

## Eigenvalues and Eigenvectors

Matrix  $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$

vector  $v = \begin{pmatrix} x \\ y \end{pmatrix}$  is an eigenvector

of  $A$  if

$$Av = \lambda v$$

for some scalar  $\lambda$ . Which scalars? There must be a nonzero solution of

$$\begin{cases} (3-\lambda)x + 4y = 0 \\ 2x + (3-\lambda)y = 0 \end{cases}$$

So determinant of coefficient matrix is zero:

$$\begin{vmatrix} 3-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

Quadratic equation

$$\lambda^2 - 6\lambda + 1 = 0$$

$$\lambda = 3 \pm 2\sqrt{2}$$

We now know the eigenvalues. They are conjugate to one another.

Let

$$\lambda_+ = 3 + 2\sqrt{2}$$

$$\lambda_- = 3 - 2\sqrt{2}$$

Thus  $\overline{\lambda_+} = \lambda_-$ .

Problem: Find eigenvector  $v_+$  associated to  $\lambda_+$ .

Solution: Solve

$$\begin{cases} (3 - \lambda_+)x + 4y = 0 \\ 2x + (3 - \lambda_+)y = 0 \end{cases}$$

$$\begin{cases} -2\sqrt{2}x + 4y = 0 \\ 2x - 2\sqrt{2}y = 0 \end{cases} \quad \leftarrow \begin{array}{l} \text{dependent} \\ \text{(proportional)} \\ \text{equations} \end{array}$$

Set  $x = 1$ , solve for  $y$ :

$$y = \sqrt{2}/2. \text{ Thus:}$$

$$v_+ = \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

Problem: Find eigenvector  $v_-$  associated to  $\lambda_-$ .

Solution: Imitate above or use pure thought:

$v_-$  is conjugate of  $v_+$  because  $\lambda_-$  is conjugate of  $\lambda_+$ :

$$v_- = \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

Problem: Express  $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$  as a linear combination of  $v_+$  and  $v_-$ .

Solution: 
$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = a \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix} + b \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\begin{cases} a + b = 3 & (1) \\ \frac{\sqrt{2}}{2}a - \frac{\sqrt{2}}{2}b = 2 & (2) \end{cases}$$

Multiply (1) by  $\sqrt{2}$ , multiply (2) by 2:

$$\sqrt{2}a + \sqrt{2}b = 3\sqrt{2}$$

$$\sqrt{2}a - \sqrt{2}b = 4$$

$$\hline 2\sqrt{2}a = 4 + 3\sqrt{2}$$

$$a = \frac{3 + 2\sqrt{2}}{2}$$

$$b = \frac{3 - 2\sqrt{2}}{2}$$

Problem: Compute  $A^{100} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ .

Solution: 
$$\begin{aligned} A^{100} \begin{pmatrix} 3 \\ 2 \end{pmatrix} &= A^{100} (a v_+ + b v_-) \\ &= a A^{100} v_+ + b A^{100} v_- \\ &= a \lambda_+^{100} v_+ + b \lambda_-^{100} v_- \end{aligned}$$

4.

$$\lambda_+ \sim 5.828$$

$$\lambda_- \sim 0.172$$

$$\lambda_+^{100} \sim 3.56 \times 10^{76}$$

$$\lambda_-^{100} \sim 3.57 \times 10^{-77}$$

$$a \sim 2.91$$

$$\begin{aligned} \text{Thus } A^{100} \begin{pmatrix} 3 \\ 2 \end{pmatrix} &\sim 2.91 \times 3.57 \times 10^{76} v_+ \\ &\sim 1.04 \times 10^{77} v_+ \end{aligned}$$

$$\text{Recall: } \begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^{n-1} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Thus for  $n \gg 0$ ,  $x_n / y_n \sim (\text{x-coord of } v_+) / (\text{y-coord of } v_+)$

$$\sim \sqrt{2} \sim 1.414.$$