Eigenvalues and Eigenvectors

Matrix $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$

vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is an <u>eigenvector</u>

of A if

$$Av = \lambda v$$

for some scalar λ. Which scalars? There must be a nonzero solution of

$$\begin{cases} (3-\lambda)x + 4y = 0 \\ 2x \quad (3-\lambda)y = 0 \end{cases}$$

So determinant of coefficient matrix is zero:

$$\begin{vmatrix} 3-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

Quadratic equation

$$\lambda^2 - 6\lambda + 1 = 0$$

$$\lambda = 3 \pm 2\sqrt{2}$$

We now know the eigenvalues. They are conjugate to one another.

Let

$$\lambda_+ = 3+2\sqrt{2}$$
$$\lambda_- = 3-2\sqrt{2}$$

Thus $\bar{\lambda}_+ = \lambda_-$.

Problem: Find eigenvector v_+ associated to λ_+.
Solution: Solve

$$\begin{cases} (3-\lambda_+)x + 4y = 0 \\ 2x + (3-\lambda_+)y = 0 \end{cases}$$

$$\begin{cases} -2\sqrt{2}x + 4y = 0 \\ 2x - 2\sqrt{2}y = 0 \end{cases}$$ ← dependent (proportional) equations

Set $x = 1$, solve for y:
$y = \sqrt{2}/2$. Thus:

$$v_+ = \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

Problem: Find eigenvector v_- associated to λ_-.
Solution: Imitate above or use pure thought: v_- is conjugate of v_+ because λ_- is conjugate of λ_+:

$$v_- = \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

Problem Express $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ as a linear combination of v_+ and v_-.

Solution: $$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = a\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix} + b\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\begin{cases} a + b = 3 & (1) \\ \frac{\sqrt{2}}{2}a - \frac{\sqrt{2}}{2}b = 2 & (2) \end{cases}$$

Multiply (1) by $\sqrt{2}$, multiply (2) by 2:

$$\begin{aligned} \sqrt{2}a + \sqrt{2}b &= 3\sqrt{2} \\ \sqrt{2}a - \sqrt{2}b &= 4 \\ \hline 2\sqrt{2}a &= 4 + 3\sqrt{2} \end{aligned}$$

$$\boxed{a = \frac{3+2\sqrt{2}}{2}}$$

$$\boxed{b = \frac{3-2\sqrt{2}}{2}}$$

Problem: Compute $A^{100}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Solution: $$\begin{aligned} A^{100}\begin{pmatrix} 3 \\ 2 \end{pmatrix} &= A^{100}(av_+ + bv_-) \\ &= aA^{100}v_+ + bA^{100}v_- \\ &= a\lambda_+^{100}v_+ + b\lambda_-^{100}v_- \end{aligned}$$

$\lambda_+ \sim 5.828$

$\lambda_- \sim 0.172$

$\lambda_+^{100} \sim 3.56 \times 10^{76}$

$\lambda_-^{100} \sim 3.57 \times 10^{-77}$

$a \sim 2.91$

Thus $A^{100}\begin{pmatrix} 3 \\ 2 \end{pmatrix} \sim 2.91 \times 3.57 \times 10^{76} \, v_+$

$\sim 1.04 \times 10^{77} \, v_+$

<u>Recall</u>: $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^{n-1}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Thus for $n >> 0$, $x_n / y_n \sim$ (x-coord of v_+)/(y-coord of v_+)

$\sim \sqrt{2} \sim 1.414$.