Hyperbolic Geometry and the Moduli Space of Real Binary Sextics

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Abstract. The moduli space of real 6-tuples in $\mathbb{C}P^1$ is modeled on a quotient of hyperbolic 3-space by a nonarithmetic lattice in Isom H^3. This is an expository note; the first part of it is an introduction to orbifolds and hyperbolic reflection groups.

These notes are an exposition of the key ideas behind our result that the moduli space $\mathcal{M}_s$ of stable real binary sextics is the quotient of real hyperbolic 3-space H^3 by a certain Coxeter group (together with its diagram automorphism). We hope they can serve as an aid in understanding our work [2] on moduli of real cubic surfaces, since exactly the same ideas are used, but the computations are easier and the results can be visualized.

These notes derive from the first author's lectures at the summer school "Algebra and Geometry around Hypergeometric Functions", held at Galatasary University in Istanbul in July 2005. He is grateful to the organizers, fellow speakers and students for making the workshop very rewarding. To keep the flavor of lecture notes, not much has been added beyond the original content of the lectures; some additional material appears in an appendix. The pictures are hand-drawn to encourage readers to draw their own.

Lecture 1

Hyperbolic space H^3 is a Riemannian manifold for which one can write down an explicit metric, but for us the following model will be more useful; it is called the upper half-space model. Its underlying set is the set of points in $\mathbb{R}^3$ with positive vertical coordinate, and geodesics appear either as vertical half-lines, or

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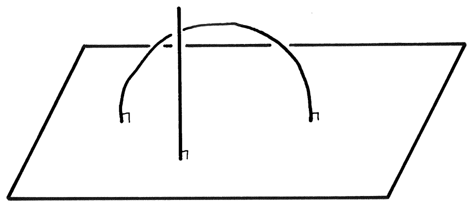
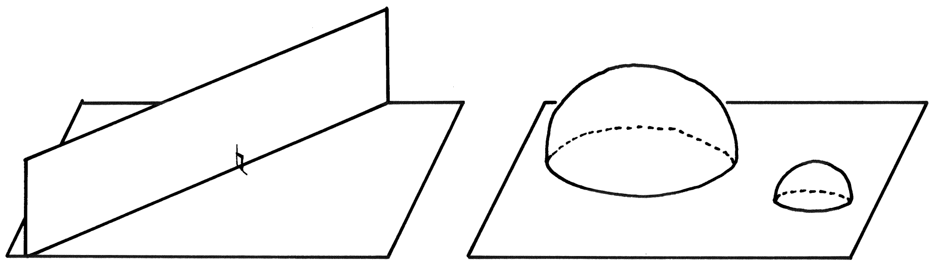
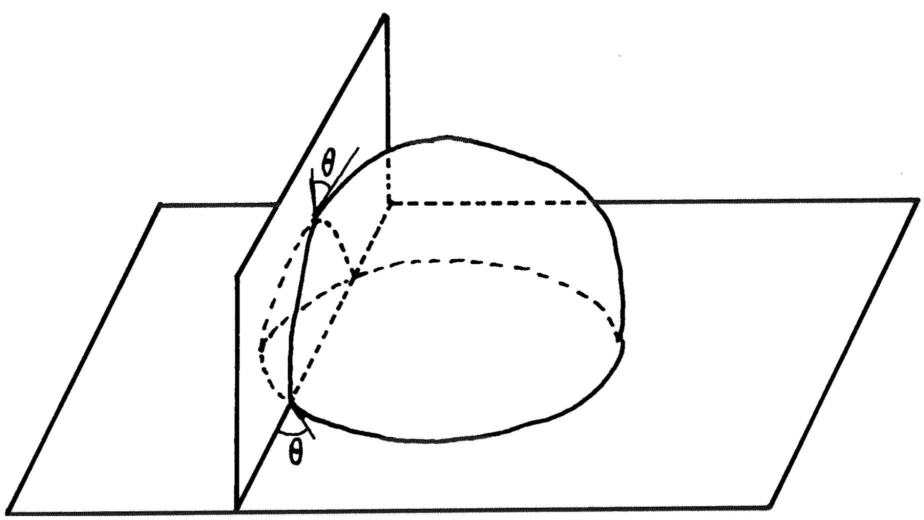
as semicircles with both ends resting on the bounding $\mathbb{R}^2$:



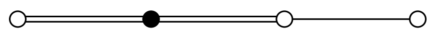
Note that the 'endpoints' of these geodesics lie in the boundary of H^3, not in H^3 itself. Planes appear either as vertical half-planes, or as hemispheres resting on $\mathbb{R}^2$:



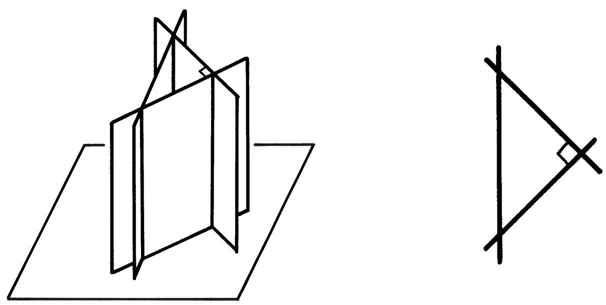
If two planes meet then their intersection is a geodesic. The most important property of the upper half-space model is that it is conformal, meaning that an angle between planes under the hyperbolic metric equals the Euclidean angle between the half-planes and/or hemispheres. For example, the following angle θ looks like a $\pi/4$ angle, so it *is* a $\pi/4$ angle:



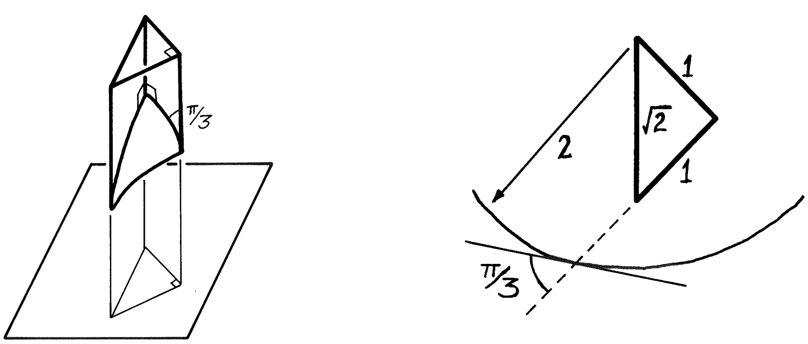
This lets us build hyperbolic polyhedra with specified angles by pushing planes around. For example, the diagram

 P_0 (1)

describes a polyhedron P_0 with four walls, corresponding to the nodes, with the interior angle between two walls being $\pi/2$, $\pi/3$ or $\pi/4$ according to whether the nodes are joined by no edge, a single edge or a double edge. For now, ignore the colors of the nodes; they play no role until theorem 2. We can build a concrete model of P_0 by observing that the first three nodes describe a Euclidean $(\pi/2, \pi/4, \pi/4)$ triangle, so the first three walls should be arranged to appear as vertical halfplanes. Sometimes pictures like this can be easier to understand if you also draw the view down from vertical infinity; here are both pictures:



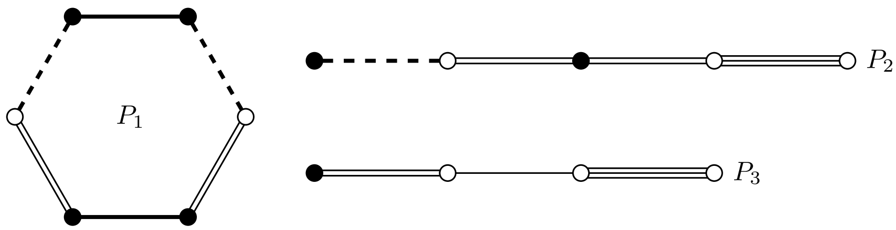
How to fit in the fourth plane? After playing with it one discovers that it cannot appear as a vertical halfplane, so we look for a suitable hemisphere. It must be orthogonal to two of our three walls, so it is centered at the foot of one of the half-lines of intersection. The size of the hemisphere is determined by the angle it makes with the remaining wall (namely $\pi/3$). We have drawn the picture so that the hemisphere is centered at the foot of the back edge. The figure should continue to vertical infinity, but we cut it off because seeing the cross-section makes the polyhedron easier to understand. We've also drawn the view from above; the boundary circle of the hemisphere strictly contains the triangle, corresponding to the fact that P_0 does not descend all the way to the boundary $\mathbb{R}^2$.



(2)

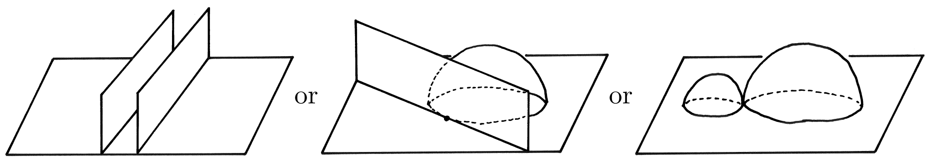
We think of P_0 as an infinitely tall triangular chimney with its bottom bitten off by a hemisphere. The dimensions we have drawn on the overhead view refer to Euclidean distances, not hyperbolic ones. The "size" of a hemisphere has no intrinsic meaning in hyperbolic geometry, since the isometry group of H^3 acts transitively on planes.

Readers may enjoy trying their hands at this by drawing polyhedra for the diagrams

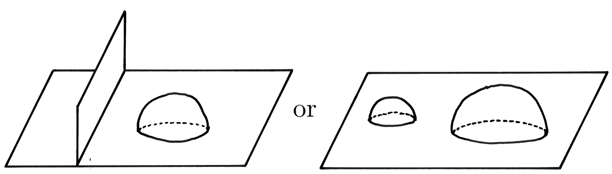


(3)

where the absent, single and double bonds mean the same as before, a triple bond indicates a $\pi/6$ angle, a heavy bond means parallel walls and a dashed bond means ultraparallel walls. In the last two cases we describe the meaning by pictures: parallelism means



and ultraparallelism means

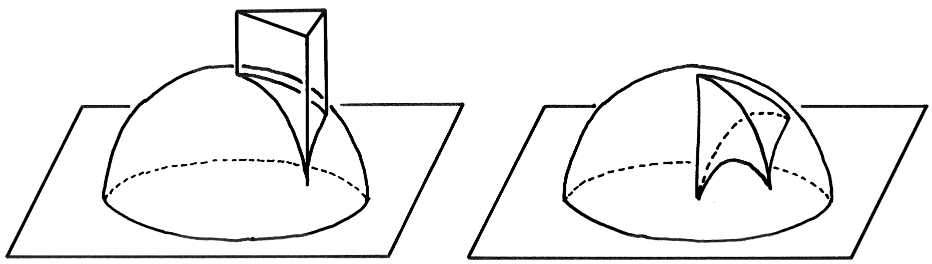


That is, when two planes do not meet in H^3, we call them parallel if they meet at the boundary of H^3 and ultraparallel if they do not meet even there.

Diagrams like (1) and (3) are called Coxeter diagrams after H. S. M. Coxeter, who introduced them to classify the finite groups generated by reflections. Given a random diagram, there is no guarantee that one can find a hyperbolic polyhedron with those angles, but if there is one then it describes a discrete group acting on H^3:

Theorem 1 (Poincaré Polyhedron Theorem). *Suppose $P \subseteq H^3$ is a polyhedron (i.e., the intersection of a finite number of closed half-spaces) with every dihedral angle of the form $\pi/(\text{an integer})$. Let Γ be the group generated by the reflections across the walls of P. Then Γ is discrete in $\operatorname{Isom} H^3$ and P is a fundamental* *domain for* Γ *in the strong sense: every point of* H^3 *is* Γ*-equivalent to exactly one point of* P.

The proof is a very pretty covering space argument; see [4] for this and for a nice introduction to Coxeter groups in general. A reflection across a plane means the unique isometry of H^3 that fixes the plane pointwise and exchanges the components of its complement. Reflection across a vertical half-plane looks like an ordinary Euclidean reflection, and reflection across a hemisphere means inversion in it; here are before-and-after pictures of an inversion.



An inversion exchanges vertical infinity with the point of $\mathbb{R}^2$ "at the center" of the hemisphere.

The data of a group Γ acting discretely on H^3 is encoded by an object called an orbifold. As a topological space it is H^3/Γ. But the orbifold has more structure: an orbifold chart on a topological space X is a continuous map from an open subset U of $\mathbb{R}^n$ to X, that factors as

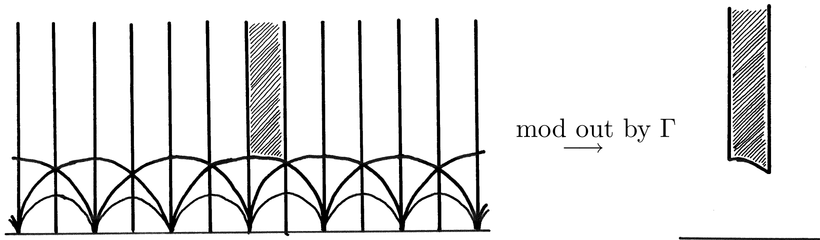
$$U \to U/\Gamma_U \to X,$$

where Γ_U is a finite group acting on U and the second map is a homeomorphism onto its image. Our H^3/Γ has lots of such charts, because with $x \in H^3$, Γ_x its stabilizer in Γ and U a sufficiently small open ball around x,

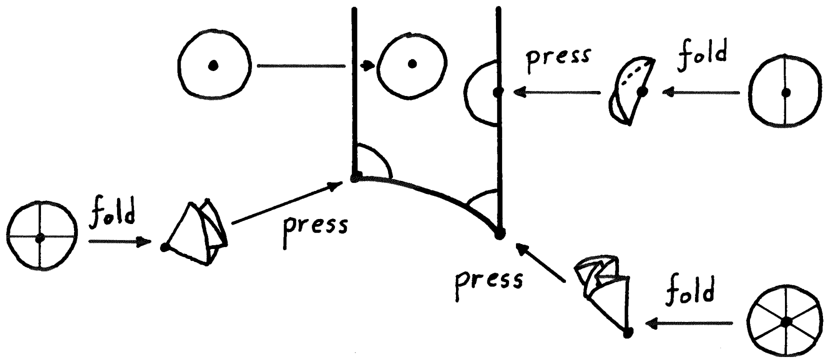
$$U \to U/\Gamma_x \to H^3/\Gamma$$

is an orbifold chart. An orbifold is a space locally modeled on a manifold modulo finite groups. Formally, an orbifold X is a hausdorff space covered by such charts, with the compatibility condition that if $x \in X$ lies in the image of charts $U \to U/\Gamma_U \to X$ and $U' \to U'/\Gamma_{U'} \to X$ then there are preimages v and v' of x in U and U' with neighborhoods V and V' preserved by $\Gamma_{U,v}$ and $\Gamma_{U',v'}$, an isomorphism $\Gamma_{U,v} \cong \Gamma_{U',v'}$ and an equivariant isomorphism $\tau_{V,V'}$ between V and V' identifying v with v'. The group $\Gamma_{U,v}$ is called the local group at x, and the nature of the isomorphisms $\tau_{V,V'}$ determines the nature of the orbifold. That is, if all the $\tau_{V,V'}$ are homeomorphisms then X is a topological orbifold, if all are real-analytic diffeomorphisms then X is a real-analytic orbifold, if all are hyperbolic isometries then X is a hyperbolic orbifold, and so on. So H^3/Γ is a hyperbolic orbifold. There is a notion of orbifold universal cover which allows one to reconstruct H^3 and its Γ-action from the orbifold H^3/Γ.

Only in two dimensions is it easy to draw pictures of orbifold charts; here they are for the quotient of the upper half-plane H^2 by the group Γ generated by reflections across the edges of the famous $(\pi/2, \pi/3, \pi/\infty)$ triangle.



Here are local orbifold charts around various points of H^3/Γ:



In three dimensions essentially the same thing happens: the local chart at a generic point of a wall is the quotient of a 3-ball by a reflection, and along an edge it is the quotient of a 3-ball by a dihedral group. One needs to understand the finite Coxeter groups in dimension 3 in order to understand the folding at the vertices, but this is not necessary here.

We care about hyperbolic orbifolds because it turns out that moduli spaces arising in algebraic geometry are usually orbifolds, and it happens sometimes that such a moduli space happens to coincide with a quotient of hyperbolic space (or complex hyperbolic space or any of the other symmetric spaces). So we can sometimes gain insight into the algebraic geometry by manipulating simple objects like tilings of hyperbolic space.

Suppose a Lie group G acts properly on a smooth manifold X with finite stabilizers. (Properly means that the map $G \times X \to X \times X$ given by $(g, x) \mapsto (g(x), x)$ is a proper map, which means that preimages of compact sets are compact; this is needed for $G\backslash X$ to be Hausdorff.) Then the quotient $G\backslash X$ is an orbifold, by the following construction. For $x \in X$ one can find a small transversal T to the orbit $G.x$, that is preserved by the stabilizer G_x. Then $T \to G_x\backslash T \to G\backslash X$ gives an orbifold chart. In particular, the local group at the image of x in $G\backslash X$ is G_x. If G acts real-analytically then $G\backslash X$ is a real-analytic orbifold.

Now we come to the case which concerns us. Let $\mathcal{C}$ be the set of binary sextics, i.e., nonzero 2-variable homogeneous complex polynomials of degree 6, modulo scalars, so $\mathcal{C} = \mathbb{C}P^6$. Let $\mathcal{C}^{\mathbb{R}}$ be the subset given by those with real coefficients, $\mathcal{C}_0$ the smooth sextics (those with 6 distinct roots), and $\mathcal{C}_0^{\mathbb{R}}$ the intersection. Then $G = \mathrm{PGL}_2\mathbb{C}$ acts on $\mathcal{C}$ and $\mathcal{C}_0$ and $G^{\mathbb{R}} = \mathrm{PGL}_2\mathbb{R}$ acts on $\mathcal{C}^{\mathbb{R}}$ and $\mathcal{C}_0^{\mathbb{R}}$. The moduli space $\mathcal{M}_0$ of smooth binary sextics is $G\backslash\mathcal{C}_0$, of 3 complex dimensions. The real moduli space $\mathcal{M}_0^{\mathbb{R}} = G^{\mathbb{R}}\backslash\mathcal{C}_0^{\mathbb{R}}$ is *not* the moduli space of 6-tuples in $\mathbb{R}P^1$; rather it is the moduli space of nonsingular 6-tuples in $\mathbb{C}P^1$ which are preserved by complex conjugation. This set has 4 components, $\mathcal{M}_{0,j}^{\mathbb{R}}$ being $G^{\mathbb{R}}\backslash\mathcal{C}_{0,j}^{\mathbb{R}}$, where j indicates the number of pairs of conjugate roots. It turns out that G acts properly on $\mathcal{C}_0$, and since the point stabilizers are compact algebraic subgroups of G they are finite; therefore $\mathcal{M}_0$ is a complex-analytic orbifold and the $\mathcal{M}_{0,j}^{\mathbb{R}}$ are real-analytic orbifolds. The relation with hyperbolic geometry begins with the following theorem:

Theorem 2. *Let Γ_j be the group generated by the Coxeter group of P_j from* (1) *or* (3)*, together with the diagram automorphism when $j=1$. Then $\mathcal{M}_{0,j}^{\mathbb{R}}$ is the orbifold H^3/Γ_j, minus the image therein of the walls corresponding to the blackened nodes and the edges corresponding to triple bonds. Here, 'is' means an isomorphism of real-analytic orbifolds.*

In the second lecture we will see that the faces of the P_j corresponding to blackened nodes and triple bonds are particularly interesting; we will glue the P_j together to obtain a real-hyperbolic description of the entire moduli space.

References. The canonical references for hyperbolic geometry and an intreoduction to orbifolds are Thurston's notes [13] and book [14]. The book is a highly polished treatment of a subset of the material in the notes, which inspired a great deal of supplementary material, e.g., [3]. For other applications of hyperbolic geometry to real algebraic geometry, see Nikulin's paper [11], which among other things describes moduli spaces of various sorts of K3 surfaces as quotients of H^n.

Lecture 2

We will not really provide a proof of theorem 2; instead we will develop the ideas behind it just enough to motivate the main construction leading to theorem 4 below. Although theorem 2 concerns smooth sextics, it turns out to be better to consider mildly singular sextics as well. Namely, let $\mathcal{C}_s$ be the set of binary sextics with no point of multiplicity 3 or higher, and let $\Delta \subseteq \mathcal{C}_s$ be the discriminant, so $\mathcal{C}_0 = \mathcal{C}_s - \Delta$. (For those who have seen geometric invariant theory, $\mathcal{C}_s$ is the set of stable sextics, hence the subscript s.) It is easy to see that Δ is a normal crossing divisor in $\mathcal{C}_s$. (In the space of *ordered* 6-tuples in $\mathbb{C}P^1$ this is clear; to get the picture in $\mathcal{C}_s$ one mods out by permutations.) Now let $\mathcal{F}_s$ be the universal branched cover of $\mathcal{C}_s$, with ramification of order 6 along each component of the preimage of Δ. $\mathcal{F}_s$ turns out to be smooth and the preimage of Δ a normal crossing divisor. More precisely, in a neighborhood of a point of $\mathcal{F}_s$ describing a sextic with k double points, the map to $\mathcal{C}_s$ is given locally by

$$(z_1, \ldots, z_6) \mapsto (z_1^6, \ldots, z_k^6, z_{k+1}, \ldots, z_6) \ ,$$

where the branch locus is the union of the hypersurfaces $z_1 = 0, \ldots, z_k = 0$. Let $\mathcal{F}_0$ be the preimage of $\mathcal{C}_0$ and let Γ be the deck group of $\mathcal{F}_s$ over $\mathcal{C}_s$. We call an element of $\mathcal{F}_s$ (resp. $\mathcal{F}_0$) a framed stable (resp. smooth) binary sextic. Geometric invariant theory implies that G acts properly on $\mathcal{C}_s$, and one can show that this G-action lifts to one on $\mathcal{F}_s$ which is not only proper but free, so $G\backslash\mathcal{F}_s$ is a complex manifold. The reason we use 6-fold branching is that in this case $G\backslash\mathcal{F}_s$ has a nice description, given by the following theorem. See the appendix for a sketch of the Hodge theory involved in the proof.

Theorem 3 (Deligne-Mostow [5]**).** *There is a properly discontinuous action of* Γ *on complex hyperbolic* 3*-space* $\mathbb{C}H^3$ *and a* Γ*-equivariant diffeomorphism* $g : G\backslash\mathcal{F}_s \to \mathbb{C}H^3$*, identifying* $G\backslash\mathcal{F}_0$ *with the complement of a hyperplane arrangement* $\mathcal{H}$ *in* $\mathbb{C}H^3$.

Complex hyperbolic space is like ordinary hyperbolic space except that it has 3 complex dimensions, and hyperplanes have complex codimension 1. There is an upper-half space model analogous to the real case, but the most common model for it is the (open) complex ball. This is analogous to the Poincaré ball model for real hyperbolic space; we don't need the ball model except to see that complex conjugation of $\mathbb{C}H^3$, thought of as the complex 3-ball, has fixed-point set the real 3-ball, which is H^3.

Given a framed stable sextic $\tilde{S}$, theorem 3 gives us a point $g(\tilde{S})$ of $\mathbb{C}H^3$. If $\tilde{S}$ lies in $\mathcal{F}_0^{\mathbb{R}}$ (the preimage of $\mathcal{C}_0^{\mathbb{R}}$), say over $S \in \mathcal{C}_0^{\mathbb{R}}$, then we can do better, obtaining not just a point of $\mathbb{C}H^3$ but also a copy of H^3 containing it. The idea is that complex conjugation κ of $\mathcal{C}_0$ preserves S and lifts to an antiholomorphic involution (briefly, an anti-involution) $\tilde{\kappa}$ of $\mathcal{F}_0$ that fixes $\tilde{S}$. This uses the facts that $\mathcal{F}_0 \to \mathcal{C}_0$ is a covering space and that $\pi_1(\mathcal{F}_0) \subseteq \pi_1(\mathcal{C}_0)$ is preserved by κ. Riemann extension extends $\tilde{\kappa}$ to an anti-involution of $\mathcal{F}_s$. Since κ normalizes G's action on $\mathcal{C}_s$, $\tilde{\kappa}$ normalizes G's action on $\mathcal{F}_s$, so $\tilde{\kappa}$ descends to an anti-involution κ' of $\mathbb{C}H^3 = G\backslash\mathcal{F}_s$. Each anti-involution of $\mathbb{C}H^3$ has a copy of H^3 as its fixed-point set, so we have defined a map $g^{\mathbb{R}}$ from $\mathcal{F}_0^{\mathbb{R}}$ to the set of pairs

$$(x \in \mathbb{C}H^3, \text{ a copy of } H^3 \text{ containing } x). \tag{4}$$

Note that $\tilde{\kappa}$ fixes every point of $\mathcal{F}_0^{\mathbb{R}}$ sufficiently near $\tilde{S}$, so all nearby framed real sextics determine the same anti-involution κ' of $\mathbb{C}H^3$. Together with the G-invariance of g, this proves that $g^{\mathbb{R}}$ is invariant under the identity component of $G^{\mathbb{R}}$. A closer study of $g^{\mathbb{R}}$ shows that it is actually invariant under all of $G^{\mathbb{R}}$. We write K_0 for the set of pairs (4) in the image $g^{\mathbb{R}}(\mathcal{F}_0^{\mathbb{R}})$. An argument relating points of $\mathcal{C}_s$ preserved by anti-involutions in $G \rtimes (\mathbb{Z}/2)$ to points of $\mathbb{C}H^3$ preserved by anti-involutions in $\Gamma \rtimes (\mathbb{Z}/2)$ shows that if $x \in \mathcal{F}_0^{\mathbb{R}}$ has image $(g(x), H)$, then every pair $(y \in H - \mathcal{H}, H)$

also lies in K_0. That is, K_0 is the disjoint union of a bunch of H^3's, minus their intersections with $\mathcal{H}$. The theoretical content of theorem 2 is that $g^{\mathbb{R}} : G^{\mathbb{R}}\backslash\mathcal{F}_0^{\mathbb{R}} \to K_0$ is a diffeomorphism.

The computational part of theorem 2 is the explicit description of K_0, in enough detail to understand $\mathcal{M}_0 = G\backslash\mathcal{F}_0^{\mathbb{R}}/\Gamma = K_0/\Gamma$ concretely. It turns out that Γ, $\mathcal{H}$ and the anti-involutions can all be described cleanly in terms of a certain lattice Λ over the Eisenstein integers $\mathcal{E} = \mathbb{Z}[\omega{=}e^{2\pi i/3}]$. Namely, Λ is a rank 4 free $\mathcal{E}$-module with Hermitian form

$$\langle a|a\rangle = a_0\bar{a}_0 - a_1\bar{a}_1 - a_2\bar{a}_2 - a_3\bar{a}_3\ . \tag{5}$$

The set of positive lines in $P(\mathbb{C}^{1,3} = \Lambda \otimes_{\mathcal{E}} \mathbb{C})$ is a complex 3-ball (i.e., $\mathbb{C}H^3$), $\Gamma = P\mathrm{Aut}\,\Lambda$, $\mathcal{H}$ is the union of the hyperplanes orthogonal to norm -1 elements of Λ, and the anti-involutions of $\mathbb{C}H^3$ corresponding to the elements of K_0 are exactly

$$\begin{aligned}
\kappa_0 &: (x_0,x_1,x_2,x_3) \mapsto (\bar{x}_0,\ \ \bar{x}_1,\ \ \bar{x}_2,\ \ \bar{x}_3)\\
\kappa_1 &: (x_0,x_1,x_2,x_3) \mapsto (\bar{x}_0,\ \ \bar{x}_1,\ \ \bar{x}_2,-\bar{x}_3)\\
\kappa_2 &: (x_0,x_1,x_2,x_3) \mapsto (\bar{x}_0,\ \ \bar{x}_1,-\bar{x}_2,-\bar{x}_3)\\
\kappa_3 &: (x_0,x_1,x_2,x_3) \mapsto (\bar{x}_0,-\bar{x}_1,-\bar{x}_2,-\bar{x}_3)
\end{aligned} \tag{6}$$

and their conjugates by Γ. We write H^3_j for the fixed-point set of κ_j.

Since $H^3_0,\dots,H^3_3$ form a complete set of representatives for the H^3's comprising K_0, we have

$$\mathcal{M}_0^{\mathbb{R}} = K_0/\Gamma = \coprod_{j=0}^{3} \left(H^3_j - \mathcal{H}\right)\Big/\text{(its stabilizer }\Gamma_j\text{ in }\Gamma)$$

Understanding the stabilizers Γ_j required a little luck. Vinberg devised an algorithm for searching for a fundamental domain for a discrete group acting on H^n that is generated by reflections [16]. It is not guaranteed to terminate, but if it does then it gives a fundamental domain. We were lucky and it did terminate; the reflection subgroup of Γ_j turns out to be the Coxeter group of the polyhedron P_j.

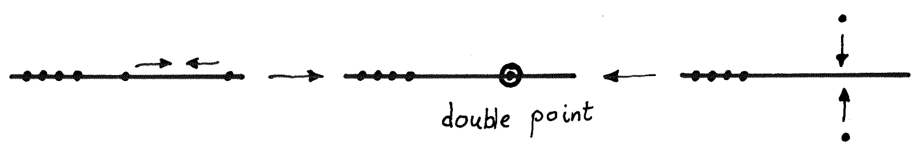
One can obtain our polyhedra by applying his algorithm to the $\mathbb{Z}$-sublattices of Λ fixed by each κ_j. For example, an element of the κ_2-invariant part of Λ has the form $(a_0, a_1, a_2\sqrt{-3}, a_3\sqrt{-3})$ with $a_0,\dots,a_3 \in \mathbb{Z}$, of norm $a_0^2 - a_1^2 - 3a_2^2 - 3a_3^2$. Similar analysis leads to the norm forms

$$\begin{aligned}
\langle a|a\rangle &= a_0^2 - \ \ a_1^2 - \ \ a_2^2 - \ \ a_3^2\\
\langle a|a\rangle &= a_0^2 - \ \ a_1^2 - \ \ a_2^2 - 3a_3^2\\
\langle a|a\rangle &= a_0^2 - \ \ a_1^2 - 3a_2^2 - 3a_3^2\\
\langle a|a\rangle &= a_0^2 - 3a_1^2 - 3a_2^2 - 3a_3^2\ .
\end{aligned}$$

in the four cases of (6). Now, Γ_j lies between its reflection subgroup and the semidirect product of this subgroup by its diagram automorphisms. After checking that the diagram automorphism of P_1 lies in Γ_1, the identification of the Γ_j is complete.

The final part of theorem 2 boils down to considering how the H^3's comprising K_0 meet the hyperplanes comprising $\mathcal{H}$. There is no big idea here; one just works out the answer and writes it down. There are essentially two ways that the H^3 fixed by an anti-involution κ of Λ can meet a hyperplane $r^\perp$, where $r \in \Lambda$ has norm -1. It might happen that $\kappa(r)$ is proportional to r, in which case $H^3 \cap r^\perp$ is a copy of H^2; this accounts for the deleted walls of the P_j. It can also happen that $\kappa(r) \perp r$, in which case $H^3 \cap r^\perp$ is a copy of H^1; this accounts for the deleted edges.

Now, the deleted faces are very interesting, and the next step in our discussion is to add them back in. By theorem 3 we know that points of $\mathcal{H}$ represent singular sextics, which occur along the boundary between two components of $\mathcal{C}_0^{\mathbb{R}}$. For example,



Varying the remaining four points gives a 2-parameter family of singular sextics which lie in the closures of both $\mathcal{C}_{0,0}^{\mathbb{R}}$ and $\mathcal{C}_{0,1}^{\mathbb{R}}$. This suggests reinstating the deleted walls of P_0 and P_1 and gluing the reinstated wall of P_0 to one of the reinstated walls of P_1. Which walls, and by what identification? There is really no choice here, because H_0^3 and H_1^3 meet along an H^2 that lies in $\mathcal{H}$, namely the locus

$$\{(a_0, a_1, a_2, a_3) \in \mathbb{C}^{1,3} \mid a_0, a_1, a_2 \in \mathbb{R} \text{ and } a_3 = 0\} .$$

This gives a rule for identifying the points of P_0 and P_1 that lie in this H^2.

Carrying out the gluing visually is quite satisfying; we will draw the pictures first and then worry about what they mean. We have indicated why P_0 and P_1 are glued; in a similar way, P_1 and P_2 are glued, as are P_2 and P_3. This uses up all the gluing walls of the various H_j^3/Γ_j because each has only two, except for P_0 and P_3 which have one each. The $j = 1$ case is interesting because P_1 has four gluing walls, but H_1^3/Γ_1 has only two because the diagram automorphism of P_1 exchanges them in pairs. So the gluing pattern is

$$P_0 \;—\; P_1/(\mathbb{Z}/2) \;—\; P_2 \;——\; P_3 \tag{7}$$

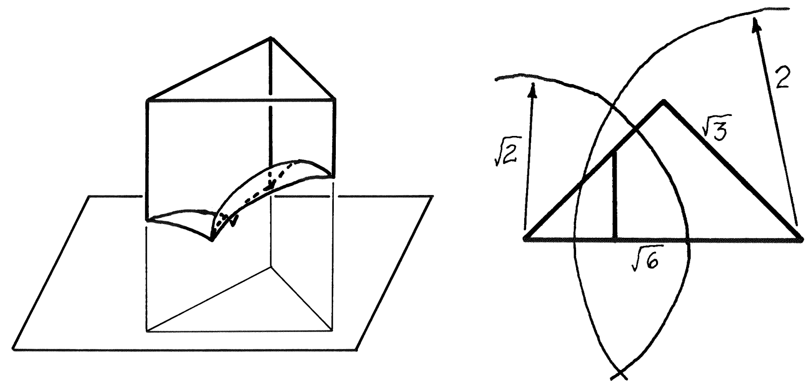
Working with polyhedra is so much simpler than working with quotients of them by isometries that we will carry out the gluing by assembling P_1 and two copies each of P_0, P_2 and P_3, according to

$$\begin{array}{ccccccc} P_0 & & & & P_2 & —— & P_3 \\ & \diagdown & & \diagup & & & \\ & & P_1 & & & & \\ & \diagup & & \diagdown & & & \\ P_0 & & & & P_2 & —— & P_3 \end{array} \tag{8}$$

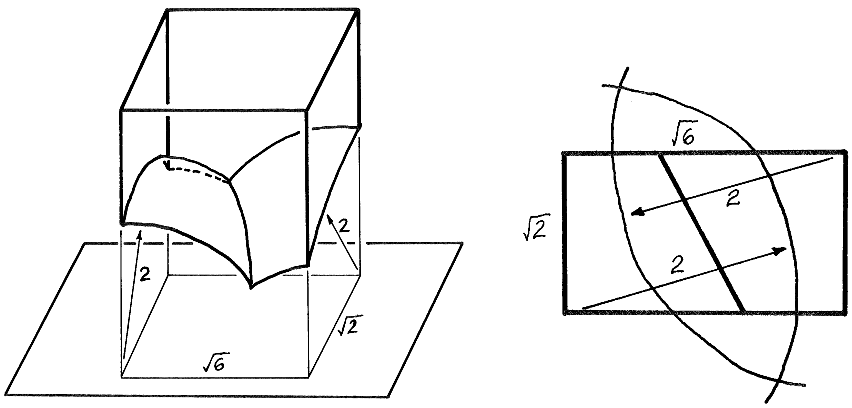
and take the quotient of the result by the diagram automorphism.

We begin by assembling P_1 and the copies of P_0 and P_2. This requires pictures of the polyhedra. P_0 appears in (2), and for the others we draw both 3-dimensional and an overhead views.

P_2



P_1



As before, length markings refer to Euclidean, not hyperbolic, distances.

There is only one way to identify isometric faces in pairs, pictured in figure 1. We wind up with a square chimney with four bites taken out of the bottom, two of radius 2 and two of radius $\sqrt{2}$. The result appears in figures 2 and 3 in overhead and 3-dimensional views.

It is time to attach the two copies of P_3. We won't use a "chimney" picture of P_3 because none of the four vertical walls in figure 3 are gluing walls; rather, the two gluing walls are the two small faces on the bottom. Happily, the region bounded by one of these walls and the extensions across it of the three walls it meets is a copy of P_3. That is, P_3 may be described as the interior of a hemisphere of radius $\sqrt{2}$, intersected with one side of a vertical half-plane and the *exteriors*

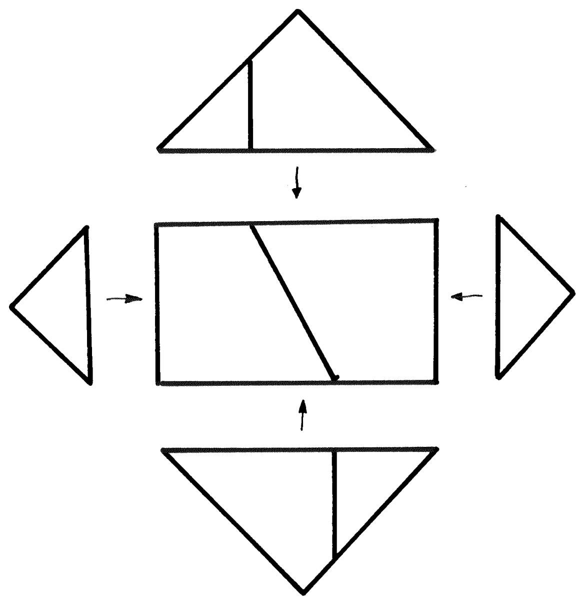


Figure 1. Overhead view of instructions for gluing P_1 to two copies of P_0 and two copies of P_2.

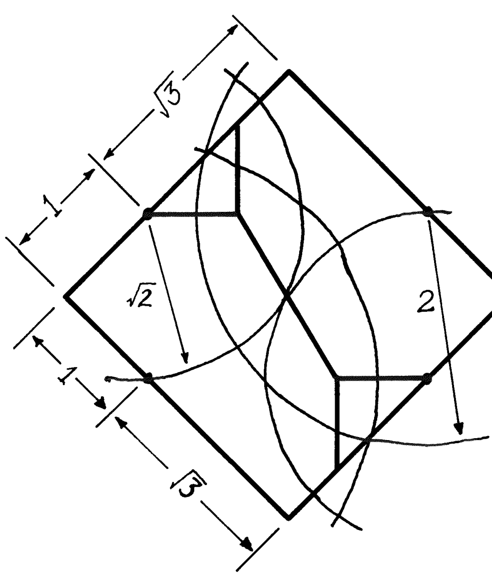


Figure 2. Overhead view of the result of gluing P_1 to two copies of P_0 and two copies of P_2.

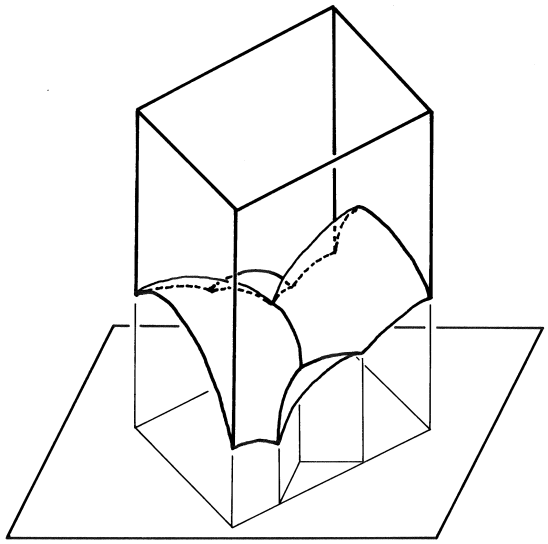
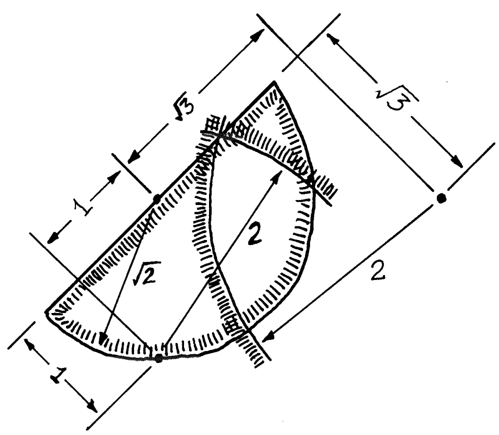
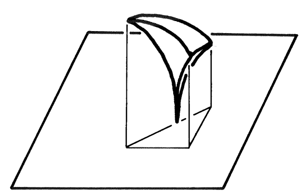


FIGURE 3. Three-dimensional view of the result of gluing P_1 to two copies of P_0 and two copies of P_2.

of two hemispheres of radius 2:





The 3-dimensional picture shows a copy of P_3 that fits neatly beneath one of the bottom walls of figure 3 (the back one). Adjoining it, and another copy of P_3 in the symmetrical way, completes the gluing described in (8). The result appears in figures 4 and 5.

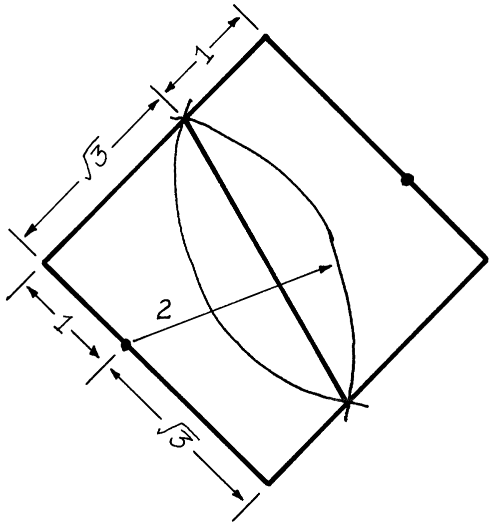


FIGURE 4. Overhead view of the final result of gluing the polyhedra according to (8).

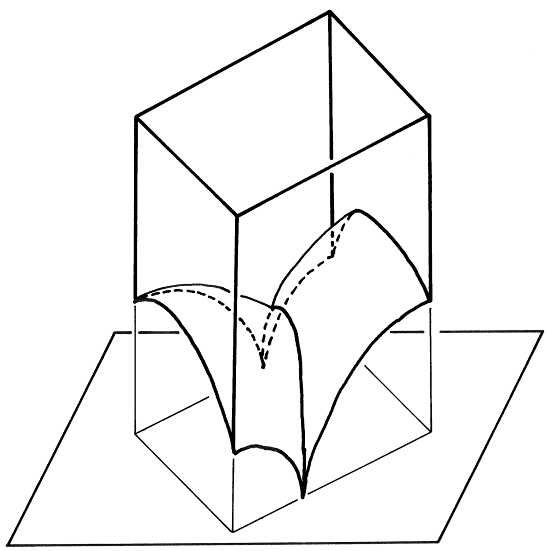
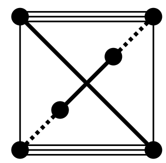


FIGURE 5. Three-dimensional view of the final result of gluing the polyhedra according to (8).

One can find its dihedral angles from our pictures; it is a Coxeter polyhedron with diagram

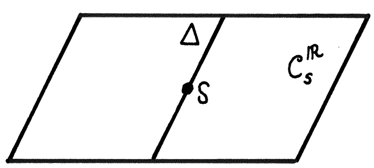


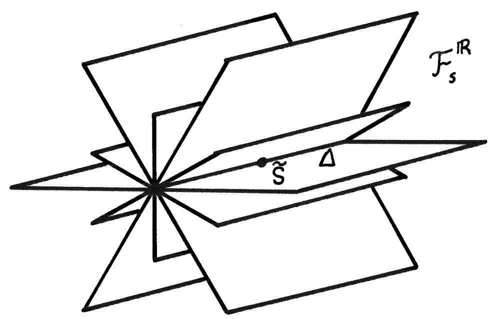
This leads to our main result; we write $\Gamma^{\mathbb{R}}$ for the group generated by this Coxeter group and its diagram automorphism, and Q for $H^3/\Gamma^{\mathbb{R}}$.

Theorem 4. *We have $\mathcal{M}_s^{\mathbb{R}} \cong Q = H^3/\Gamma^{\mathbb{R}}$, where "$\cong$" means the following:*

(*i*) *$\mathcal{M}_s^{\mathbb{R}} \to Q$ is a homeomorphism;*
(*ii*) *$\mathcal{M}_s^{\mathbb{R}} \to Q$ is an isomorphism of topological orbifolds if the orbifold structure of Q is changed along the edges associated to triple bonds, by replacing the dihedral group D_6 of order 12 by $\mathbb{Z}/2$ (see below);*
(*iii*) *$\mathcal{M}_s^{\mathbb{R}} \to Q$ is an isomorphism of real-analytic orbifolds if Q is altered as in* (*ii*) *and also along the loci where the P_j are glued together.*

For the rest of the lecture we will focus on the perhaps-surprising subtlety regarding the orbifold structures of $\mathcal{M}_s^{\mathbb{R}}$ and Q. We take $\mathcal{F}_s^{\mathbb{R}}$ to be the preimage of $\mathcal{C}_s^{\mathbb{R}}$, or equivalently the closure of $\mathcal{F}_0^{\mathbb{R}}$. Now, $\mathcal{F}_s^{\mathbb{R}}$ is not a manifold because of the branching of the cover $\mathcal{F}_s \to \mathcal{C}_s$. One example occurs at $\tilde{S} \in \mathcal{F}_s^{\mathbb{R}}$ lying over a sextic $S \in \mathcal{C}_s^{\mathbb{R}}$ with a single double point, necessarily real. In a neighborhood U of S, $\mathcal{C}_s^{\mathbb{R}}$ is a real 6-manifold meeting the discriminant (a complex 5-manifold) along a real 5-manifold. A neighborhood of $\tilde{S}$ is got by taking a 6-fold cover of U, branched along Δ. Therefore near $\tilde{S}$, $\mathcal{F}_s^{\mathbb{R}}$ is modeled on 12 half-balls of dimension 6 meeting along their common 5-ball boundary. Here are pictures of the relevant parts of $\mathcal{C}_s^{\mathbb{R}}$ and $\mathcal{F}_s^{\mathbb{R}}$:





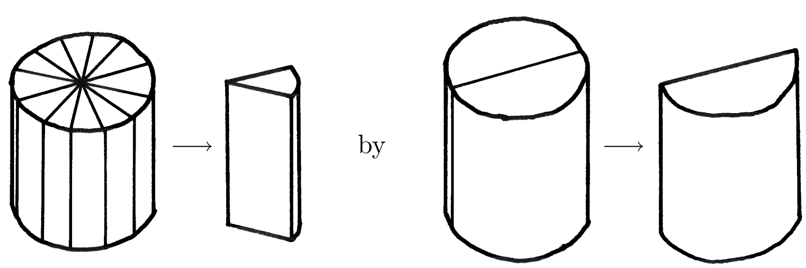
To get an orbifold chart around the image of S in $\mathcal{M}_s^{\mathbb{R}}$, we take a small transversal to $G^{\mathbb{R}}.S$ and mod out by the stabilizer of S in $G^{\mathbb{R}}$, as explained in lecture 1. To get an orbifold chart around the image of $\tilde{S}$ in $G^{\mathbb{R}}\backslash\mathcal{F}_s^{\mathbb{R}}/\Gamma$ we do the following, necessarily more complicated than before because $\mathcal{F}_s^{\mathbb{R}}$ isn't a manifold. We choose a transversal to $G^{\mathbb{R}}.\tilde{S}$, which is identified under g with a neighborhood of $g(\tilde{S})$ in the union X of six H^3's meeting along an H^2. We take the quotient of X by the stabilizer $\mathbb{Z}/6$ of $\tilde{S}$ in Γ. The result is isometric to H^3, and we take an open set in *this* H^3 as the domain for the orbifold chart, mapping to $G^{\mathbb{R}}\backslash\mathcal{F}_s^{\mathbb{R}}/\Gamma$ by taking the quotient of it by

$$(\text{the stabilizer of } g(\tilde{S}) \text{ and } X \text{ in } \Gamma) \big/ (\mathbb{Z}/6) \cong (\text{the stabilizer of } S \text{ in } G^{\mathbb{R}})\ .$$

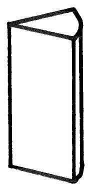
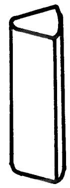
Identifying $\mathcal{M}_s^{\mathbb{R}}$ with $G^{\mathbb{R}}\backslash\mathcal{F}_s^{\mathbb{R}}/\Gamma$ leads to two orbifold charts around the same point. One can check that these charts define the same topological orbifold structure but different real-analytic structures. This leads to (*iii*) in theorem 4.

A slightly different phenomenon leads to (*ii*). Another possibility for how Δ meets $\mathcal{C}_s^{\mathbb{R}}$ is at a sextic S with two complex conjugate double points. Then in a neighborhood U of S, Δ has two branches through S, meeting transversely. The real 6-manifold $\mathcal{C}_s^{\mathbb{R}}$ meets Δ along a real 4-manifold lying in the intersection of these two branches. Since Δ has two branches through S, there is not 6-to-1 but 36-to-1 branching near $\tilde{S} \in \mathcal{F}_s^{\mathbb{R}}$ lying over S. It turns out that a neighborhood $\tilde{U}$ of $\tilde{S}$ in $\mathcal{F}_s^{\mathbb{R}}$ may be taken to be the union of six real 6-balls meeting along a common 4-ball, with each of the 6-balls mapping to U as a 6-to-1 cover branched over the 4-ball. We get an orbifold chart around the image of $\tilde{S}$ in $G^{\mathbb{R}}\backslash\mathcal{F}_s^{\mathbb{R}}/\Gamma$ as follows. Choose a transversal to $\Gamma^{\mathbb{R}}.\tilde{S}$, which maps bijectively to its image in $\mathbb{C}H^3$, which can be described as a neighborhood of $g(\tilde{S})$ in the union of six H^3's meeting along an H^1. Choose *one* of these H^3's and take the quotient of it by the subgroup of Γ which carries both it and $g(\tilde{S})$ to themselves. Generically this subgroup is D_6, because of the $\mathbb{Z}/6$ coming from the branching and the fact that S has a $\mathbb{Z}/2$ symmetry exchanging its double points. This gives an orbifold chart $U \to U/D_6 \to G^{\mathbb{R}}\backslash\mathcal{F}_s^{\mathbb{R}}/\Gamma$. (The idea also applies if S has more symmetry than the generic $\mathbb{Z}/2$.)

Now, this *cannot* be a valid description of the orbifold $\mathcal{M}_s^{\mathbb{R}}$, because the symmetry group of S is $\mathbb{Z}/2$ and so the local group at the image of S in $\mathcal{M}_s^{\mathbb{R}}$ should be $\mathbb{Z}/2$ not D_6. The problem is that the $\mathbb{Z}/6$ coming from the branching is an artifact of our construction. To eliminate it, we take the quotient of the chart by the $\mathbb{Z}/6$, obtaining a topological ball, and use this ball rather than the original one as the domain for the orbifold chart, with local group $D_6\big/(\mathbb{Z}/6) = \mathbb{Z}/2$. The effect of this operation is to replace the orbifold chart



We may picture this as a smoothing of the crease:

 becomes  .

Therefore $\mathcal{M}_s^{\mathbb{R}}$'s topological orbifold structure can be completely visualized by taking the hyperbolic polyhedron in figure 5 and smoothing two of its edges in this manner.

Appendix

We will give a sketch of the Hodge theory behind theorem 3 and then make a few remarks.

Theorem 3 is due to Deligne and Mostow [5], building on ideas of Picard; our approach is more explicitly Hodge-theoretic, along the lines of our treatment of moduli of cubic surfaces in [1]. Let $S \in \mathcal{C}_0$ be a smooth binary sextic, defined by $F(x_0, x_1) = 0$, and let C be the 6-fold cyclic cover of $\mathbb{C}P^1$ defined in $\mathbb{C}P^2$ by $F(x_0, x_1) + x_2^6 = 0$, which is a smooth curve of genus 10. It has a 6-fold symmetry $\sigma : x_2 \to -\omega x_2$, where ω is our fixed cube root of unity. Now, σ^* acts on $H^1(C; \mathbb{C})$ and its eigenspaces refine the Hodge decomposition because σ acts holomorphically. One finds $H^1_\omega(C; \mathbb{C}) = H^{1,0}_\omega(C) \oplus H^{0,1}_\omega(C)$, the summands having dimensions 1 and 3 respectively. In fact, $H^{1,0}_\omega(C)$ is generated by the residue of the rational differential

$$\frac{(x_0\, dx_1 \wedge dx_2 + x_1\, dx_2 \wedge dx_0 + x_2\, dx_0 \wedge dx_1)\, x_2^3}{F(x_0, x_1) + x_2^6} \, .$$

We remark that our construction really uses the 3-fold cover of $\mathbb{C}P^1$ rather than the 6-fold cover, because we are working with the ω-eigenspace. We have used the 6-fold cover because the residue calculus is less fussy in projective space than in weighted projective space.

The Hermitian form

$$\langle \alpha | \beta \rangle = i\sqrt{3} \int_C \alpha \wedge \bar{\beta} \tag{9}$$

on $H^1(C; \mathbb{C})$ is positive-definite on $H^{1,0}$ and negative-definite on $H^{0,1}$. Therefore $H^{1,0}_\omega(C) \hookrightarrow H^1_\omega(C; \mathbb{C})$ is an inclusion of a positive line into a Hermitian vector space of signature $(1, 3)$, i.e., a point of the complex 3-ball consisting of all such lines in $P(H^1_\omega(C; \mathbb{C}))$. The $\sqrt{3}$ in (9) is not very important; it makes the map Z defined below be an isometry.

To identify this ball with a single fixed complex 3-ball we need an additional structure, namely a choice of basis for the relevant part of $H^1(C;\mathbb{Z})$, that is suitably compatible with σ. Let $\Lambda(C)$ be the sublattice of $H^1(C;\mathbb{Z})$ where σ^* has order 3, together with the 0 element. Then $\Lambda(C)$ is an $\mathcal{E}$-module, with ω acting as σ^*. The projection

$$Z : \Lambda(C) \otimes_{\mathcal{E}} \mathbb{C} = \Lambda(C) \otimes_{\mathbb{Z}} \mathbb{R} \to H^1_\omega(C;\mathbb{C})$$

is an isomorphism of complex vector spaces. The $\mathcal{E}$-module structure and the intersection pairing Ω together define a Hermitian form on $\Lambda(C)$, namely

$$\langle x|y\rangle = -\frac{\Omega(\theta x, y) + \theta\Omega(x,y)}{2} ,$$

where $\theta = \omega - \bar{\omega}$. This turns out to be a copy of Λ, the lattice from (5). A framing of S is a choice of isometry $\phi : \Lambda(C) \to \Lambda$, taken modulo scalars. (The term 'marking' is already taken, usually indicating an ordering of the six points of S.) It turns out that Z is an isometry, so together with ϕ it identifies the ball in $P(H^1_\omega(C;\mathbb{C}))$ with the standard one, i.e., the one in $P(\mathbb{C}^{1,3} = \Lambda \otimes_{\mathcal{E}} \mathbb{C})$. This defines a holomorphic map $g : \mathcal{F}_0 \to B^3$. One constructs an extension of the covering space $\mathcal{F}_0 \to \mathcal{C}_0$ to a branched covering $\mathcal{F}_s \to \mathcal{C}_s$ and extends g to $\mathcal{F}_s$; g is then the isomorphism of theorem 3. One can show (see, e.g., [1, lemma 7.12]) that the monodromy homomorphism $\pi_1(\mathcal{C}_0, S) \to P\mathrm{Aut}\,\Lambda(C)$ is surjective, and it follows that $\mathcal{F}_0$ and $\mathcal{F}_s$ are connected, with deck group $\Gamma = P\mathrm{Aut}\,\Lambda$, and that g is Γ-equivariant.

The reason that $\mathcal{F}_s \to \mathcal{C}_s$ has 6-fold branching along each component of the preimage of Δ is that one can use [12] to work out the monodromy in $P\mathrm{Aut}\,\Lambda$ of a small loop encircling Δ at a general point of Δ; it turns out to have order 6.

We close with some remarks relevant but not central to the lectures.

Remark 1. We have treated moduli of unordered real 6-tuples in $\mathbb{C}P^1$, which at first might sound like only a slight departure from the considerable literature on the hyperbolic structure on the moduli space of ordered 6-tuples in $\mathbb{R}P^1$. Briefly, Thurston [15, pp. 515–517] developed his own approach to theorem 3, and described a component of $G^{\mathbb{R}}\backslash\big((\mathbb{R}P^1)^6 - \Delta\big)$ as the interior of a certain polyhedron in H^3. Using hypergeometric functions, Yoshida [17] obtained essentially the same result, described the tessellation of $G^{\mathbb{R}}\backslash\big((\mathbb{R}P^1)^6 - \Delta\big)$ by translates of this open polyhedron, and discussed the degenerations of 6-tuples corresponding to the boundaries of the components. See also [7] and [9]. The relation to our work is the following: the space $G^{\mathbb{R}}\backslash\big((\mathbb{R}P^1)^6 - \Delta\big)$ is the quotient of $H^3_0 - \mathcal{H}$ by the level 3 principal congruence subgroup $\Gamma_{0,3}$ of Γ_0. A component C of $H^3_0 - \mathcal{H}$ is a copy of Thurston's open polyhedron, its stabilizer in Γ_0 is $S_3 \times \mathbb{Z}/2$, and the quotient of C by this group is the Coxeter orbifold P_0, minus the wall corresponding to the blackened node of the Coxeter diagram. There are $|S_6|/|S_3 \times \mathbb{Z}/2| = 60$ components of $G^{\mathbb{R}}\backslash\big((\mathbb{R}P^1)^6 - \Delta\big)$, permuted by S_6. The S_6 action is visible because the κ_0-invariant part of Λ is $\mathbb{Z}^{1,3}$, and $\Gamma_0/\Gamma_{0,3}$ acts on the $\mathbb{F}_3$-vector space $\mathbb{Z}^{1,3}/3\mathbb{Z}^{1,3}$. Reducing inner products of lattice vectors modulo 3 gives a quadratic form on this vector space, and S_6 happens to be isomorphic to the corresponding projective orthogonal group.

In a similar way, one could consider the moduli space of ordered 6-tuples of distinct points in $\mathbb{C}P^1$ such that (say) points 1 and 2 are conjugate and points $3, \ldots, 6$ are all real. This moduli space is a quotient of $H_1^3 - \mathcal{H}$ by a subgroup of Γ_1. Other configurations of points give quotients by subgroups of the other Γ_j. It is only by considering *unordered* 6-tuples that one sees all four types of 6-tuples occurring together, leading to our gluing construction. One way that our results differ from earlier ones is that the gluing leads to a nonarithmetic group acting on H^3 (see remark 5 below), whereas the constructions using ordered 6-tuples lead to arithmetic groups.

Remark 2. Γ has a single cusp in $\mathbb{C}H^3$, corresponding to the 6-tuple consisting of two triple points; this is the unique minimal semistable orbit (in the sense of geometric invariant theory) in $\mathcal{C}$. The two cusps of $\Gamma^{\mathbb{R}}$ correspond to the two possible real structures on such a 6-tuple—the triple points can be conjugate, or can both be real.

Remark 3. Part (*ii*) of theorem 4 lets us write down the orbifold fundamental group $\pi_1^{\text{orb}}(\mathcal{M}_s^{\mathbb{R}})$. The theory of Coxeter groups shows that the reflection subgroup R of $\Gamma^{\mathbb{R}}$ is defined as an abstract group by the relations that the six generating reflections are involutions, and that the product of two has order n when the corresponding walls meet at angle π/n. The modification of orbifold structures amounts to setting two of the generators equal if their walls meet at angle $\pi/6$. This reduces R to $D_\infty \times \mathbb{Z}/2$ where D_∞ denotes the infinite dihedral group. Adjoining the diagram automorphism gives $\pi_1^{\text{orb}}(\mathcal{M}_s^{\mathbb{R}}) \cong (D_\infty \times \mathbb{Z}/2) \rtimes (\mathbb{Z}/2)$, where the $\mathbb{Z}/2$ acts on $D_\infty \times \mathbb{Z}/2$ by exchanging the involutions generating D_∞. This larger group is also isomorphic to $D_\infty \times \mathbb{Z}/2$, so we conclude $\pi_1^{\text{orb}}(\mathcal{M}_s^{\mathbb{R}}) \cong D_\infty \times \mathbb{Z}/2$. This implies that $\mathcal{M}_s^{\mathbb{R}}$ is not a good orbifold in the sense of Thurston [14].

Remark 4. One can work out the volumes of the P_j by dissecting them into suitable simplices, whose volumes can be expressed in terms of the Lobachevsky function $\Lambda(z)$. For background see [8] and [10]. The results are

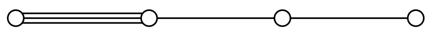
j	**covolume(Γ_j)**			**fraction of total**
0	$\Lambda(\pi/4)/6$	=	.07633...	$\sim$ 8.66 %
1	$15\Lambda(\pi/3)/16$	=	.31716...	$\sim$ 36.01 %
2	$5\Lambda(\pi/4)/6$	=	.38165...	$\sim$ 43.33 %
3	$5\Lambda(\pi/3)/16$	=	.10572...	$\sim$ 12.00 %

These results suggest that Γ_0 and Γ_2 are commensurable, that Γ_1 and Γ_3 are commensurable, and that these two commensurability classes are distinct. We have verified these statements.

Remark 5. The group $\Gamma^{\mathbb{R}}$ is nonarithmetic; this is suggested by the fact that we built it by gluing together noncommensurable arithmetic groups in the spirit of Gromov and Piatetski-Shapiro's construction of nonarithmetic lattices in $O(n,1)$.

(See [6].) Their results do not directly imply the nonarithmeticity of $\Gamma^{\mathbb{R}}$, so we used 12.2.8 of [5]. verified this directly. That is, we computed the trace field of $\Gamma^{\mathbb{R}}$, which turns out to be $\mathbb{Q}(\sqrt{3})$, showed that $\Gamma^{\mathbb{R}}$ is a subgroup of the isometry group of the quadratic form $\mathrm{diag}[-1,+1,+1,+1]$ over $\mathbb{Z}[\sqrt{3}]$, and observed that the Galois conjugate of this group is noncompact over $\mathbb{R}$.

Remark 6. The anti-involutions (6) and their Γ-conjugates do not account for all the anti-involutions of $\mathbb{C}H^3$ in $\Gamma \rtimes (\mathbb{Z}/2)$: there is exactly one more conjugacy class. Pick a representative κ_4 of this class and write H^3_4 for its fixed-point set. The points of H^3_4 correspond to 6-tuples in $\mathbb{C}P^1$ invariant under the non-standard anti-involution of $\mathbb{C}P^1$, which can be visualized as the antipodal map on the sphere S^2. A generic such 6-tuple cannot be defined by a sextic polynomial with real coefficients, but does represent a real point of $\mathcal{M}^{\mathbb{R}}_0$. One can show that the stabilizer Γ_4 of H^3_4 in Γ is the Coxeter group



and that the moduli space of such 6-tuples is H^3_4/Γ_4, minus the edge corresponding to the triple bond.

Remark 7. When discussing the gluing patterns 7 and 8 we did not specify information such as which gluing wall of P_2 is glued to the gluing wall of P_3. It turns out that there is no ambiguity because the only isometries between walls of the P_j are the ones we used. But for the sake of explicitness, here are the identifications. The gluing wall of P_0 is glued to one of the top gluing walls of P_1, the gluing wall of P_3 is glued to the left gluing wall of P_2, and the other gluing wall of P_2 is glued to one of the bottom gluing walls of P_1. The words 'left', 'right', 'top' and 'bottom' refer to the Coxeter diagrams (1) and (3), not to the pictures of the polyhedra.

Remark 8. In these notes we work projectively, while in [2] we do not. This means that our space $\mathcal{C}$ is analogous to the $\mathbb{C}P^{19}$ of cubic surfaces in $\mathbb{C}P^3$, which is the projectivization of the space called $\mathcal{C}$ in [2], and similarly for the various versions of $\mathcal{F}$. The group in [2] analogous to G here is the projectivization of the group called G there, and similarly for $G^{\mathbb{R}}$, Γ, Γ_j and $\Gamma^{\mathbb{R}}$.

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