B(\sigma)$, let $H_{\mathbf{C}} = \oplus_{p,q\geq 0} I^{p,q}$ be the generalized Hodge decomposition. Set

$$U_1 = (I^{1,0} \oplus I^{0,1})_{\mathbf{R}}, \quad U_2 = I^{1,1}.$$

We need to check that $\phi(I^{1,0}, \overline{I^{1,1}}) = 0$, but $I^{1,0} = F^1 \cap W_1(\sigma)$ and $\overline{I^{1,1}} = \overline{F^1} \cap (F^1 + W_0(\sigma))$ and $\phi(W_0(\sigma), W_1(\sigma)) = 0$.

Conversely, given $(U_1, U_2) \in A$ and $\Omega \in \mathbf{B}_S(W_0(\sigma)) = \mathscr{H}(Gr_1(W_*(\sigma)), \tilde{\phi})$, let $I^{1,0} \subset W_1(\sigma)$ be the unique subspace such that

$$U_1 = (I^{1,0} \oplus \overline{I^{1,0}})_{\mathbf{R}} \text{ and } (I^{1,0} + W_0(\sigma))/W_0(\sigma) = \Omega.$$

We remark that if $(\Omega, U_1, U_2) \in B(\sigma)$ and $g \in \exp(\sigma_{\mathbf{C}})$ then $g(\Omega, U_1, U_2) = (\Omega, U_1, g(U_2))$, hence

$$\mathbf{B}(\sigma) \xrightarrow{\zeta} \mathbf{B}_S(W_0(\sigma))$$

is also a trivial fibering, with fiber $A/\exp(\sigma_{\mathbf{C}})$.

We end this section with an example which shall illustrate some of the objects introduced above. Geometrically, the situation described below corresponds to that discussed in (3.5). Let $H_{\mathbf{Z}}$ be a lattice of rank $2g$, $\xi = (e_1, \ldots, e_g, f_1, \ldots, f_g)$ a $\mathbf{Z}$-basis of $H_{\mathbf{Z}}$ and ϕ the skew-symmetric form in $H_{\mathbf{Z}}$ defined by

$$\begin{aligned} \phi(e_i, e_j) =& \phi(f_i, f_j) = 0 \\ \phi(e_i, f_j) =& -\phi(f_i, e_j) = -\delta_{ij}. \end{aligned} \tag{4}$$

We set $S_i = \operatorname{span}_{\mathbf{R}}\{e_1, \ldots, e_i\}$ and let $N_j \in \operatorname{Cl}(\eta_g^*)$ be given, relative to ξ, by

$$N_j = \left[\begin{array}{c|c} 0 & \Lambda_j \\ \hline 0 & 0 \end{array}\right]$$

where Λ_j is the diagonal matrix with 1 in the (j,j) position and zeroes elsewhere. The set

$$\sigma_r = \{N \in \operatorname{Cl}(\eta_g^+) \,:\, N = \sum_{j=1}^{r} \lambda_j N_j, \;\; \lambda_j \geq 0\}.$$

is then a rational polyhedral cone in $\operatorname{Cl}(\eta_g^+)$, with monodromy weight filtration:

$$W_0(\sigma_r) = S_r, \;\; W_1(\sigma_r) = S_r^{\perp} = \operatorname{span}_{\mathbf{R}}\{e_1, \ldots, e_g, f_1, \ldots, f_g\}.$$

If $f^1 \in \check{\mathscr{H}}$, then it may be represented as the row space of a matrix

$$P = [P_1, P_2]$$

where P_1 and P_2 are $g \times g$ complex matrices, and P is well defined up to left-action by $\mathbf{GL}(g, \mathbf{C})$. If in addition $F^1 \in B(\sigma_r)$, then for any $N \in \operatorname{Int}(\sigma_r)$, $\exp(zN)\cdot$

$F^1 \in \mathscr{H}_g$ for $\mathrm{Im}(z)$ sufficiently large, and this is easily seen to imply that P_1 must be non-singular. Hence F^1 can be (uniquely) represented as

$$F^1 = [I, Z]$$

and since $F^1 \in \check{\mathscr{H}}_g$ we must have that Z is a symmetric matrix whose imaginary part is positive semi-definite. Writing

$$F^1 = \begin{bmatrix} I_r & 0 & Z_{11} & {}^tZ_{21} \\ 0 & I_{g-r} & Z_{21} & Z_{22} \end{bmatrix}.$$

We see that the Hodge structure induced by F^1 in $Gr_1(W_*(\sigma_R))$ has a period matrix

$$[I_{g-r}, Z_{22}]. \tag{5}$$

Hence $\mathrm{Im}(Z_{22}) > 0$, and the projection $\hat{\zeta}$ defined in (2) is given by $\hat{\zeta}(Z) = Z_{22}$.

The terms in the generalized Hodge decomposition are defined by

$$I^{0,0} = W_0(\sigma) = S_r$$

$$I^{1,0} = \text{row space of } [0, I_{g-r}, Z_{21}, Z_{22}],\ \ I^{0,1} = \overline{I^{1,0}}$$

$$I^{1,1} = \text{row space of } [Z_{11}, \mathrm{Re}({}^tZ_{21}), I_r, -\mathrm{Im}({}^tZ_{21})]$$

and these in turn determine the pair (U_1, U_2) as in the proof of (4.5). Finally, note that $\exp(\sigma_{\mathbf{C}})$ acts on $B(\sigma)$ by

$$\exp\left(\sum_{i=1}^{r} z_i N_i\right) \cdot Z = Z + D(z_1, \ldots, z_r)$$

where

$$D(z_1, \ldots, z_r) = \left[\begin{array}{ccc|c} z_1 & & & \\ & \ddots & & 0 \\ & & z_r & \\ \hline & 0 & & 0 \end{array}\right]$$

and thus, it is the information contained in the terms that are off the diagonal of Z_{11} that passes on to the boundary component $\mathbf{B}(\sigma_r)$, while the Satake component contains only the element Z_{22}.

5 Group actions

In this section we exhibit the boundary components in the form defined by Mumford in [1] and [10].

Let $\sigma = \sigma(N_1, \ldots, N_r) \subset \mathrm{Cl}(\eta_i^+)$ be a rational polyhedral cone and consider the subgroup of G_Λ

$$\mathrm{Norm}_\Lambda(\sigma) = \{g \in G_\Lambda \,:\, \mathrm{Ad}(g)N_j,\ 1 \le j \le r\}$$

and the subalgebra of $\mathfrak{g}_{\mathbf{C}}$

$$L = L(\mathrm{Norm}_{\mathbf{C}}(\sigma)) = \{X \in \mathfrak{g}_{\mathbf{C}} : [X, N_j] = 0,\ 1 \leq j \leq r\}.$$

In particular, if $X \in L$ it must preserve the monodromy weight filtration $W_*(\sigma)$. We then set for $l \geq 0$

$$W_{-l} = W_{-l}(L) = \{X \in L\ :\ X(W_j(\sigma)) \subset W_{j-l}(\sigma)\}.$$

This defines a "weight" filtration

$$\{0\} = W_{-3} \subset W_{-2} \subset W_{-1} \subset W_0 = L$$

on L which is compatible with the Lie algebra structure. More precisely

$$[W_{-r}, W_{-s}] \subset W_{-(r+s)}$$

and hence it follows that W_{-l} consists of nilpotent elements for $l \geq 1$. By exponentiation we can then define a corresponding increasing filtration $W_{-l}(\mathrm{Norm}_{\mathbf{C}}(\sigma))$ in $\mathrm{Norm}_{\mathbf{C}}(\sigma)$. Finally, we set

$$V(\sigma) = \mathrm{Norm}_{\mathbf{R}}(\sigma) \cdot W_{-2}(\mathrm{Norm}_{\mathbf{C}}(\sigma)),$$
$$U(\sigma) = V(\sigma) \cap W_{-1}(\mathrm{Norm}_{\mathbf{C}}(\sigma)).$$

It is easy to check that $U(\sigma)$ is the unipotent radical of $V(\sigma)$ and, thus, a normal subgroup of $V(\sigma)$. Moreover, since $N_j(W_l(\sigma)) \subset W_{l-2}(\sigma)$, we have

$$\exp(\sigma_{\mathbf{C}}) \subset W_{-2}(\mathrm{Norm}_{\mathbf{C}}(\sigma)) \subset U(\sigma).$$

In particular we have (cf. section 4) that

$$B(\sigma) = W_{-2}(\mathrm{Norm}_{\mathbf{C}}(\sigma)) \cdot \mathscr{H}_g,$$

and a linear algebra argument shows

Theorem 5.1. *The group $U(\sigma)$ acts simply transitively on the fibers of the fibration (4.5).*

If $S(\sigma)$ is the subgroup of $V(\sigma)$ leaving the subspaces U_i of (4.5) stable, then $S(\sigma)$ is semisimple and $V(\sigma) = S(\sigma) \cdot U(\sigma)$. We then obtain,

Corollary 5.2. (i) The group $V(\sigma)$ acts transitively on the boundary precomponent $B(\sigma)$.

(ii) There are smooth identifications $B(\sigma) \cong \mathbf{B}_S(W_0(\sigma)) \times U(\sigma)$ and $\mathbf{B}(\sigma) = \mathbf{B}_S(W_0(\sigma)) \times (U(\sigma)/\exp(\sigma_{\mathbf{C}}))$, where $\mathbf{B}_S(W_0(\sigma)) = \mathscr{H}(Gr_1(W_*(\sigma)), \tilde{\phi})$ is the Satake boundary component associated to the rational, ϕ-isotropic subspace $W_0(\sigma)$.

Remark 5.3. We have already noted that

$$B(\sigma) = \exp W_{-2}(\mathrm{Norm}_{\mathbf{C}}(\sigma)) \cdot \mathscr{H}_g;$$

in particular, we see that $B(\sigma)$ agrees with the open set $D(F)$ defined by Mumford in [1], and the second identity in (ii) of (5.2) is just the decomposition in ([1, p. 235]). The group $\mathrm{Norm}_{\mathbf{Z}}(\sigma)$ acts properly discontinuously on $B(\sigma)$ and $\mathbf{B}(\sigma)$ and we can consider the fibration

$$\exp(\sigma_{\mathbf{C}})/\exp(\sigma_{\mathbf{Z}}) \longrightarrow B(\sigma)/\mathrm{Norm}_{\mathbf{Z}}(\sigma) \longrightarrow \mathbf{B}(\sigma)/\mathrm{Norm}_{\mathbf{Z}}(\sigma).$$

Since $\sigma_{\mathbf{C}}$ is abelian, this is a toroidal fibration and it is the basic ingredient in Mumford's construction.

Finally, we shall again use the example discussed at the end of Section 4 to clarify some of the groups introduced above. In order to simplify the notation we shall make a slight change in the basis ξ by exchanging the subsets $\{f_1, \ldots, f_r\}$ and $\{f_{r+1}, \ldots, f_g\}$. In the new basis then

$$\phi = \left[\begin{array}{c|c|c} & & -I_r \\ \hline & \tilde{\phi} & \\ \hline I_r & & \end{array}\right] \qquad \tilde{\phi} = \left[\begin{array}{c|c} & -I_{g-r} \\ \hline I_{g-r} & \end{array}\right].$$

We then have that $\mathrm{Norm}_{\mathbf{C}}(\sigma)$ consists of complex matrices of the form:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ & A_{22} & A_{23} \\ & & {}^tA_{11}^{-1} \end{bmatrix}$$

satisfying the following conditions

(i) $A_{11}^{-1}\Lambda_j = \Lambda_j^t A_{11} \quad j = 1, \ldots, r,$

(ii) $A_{22} \in \mathrm{Sp}(g - r, \mathbf{R}),$

(iii) $A_{11}^{-1}A_{12} + {}^tA_{23}\tilde{\phi}A_{22} = 0,$

(iv) $A_{11}^{-1}A_{13} + {}^tA_{23}\tilde{\phi}A_{23} - {}^tA_{13}{}^tA_{11} = 0.$

Notice that the subgroup of $\mathbf{GL}(r, \mathbf{C})$ of matrices A_{11} satisfying (i) above, is finite. Now, $V(\sigma)$ consists of those matrices in $\mathrm{Norm}_{\mathbf{C}}$, for which A_{11}, A_{12} and hence A_{23} are real, while $U(\sigma)$ is the subgroup of $V(\sigma)$ defined by $A_{11} = I_r$, $A_{22} = I_{2(g-r)}$. Finally, the isomorphism in (5.2.ii) is defined in the following way: let $F^1 \in B(\sigma)$ and write as in (5), $F^1 = [I, Z]$ with

$$Z = \begin{bmatrix} Z_{11} & {}^tZ_{21} \\ Z_{21} & Z_{22} \end{bmatrix}.$$

Then $F^1 \leftrightarrow (Z_{22}, g)$, where $Z_{22} \in \mathbf{B}_S(W_0(\sigma_r))$ and

$$g = \left[\begin{array}{c|c|c} I & \operatorname{Im}(Z_{21})\,\operatorname{Re}(Z_{21}) & Z_{11} \\ \hline & I & \operatorname{Re}({}^tZ_{21}) - \operatorname{Im}({}^tZ_{21}) \\ \hline & & I \end{array}\right] \in U(\sigma).$$

6 Construction of the compactification

Here we shall present, in its bare outlines, the construction of a compactification of $\mathscr{H}_g/\Gamma$, $\Gamma = G_{\mathbf{Z}}$, whose boundary components are discrete quotients of the spaces described in sections 4 and 5. This is done in the spirit of Satake's original compactification [12, 13], the end result is Mumford's compactification but with the "toroidal embeddings", by which the boundary components are "glued" to $\mathscr{H}_g/\Gamma$, made somewhat more explicit. We refer the reader to [1] and [10] for the details of Mumford's more general construction. Let us also remark that Namikawa [11] has considered the compactification resulting from the Delony-Voronoi decomposition and its applications to the study of degenerations of abelian varieties.

For simplicity we shall assume that the polarizing form ϕ admits an integral symplectic basis; this is of course the case in the geometric situation.

Let $\xi = (e_1, \dots, e_g, f_1, \dots, f_g)$ be a basis of $H_{\mathbf{Z}}$, such that ϕ satisfies conditions (4), and

$$\{0\} = S_0 \subset S_1 \subset \cdots \subset S_g$$

the maximal flag of rational totally isotropic subspaces defined by

$$S_r = \operatorname{span}_{\mathbf{R}}\{e_1, \dots, e_r\}$$

If $\operatorname{Norm}_{\mathbf{Z}}(S_r) = \{g \in G_{\mathbf{Z}} : g(S_r) = S_r\}$, then its adjoint action on $\mathfrak{g}_0$ restricts to an action on η_r which preserves the cone η_r^+: indeed if $N \in \eta_r$,

$$\operatorname{Im}(\operatorname{Ad}(g)N) = \operatorname{Im}(g^{-1}N) = g^{-1}(\operatorname{Im}(N)) \subset g^{-1}(S_r) = S_r$$

and thus $\operatorname{Ad}(g)N \in \eta_r$. On the other hand, if $N \in \eta_r^+$, then $\phi_N(x, y)$ is positive definite in $H_{\mathbf{R}}/S_r^{\perp}$ but then so is the form $\phi(gx, gy) = \phi_{\operatorname{Ad}(g)N}(x, y)$ which implies that $\operatorname{Ad}(g)N \in \eta_r^+$.

Let $\operatorname{Cent}_{\mathbf{Z}}(S_r) = \{g \in \operatorname{Norm}_{\mathbf{Z}}(S_r) : g|_{S_r} = I\}$ and

$$G_{\mathbf{Z}}(S_r) = \operatorname{Norm}_{\mathbf{Z}}(S_r)/\operatorname{Cent}_{\mathbf{Z}}(S_r).$$

If we identify η_r^+ (via the basis ξ) with the set $\mathcal{M}$ of $r \times r$ positive definite real symmetric matrices then the action of $G_{\mathbf{Z}}(S_r)$ on η_r^+ corresponds to the natural action of $SL(r, \mathbf{Z})$ on $\mathcal{M}$. Recall also the construction of Siegel sets in $\mathcal{M}$. Given

$X \in \mathcal{M}$ we can write (Babylonian reduction) $X =^t WDW$, where W is an upper triangular unipotent matrix and D is diagonal. The set

$$\mathcal{W}_u = \{X = {}^tWDW \in \mathcal{M} : |w_{ij}| < u,\, 0 < d_i < ud_{i+1}\}$$

is called a *Siegel set*, and for u sufficiently large it is a fundamental set for the action of $SL(r, \mathbf{Z})$ on $\mathcal{M}$. We shall carry over these notions to η_r^+ and the $G_{\mathbf{Z}}(S_r)$-action.

We can now define one of the basic objects for the construction of the compactification (cf. [1]).

Definition 6.1. *A collection $\{\sigma_\alpha^{(r)}\}$ of rational polyhedral cones in $Cl(\eta_r^+)$ is said to be a $G_{\mathbf{Z}}(S_r)$-admissible rational polyhedral decomposition of η_r^+ if:*

(i) *If $\sigma \in \{\sigma_\alpha^{(r)}\}$ and $\sigma' < \sigma$, then $\sigma' \in \{\sigma_\alpha^{(r)}\}$.*

(ii) *If $\sigma, \sigma' \in \{\sigma_\alpha^{(r)}\}$ then $\sigma \cap \sigma' < \sigma$ and $\sigma \cap \sigma' < \sigma'$.*

(iii) *If $\sigma \in \{\sigma_\alpha^{(r)}\}$, $\gamma \in G_{\mathbf{Z}}(S_r)$ then $\gamma(\sigma) \in \{\sigma_\alpha^{(r)}\}$.*

(iv) *There exist finitely many cones $\sigma_1, \ldots, \sigma_n \in \{\sigma_\alpha^{(r)}\}$ such that for any $\sigma \in \{\sigma_\alpha^{(r)}\}$, there is $\gamma \in G_{\mathbf{Z}}(S_r)$ with $\gamma(\sigma) = \sigma_i$ for some $i = 1, \ldots, n$.*

(v) $\eta_r^+ = \cup(\sigma_\alpha^{(r)} \cap \eta_r^+)$.

(vi) *There exist a Siegel set $\mathcal{W}_u \subset \eta_r^+$, such that for any $\sigma \in \{\sigma_\alpha^{(r)}\}$ there exists $\gamma \in G_{\mathbf{Z}}(S_r)$ with $\sigma_\alpha \cap \eta_n^+ \subset \gamma(\mathcal{W}_u)$.*

If $r < t$, then $S_r \subset S_i$, $\eta_r \subset \eta_t$ and $\mathrm{Cl}(\eta_r^+) \subset \mathrm{Cl}(\eta_t^+)$. Hence if $\sigma \subset \mathrm{Cl}(\eta_t^+)$ is a rational polyhedral cone, then so is $\sigma \cap \mathrm{Cl}(\eta_r^+)$. Notice moreover that if $\sigma \subset \mathrm{Cl}(\eta_t^+)$, $\sigma' < \sigma$ and $\sigma' \cap \mathrm{Cl}(\eta_r^+) \neq \emptyset$, then $\sigma' \subset \mathrm{Cl}(\eta_r^+)$.

Definition 6.2. *A collection $\{\sigma_\alpha\}$ of rational polyhedral cones in $\mathrm{Cl}(\eta_g^+)$ is said to be Γ-admissible if:*

(i) *For any $r = 1, \ldots, g$, the collection $\{\sigma_\alpha^{(r)}\} = \{\sigma_\alpha \cap \mathrm{Cl}(\eta_r^+)\}$ is a $G_{\mathbf{Z}}(S_r)$-admissible rational polyhedral decomposition of η_r^+.*

(ii) *Given $r, t\,(1 \le r < t \le g)$, then*

$$\{\sigma_\alpha^{(r)}\} = \{\sigma_\alpha^{(t)} \cap \mathrm{Cl}(\eta_r^+)\}.$$

The existence of Γ-admissible polyhedral decompositions has ben proved by A. Ash in [1].

It is well known that any two totally isotropic rational subspaces of $H_{\mathbf{R}}$, of the same dimension, are conjugate under the action of $\Gamma = \mathrm{Sp}(n, \mathbf{Z})$. Moreover, the same is true of maximal flags of such subspaces. In particular, if

$$\{0\} = S_0' \subset S_1' \subset \cdots \subset S_g'$$

is a maximal flag, then there exists $\gamma \in \Gamma$, such that

$$\{\gamma\sigma_\alpha \subset \mathrm{Cl}(\eta')_g^+\}$$

is a Γ-admissible polyhedral decomposition.

Definition 6.3. *Given a Γ-admissible polyhedral decomposition $\{\sigma_\alpha,\ \alpha \in A\}$, we will denote by $\mathcal{H}_g^*$ th union of the boundary components $\gamma\sigma_\alpha$ for $\gamma \in \Gamma$, $\alpha \in A$ (we identify $\mathcal{H}_g$ with the boundary component $\mathbf{B}(\{0\})$).*

The next step is to construct a fundamental set for the action of Γ on $\mathscr{H}_g^*$. Given $\sigma \in \{\gamma\sigma_\alpha : \gamma \in \Gamma,\ \alpha \in A\}$ we shall denote by

$$\Gamma_V(\sigma) = \Gamma \cap V(\sigma) = \mathrm{Norm}_{\mathbf{Z}}(\sigma), \quad \Gamma_U(\sigma) = \Gamma \cap U(\sigma)$$
$$\Gamma_S(\sigma) = \Gamma \cap S(\sigma)$$

where $V(\sigma)$, $U(\sigma)$ and $S(\sigma)$ are the subgroups of $\mathrm{Norm}_{\mathbf{C}}(\sigma)$ defined in section 5. Note that $\Gamma_V(\sigma)$ acts almost effectively and properly discontinuously on the pre-boundary component $B(\sigma)$, with kernel $Z(\sigma) \subset S(\sigma)$. We can identify $S(\sigma)/Z(\sigma)$ with a subgroup $\tilde{S}(\sigma) \subset S(\sigma)$ and $\tilde{S}(\sigma) \cong \mathrm{Sp}(g-r, \mathbf{R})$, $r = \dim W_0(\sigma)$. We know that $\Gamma_V(\sigma)$ can be expressed as a semi-direct product

$$\Gamma_V(\sigma) = \Gamma_S(\sigma) \cdot \Gamma_U(\sigma)$$

and, if we denote by $\Gamma_{-2}(\sigma) = \Gamma_V(\sigma) \cap W_{-2}(\mathrm{Norm}_{\mathbf{C}}(\sigma))$ then $\Gamma_{-2}(\sigma)$ is a normal subgroup of $\Gamma_V(\sigma)$ contained in $\Gamma_U(\sigma)$, and the quotient $\Gamma_U(\sigma)/\Gamma_{-2}(\sigma)$ is abelian. We also note that $\Gamma_{-2}(\sigma)$ contains the abelian group $\exp(\sigma_{\mathbf{Z}})$ and that

$$\Gamma(F(\sigma)) = \Gamma_V(\sigma)/Z(\sigma) \cdot \exp(\sigma_{\mathbf{Z}})$$

acts effectively and properly discontinuously on $\mathbf{B}(\sigma)$. Moreover, if $\Omega_S(\sigma) \subset \mathbf{B}_S(W_0(\sigma))$, $\Omega_U(\sigma) \subset U(\sigma)$ are fundamental sets for the action of $\Gamma_S(\sigma)$ and $\Gamma_U(\sigma)$, respectively, then

$$\Omega(\sigma) = \Omega_U(\sigma) \times \Omega_S(\sigma)$$

is a fundamental set for the action of $\Gamma_V(\sigma)$ on $B(\sigma)$. Furthermore $\Omega_U(\sigma)$ may be constructed as a product

$$\Omega_U(\sigma) = \Omega_{-2}(\sigma) \times j(\Omega_{-1}(\sigma))$$

where $\Omega_{-2}(\sigma) \subset W_{-2}(\sigma)$, $\Omega_{-1}(\sigma) \subset U(\sigma)/W_{-2}(\sigma)$ are fundamental sets for the action of $\Gamma_{-2}(\sigma)$ and $\Gamma_U(\sigma)/\Gamma_{-2}(\sigma)$, respectively, and $j : U(\sigma)/W_{-2}(\sigma) \to U(\sigma)$ is a global section. The projection

$$\tilde{\Omega}(\sigma) = (\Omega_U(\sigma)/\exp(\sigma_{\mathbf{C}})) \times \Omega_S(\sigma) \subset \mathbf{B}(\sigma)$$

is, moreover, a fundamental set for the action of $\Gamma(\mathbf{B}(\sigma))$.

We fix now a Γ-admissible polyhedral decomposition $\{\sigma_\alpha\}$ as in (6.2), and let $\sigma_1, \ldots, \sigma_n \in \{\sigma_\alpha\}$ be a set of representatives of $G_{\mathbf{Z}}(S_g)$-equivalence classes. We can then write

$$\overline{(\mathscr{H}_g/\Gamma)} = \mathscr{H}_g^*/\Gamma = \bigcup_{i=1}^{n} \big(\mathbf{B}(\sigma_i)/\Gamma(\mathbf{B}(\sigma_i))\big)$$

Associated to the choice of a maximal flag of rational totally isotropic subspaces, there is a choice of Siegel sets in $\mathscr{H}_g$; namely, if as in (5) we view $\mathscr{H}_g$ as

$$\{Z \in \mathcal{M}_g(\mathbf{C}) : {}^tZ = Z,\ \operatorname{Im}(Z) > 0\}$$

and we write $X = \operatorname{Re}(Z)$, $Y = \operatorname{Im}(Z) = {}^tWDW$, where W is upper triangular unipotent and D is diagonal, then the subset $\Omega_t \subset \mathscr{H}_g$ defined by the conditions

$$|x_{ij}| \le t,\ \ |w_{ij}| \le t,\ \ 1 \le td_1,\ \ d_i \le td_{i+1}$$

is a Siegel set in $\mathscr{H}_g$ and for $t > 0$, sufficiently large, Ω_t is a Γ-fundamental set in $\mathscr{H}_g$. Moreover, if $p_i : \mathscr{H}_g \to \mathbf{B}(\sigma_i)$, $\zeta_i : \mathscr{H}_g \to \mathbf{B}_S(W_0(\sigma_i))$ are the projections defined in (4.4) and (3), then:

6.4. *There exist a Siegel set $\Omega_t \subset \mathscr{H}_g$ such that for $i = 1, \ldots, n$, $\zeta_i(\Omega_t)$ is a Siegel set in $\mathbf{B}_S(W_0(\sigma_i))$ and $p_i(\Omega_t)$ is a $\Gamma(\mathbf{B}(\sigma_i))$-fundamental set in $\mathbf{B}(\sigma_i)$.*

In particular the set

$$\Omega^* = \Omega_t \cup \left(\bigcup_{i=1}^{n} p_i(\Omega_t)\right)$$

where $t > 0$ is large enough so that Ω_t is a Γ-fundamental set in $\mathscr{H}_g$ and (6.4) is satisfied, is a Γ-fundamental set in $\mathscr{H}_g^*$.

We can now define a topology in Ω^* in the following way: Let $x \in p_j(\Omega_t)$, U_j an open neighborhood of x in $p_j(\Omega_t)$ relative to the natural topology of $\mathbf{B}(\sigma_j)$. For each $j = 1, \ldots, n$, let $I(j) = \{i : \sigma_i < \sigma_j\}$ and for $i \in I(j)$ let

$$p_{ij} : \mathbf{B}(\sigma_i) \longrightarrow \mathbf{B}(\sigma_j)$$

be the projection in (4.4). Finally, for each $\lambda > 0$, let $\sigma_i(\lambda) = \{N = \sum \lambda_k N_k \in \sigma_i : \lambda_k > \lambda\}$ and set

$$U_i(\lambda) = p_{ij}^{-1}(U_j) \cap (\exp(\sigma_j(\lambda)) \cdot p_i(\Omega_t)) \subset \mathbf{B}(\sigma_i);\ i \in I(j)$$
$$U_0(\lambda) = p_j^{-1}(U_j) \cap (\exp(\sigma_j(\lambda)) \cdot \Omega_t) \subset \mathscr{H}_g.$$

Then the sets

$$\mathscr{V}(U_j, \lambda) = U_0(\lambda) \cup \left(\bigcup_{i \in I(j)} U_i(\lambda)\right)$$

where U_j varies on a basis of open sets in $\mathbf{B}(\sigma_j)$ and $\lambda \in \mathbf{R}$, $\lambda > 0$, define a basis of open sets for a topology $\mathscr{T}$ in Ω^*. Moreover, if we denote by Ω^*_S the fundamental set in Satake's extended space [12]:

$$\Omega^*_S = \Omega_t \cup \left(\bigcup_{i=1}^{n} \zeta_i(\Omega_t) \right)$$

then we have a continuous map

$$\zeta : \Omega^* \longrightarrow \Omega^*_S$$

where Ω^*_S is endowed with the Satake topology $\mathscr{T}_S$ [12]. It is then possible to carry the basic properties of $(\Omega^*_S, \mathscr{T}_S)$ to $(\Omega^*, \mathscr{T})$. In particular

(i) $(\Omega^*, \mathscr{T})$ *is compact and Hausdorff.*

(ii) $\mathscr{T}$ *induces the natural topology in* Ω_t *and* $P_i(\Omega_t)$, $i = 1, \ldots, n$.

(iii) *If* $\gamma \in \Gamma$, $x \in \Omega^*$ *and* $\gamma x \in \Omega^*$, *then for any* $\mathscr{T}$*-neighborhood* $\mathscr{V}'4$, *of* γx, *there exists a* $\mathscr{T}$*-neighborhood* $\mathscr{V}$ *of* x *such that* $\gamma\mathscr{V} \cap \Omega^* \subset \mathscr{V}'$.

(iv) *If* $\gamma \in \Gamma$, $x \in \Omega^*$ *and* $\gamma x \notin \Omega^*$, *then there exists a* $\mathscr{T}$*-neighborhood* $\mathscr{V}$ *of* x *such that* $\gamma\mathscr{V} \cap \Omega^* = \emptyset$

We can now define a Satake topology in $\mathscr{H}_g^*$: a fundamental system of neighborhoods of $x \in \mathscr{H}_g^*$ consists of all sets $\mathfrak{U} \subset \mathscr{H}_g^*$ which are invariant under the action of $\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}$ and satisfy:

If $\gamma x \in \Omega^*, \gamma \in \Gamma$, *then* $\gamma\mathfrak{U} \cap \Omega^*$ *is a* $\mathscr{T}$*-neighborhood of* γx *in* Ω^*.

It then follows from Theorem 1′ in [13] that the quotient topology in $\overline{(\mathscr{H}_g/\Gamma)} = \mathscr{H}_g^*/\Gamma$ is the unique topology satisfying:

(i) *It induces the natural topology on* Ω_t *and on* $p_i(\Omega_t)$, $i = 1, \ldots, n$.

(ii) *The operations of* $\gamma \in \Gamma$ *are continuous.*

(iii) $\overline{(\mathscr{H}_g/\Gamma)} = \mathscr{H}_g/\Gamma$ *is compact and Hausdorff.*

(iv) *For each* $x \in \mathscr{H}_g^*$, *there exists a fundamental system of neighborhoods* $\{\mathfrak{U}\}$ *of* x *such that* $\gamma\mathfrak{U} = \mathfrak{U}$ *for* $\gamma \in \Gamma_x$, $\gamma\mathfrak{U} \cap \mathfrak{U} = \emptyset$ *for* $\gamma \neq \Gamma_x$.

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