On Fundamental Groups of Class VII Surfaces

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January 25, 1995
Revised August 7, 1995

1. Introduction

The purpose of this note is to obtain a restriction on the fundamental groups of non-elliptic compact complex surfaces of class VII in Kodaira's classification [**9**]. We recall that these are the compact complex surfaces with first Betti number one and no non-constant meromorphic functions. This seems to be the class of compact complex surfaces whose structure is least understood. The first and simplest examples are the general Hopf surfaces [**9**], III. Then there are various classes of examples found by Inoue [**5**,**6**], and which have been studied in more detail in [**11**]. The only known topological restriction beyond the first Betti number is that intersection form in two-dimensional homology is negative definite. There seems to be little known as to how wide this class of surfaces is. We prove the following theorem.

1.1. Theorem *Let M be a non-elliptic compact complex surface of class VII, and let N be a compact Riemannian manifold of constant negative curvature. Let $\phi : \pi_1(M) \to \pi_1(N)$ be a homomorphism. Then the image of ϕ is either trivial or an infinite cyclic group.*

Remark: It follows from our proof and the work of Hernández [**4**] that the theorem also remains true under the hypothesis that the curvature of N be negative and pointwise strictly 1/4-pinched. It also follows that it remains true under the more general, but more technical hypothesis, that the curvature of N be strictly negative and the Hermitian curvature of N be non-positive [**12**, **13**]. In particular, the theorem also holds for N a compact quotient of the unit ball in $\mathbb{C}^2$.

We observe that all the known examples of non-elliptic surfaces of class VII have solvable fundamental group. Thus it would be natural to conjecture that if M is as in the theorem, then the same conclusion should be true for N any compact manifold of strictly negative curvature, or more generally for $\pi_1(N)$ replaced by any word hyperbolic group.

Our original interest in this question was to extended a previous restriction we had found on fundamental groups of compact Kähler manifolds to fundamental groups of compact complex surfaces. Namely, our theorem, combined with some facts on classifiication of surfaces, has the following corollaries. (In both corollaries the assumption of constant negative curvature can be relaxed to pointwise strict 1/4-pinching as in the above remark.)

1.2. Corollary *Let N be a compact manifold of constant negative curvature and dimension at least 3. Then $\pi_1(N)$ is not isomorphic to the fundamental group of any compact complex surface.*

In this connection, we recall that Taubes [**14**] has proved that any finitely presented group is isomorphic to the fundamental group of a compact complex manifold of complex dimension 3. Also, if N has constant negative curvature and even dimension, then its twistor space is a complex manifold fibered over N with simply connected fiber, and hence the same fundamental group. Thus for such N there is a natural complex manifold with the same fundamental group. If $dim(N) = 4$ then the twistor space is of complex dimension 3, but for N of higher dimension the twistor space has dimension larger than 3 and there does not seem to be a construction of a 3-dimensional complex manifold with the same fundamental group except as given by the theorem of Taubes.

An immediate consequence of this corollary is the following following consequence of results of Wall and Kotschick (corollary of Theorem 10.2 of [**15**], corrected by Proposition 2 of [**10**]). In this connection we mention that Y-T. Siu pointed out to us that it not known whether compact, even dimensional constant negative curvature manifolds of dimension at least 6 admit complex structures.

1.3. Corollary *Let N be a compact manifold of constant negative curvature and real dimension 4. Then N has no complex structure.*

The proof of our theorem is based on the existence theorem for "Hermitian harmonic maps" recently proved by Jost and Yau in [**7**]. This is a modification of the harmonic map equation for maps $f : M \to N$ from Hermitian to Riemannian manifolds that has the following vitues. First, under suitable restrictions on N and on a given continuous map $g : M \to N$, there exists a Hermitian harmonic map f homotopic to g. Furthermore, pluriharmonic maps (which are not necessarily harmonic for non-Kähler metrics) are Hermitian harmonic. Finally there is a converse implication that, under suitable restrictions on M and N, the Siu - Sampson integration by parts technique holds, and any Hermitian harmonic map $f : M \to N$ is pluriharmonic.

It is easy to check that all the needed hypothesis to apply the work of Jost and Yau hold under our assumptions, and our theorem can then be proved by arguments similar to our previous arguments in the Kähler case. We remark that in the case that the dimension of N is 3 or 4 we knew several years ago how to prove Corollary (1.2) by standard harmonic maps techniques combined with surface classification. But if the dimension of N is at least 5, we know of no other proof than the one presented here. Thus our theorem provides a natural application of Hermitian harmonic maps.

We thank Dieter Kotschick and the referee for improvements on the first version of this paper.

2. Proof of the Theorem

We begin by recalling the material we need from [**7**]. Let M be a (compact) Hermitian manifold of complex dimension n, with complex structure J, Hermitian metric $< \, , \, >$ and fundamental two form ω defined by $\omega(X,Y) = < JX, Y >$. Thus M is Kähler if and only if $d\omega = 0$. Recall that the Hodge $*$-operator on one-forms $* : \Lambda^1(M) \to \Lambda^{2n-1}(M)$ is related to ω and J (up to a factor depending only on n) by the equation $*\alpha = \omega^{n-1} \wedge J\alpha$.

Now let N be a (compact) Riemannian manifold with Levi-Civita connection ∇, and let $f : M \to N$ be a smooth map. We use ∇ to denote the induced connection on f^*TN, and d_∇ to denote the exterior differentiation operator on forms on M with coefficients in f^*TN. Finally, $d^c f$ will denote the f^*TN-valued one-form Jdf.

In this notation, the harmonic equation $d_\nabla * df = 0$ for a smooth map $f : M \to N$ becomes $d_\nabla(\omega^{n-1} \wedge Jdf) = 0$, which we write in the equivalent form

$$d_\nabla(\omega^{n-1} \wedge d^c f) = 0. \tag{2.1}$$

Jost and Yau call a smooth map $f : M \to N$ *Hermitian harmonic* if it satisfies the equation

$$\omega^{n-1} \wedge d_\nabla d^c f = 0, \tag{2.2}$$

which differs from the harmonic equation (2.1) by the first order term $d\omega^{n-1} \wedge d^c f$. Thus the two equations agree for Kähler manifolds.

Recall that a map $f : M \to N$ is called *pluriharmonic* if it satisfies

$$d_\nabla d^c f = 0. \tag{2.3}$$

This is equivalent to the requirement that the restriction of f to any germ of a holomorphic curve in M be a harmonic map. This condition depends on the complex structure of M, but is independent of the Hermitian metric. Clearly holomorphic maps are pluriharmonic, and pluriharmonic maps satisfy (2.2), but need not satisfy (2.1) if the Hermitian metric is not a Kähler metric.

We will need the following two properties of Hermitian harmonic maps proved by Jost and Yau in [**7**].

2.1. Theorem *Let M be a compact Hermitian manifold whose fundamental two-form ω satisfies $dd^c\omega^{n-2} = 0$. Let N be a Riemannian manifold of non-positive Hermitian sectional curvature, in the sense that $< R(X,Y)\bar{X}, \bar{Y} > \leq 0$ holds for all $X, Y \in TN \otimes \mathbb{C}$. Then any Hermitian harmonic map $f : M \to N$ is pluriharmonic. Moreover $R(X,Y) = 0$ for all $X, Y \in df(T^{1,0}M)$.*

Note that the hypothesis on M is vacuously satisfied if the complex dimension of M is two. The proof of this theorem (cf. Lemma 7 of [**7**]) is simply the observation that the integration by parts technique of Siu [**13**], as extended by Sampson [**12**], requires just the assumption $dd^c\omega^{n-1} = 0$, rather than the stronger assumption $d\omega = 0$ used by these authors. See [**1**,**12**] for discussion of the curvature condition on N.

2.2. Theorem *Let M be a compact Hermitian manifold, let N be a compact Riemannian manifold of strictly negative curvature, and let $g : M \to N$ be a smooth map that is not homotopic to a constant map nor to a map onto a closed geodesic. Then there exists a Hermitian harmonic map $f : M \to N$ homotopic to g.*

This is Theorem 1 of [**7**], where related existence theorems are also proved.

The proof of Theorem (1.1) proceeds as follows. Suppose there exists a homomorphism $\phi : \pi_1(M) \to \pi_1(N)$ whose image is not a cyclic group. Since N is an aspherical manifold, there exists a smooth map $g : M \to N$ inducing ϕ on fundamental groups, and since the image of ϕ is not cyclic, g is not homotopic to a constant map nor to a map onto a closed geodesic. By Theorem (2.2) there exists a Hermitian harmonic map $f : M \to N$ homotopic to g. By Theorem (2.1) f is pluriharmonic. We are therefore in the position conclude the proof by using the techniques of §7 of [**1**] for studying pluriharmonic maps to constant curvature targets.

More precisely, first we have that $R(X,Y) = 0$ for $X, Y \in TN \otimes \mathbb{C}$ can only hold if $X \wedge Y = 0$. Thus, if $d'f : T^{1,0}M \to f^*TN \otimes \mathbb{C}$ denotes the restriction of df to $T^{1,0}M$, then the image of $d'f$ has rank at most one. Thus the image of df has real rank at most two, and, under the assumptions of the theorem, the generic rank must be two, since otherwise f is either constant or a map onto a closed geodesic.

Choose an open set $U \subset N$ with the property that $V = U \cap f(M)$ is not empty and for all $q \in V$, $f^{-1}(q)$ is a smooth submanifold of M (of real codimension 2), and such that the real rank of $d_pf = 2$ for all $p \in f^{-1}(q)$. Such an open set exists by the appropiate version of Sard's theorem, cf. Theorem 3.4.3 of [**2**], or Theorem 1.3 of [**16**]. Namely, since the generic rank of f is 2 and f is a smooth map, $f(M)$ has Hausdorff dimension 2. Let S denote the set of those $p \in M$ for which the rank of d_pf is at most 1. Then $f(S)$ has Hausdorff dimension at most 1, and by the compactness of M is a closed subset of $f(M)$.

Thus $f(M) - f(S)$ is a non-empty relatively open subset of $f(M)$, and our desired open set U is $N - f(S)$.

At each $p \in M$ where $rank(d_p f) = 2$ we have that $ker(d_p f)$ is invariant under J (cf. the proof of Lemma(7.3) of [**1**]). Thus for each $q \in V$, $f^{-1}(q)$ is a compact complex submanifold of M. Thus the collection $f^{-1}(q), q \in V$ gives an infinite collection of compact complex curves in M. But if a surface has no non-constant meromorphic function, then it has only finitely many compact complex curves [**8**], Theorem 5.1. This contradicts that M is a non-elliptic surface of class VII, thus such a homomorphism cannot exist and Theorem (1.1) is proved.

Remark: If we only assume that N is strictly pointwise 1/4-pinched, the conclusion of Theorem(1.1) still holds, since the above arguments are still valid in this context thanks to [**4**]. The same remark applies if N has strictly negative curvature and non-positive Hermitian curvature, thus justifying the remark immediately following the statement of Theorem (1.1).

3. Proof of Corollary (1.2)

We now prove Corollary (1.2). For this we need some facts about the classification of surfaces, for which we refer to [**15**] and parts I and II of [**9**], and facts on elliptic surfaces for which we refer to [**3**]. First we need only consider minimal surfaces (i.e., free of exceptional curves). We consider three mutually exclusive cases that together exhaust all minimal surfaces: (1) Kähler surfaces (classes I_0 to V_0 in [**9**]; equivalently minimal surfaces with even first Betti number); (2) minimal elliptic surfaces with odd first Betti number (class VI_0 and elliptic surfaces of class VII_0); (3) non-elliptic surfaces of class VII_0. (In part I of [**9**] it is proved that any surface with even first Betti number is a deformation of an algebraic surface, and this suffices for our arguments; the fact that these are all Kähler follows from later work of Miyaoka and Siu, cf. [**15**] for references.)

We have already proved the statement of Corollary (1.2) for Kähler manifolds in [**1**], hence case (1), and case (3) follows form Theorem (1.1). It only remains to consider case (2). But an elliptic surface with odd first Betti number cannot have singular fibers except smooth multiple ones. Namely, the presence of singular fibers (by monodromy and vanishing cycle considerations) forces the fundamental group of the surface to be isomorphic to the (orbifold) fundamental group of the base curve, hence the first Betti number to be even, cf. §2.2.1 of [**3**]. The absence of singular fibers (other than multiple smooth ones) gives an exact sequence of orbifold homotopy groups ¿From which one sees that one of the following two (not mutually exclusive) possibilities must hold: (i) the fundamental group contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, or (ii) the fundamental group contains an abelian subgroup of finite index. (for more details see §2.7.2 of [**3**]; (i) corresponds to the case of flat or hyperbolic base orbifold, (ii) for spherical base orbifold or for bad base orbifold). In either case we see that the fundamental group cannot be isomorphic to the fundamental group of a compact manifold of strictly negative curvature, and the proof is complete.

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