Square-Triangular Numbers

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1 Introduction

We begin with a puzzle. Its solution will lead us deep into the land of number theory:

Problem 1 *Is it possible to arrange a collection of stones in both a square and a triangular shape?*

To make sure we understand the problem, let's imagine that we have four stones. They can certainly be arranged as a square:

```
* *
* *
```

But can we arrange them as a triangle? It turns out we can't, as the next figure shows:

```
*          *                   *               etc.
         *   *               *   *
                           *   *   *
```

But perhaps there is some special number of stones — let's call it S — that can be arranged in both a square and a triangular shape:

Problem 2 *Is there a number S that is both square and triangular?*

Comments: *Make two lists, compare them. Try brute-force search. Do square-triangular numbers exist? How frequent are square-triangular numbers? Are there infinitely many of them?*

2 Existence

One way to answer the previous question — *do square-triangular numbers exist?* — is to make two lists. One list is for the square numbers 1, 4, 9, 16, etc., and another is for the triangular numbers 1, 3, 6, 10, etc. A number that appears in both lists solves the puzzle. You probably found 36 in both lists. This is the first square-triangular number. We write it as

$$S_1 = 36.$$

This raises the question: *are there any other square-triangular numbers?* So let's set a small challenge:

Problem 3 *Is there a square-triangular number $S > 100$?*

You may be able to find more square-triangular numbers by comparing lists, but we should pause to ask: *Are there a better methods?* Better methods will be needed to solve more difficult questions. Indeed, let us set the following challenge:

Problem 4 *(a) Is there a square triangular number $S > 10^6$? (b) What about $S > 10^{12}$? (c) And $S > 10^{24}$?*

One way to find square-triangular numbers might be to find a formula for them. Another might be to find an equation whose solutions are exactly the square-triangular numbers. Still another way is to write a computer program to search for square-triangular numbers. We will explore each these ideas below, and will eventually answer the fundamental question

How many square-triangular numbers are there?

3 Pell's equation

We now ask if there is a more algebraic way of approaching the problem of square-triangular numbers:

Problem 5 *(a) Is there an equation whose solution gives all square triangular numbers. (b) Is there a formula for these numbers?*

Let's see if we can solve part (a). A number S is square if it can be written as $S = m^2$ for some m. A number is triangular if it can be written as $n(n+1)/2$ for some n. This statement requires a proof:

Problem 6 *Show that every triangular number can be written as $n(n+1)/2$ for some n.*

Comments: *Two proofs. One, geometric, using a square array of dots. The other, algebraic, using induction.*

If S is both square and triangular, then it can be written as m^2 for some m and as $n(n+1)/2$ for some n. Thus $S = m^2 = n(n+1)/2$ is given by solutions to the equation

$$m^2 = n(n+1)/2 \tag{1}$$

Problem 7 *Find several positive integer solutions to the equation $m^2 = n(n+1)/2$. What are the corresponding square-triangular numbers?*

An equation like $m^2 = n(n+1)/2$ for which we demand *integer* solutions is called a *Diophantine equation*. Diophantus was a Greek mathematician who worked in Alexandria on questions like this around 250 AD. To solve our equation we will put it in simpler form. Begin by clearing denominators:

$$2m^2 = n^2 + n.$$

Now it is an equation with integer coefficients, the usual form of a Diophantine equation. Next, complete the square:

$$2m^2 = (n + 1/2)^2 - 1/4$$

Clearing denominators again, we find

$$8m^2 = (2n+1)^2 - 1,$$

or

$$(2n+1)^2 - 2(2m)^2 = 1.$$

Introduce new variables

$$x = 2n + 1 \text{ and } y = 2m. \tag{2}$$

Then our equation becomes

$$x^2 - 2y^2 = 1 \tag{3}$$

This is Pell's equation, named for the English mathematician John Pell (1611-1685), but studied in antiquity — it occurs in Archimedes' Cattle Problem and was studied by Brahmagupta (c. 598-670 BC) and Bhaskara (1111-1185) in India. The French mathematician Lagrange (1736-1813) developed a detailed theory for Pell's equation. In any case, we are led to the following result:

Theorem 1 *If x, y is a solution of Pell's equation, then $N = (y/2)^2$ is a a number that is both square and triangular. The square has $y/2$ dots on a side, the triangle has $(x-1)/2$ dots on the bottom row.*

The set of all *real* solutions of Pell's equation $x^2 - 2y^2 = 1$ forms a hyperbola. Pairs of integers (x, y) like $(5, 7)$ form a *lattice* of equally spaced points in the plane. Solutions of Pell's equation are the same as lattice points which lie on the hyperbola.

Problem 8 *Draw a picture of the lattice points in the box $|x| < 5$, $|y| < 5$ and of the hyperbola $x^2 - 2y^2 = 1$. Indicate the solutions of Pell's equation.*

Problem 9 *How many solutions of Pell's equation $x^2 - 2y^2 = 1$ can you find (in five minutes)? How many solutions do you think there are?*

4 Solutions by brute force

Let us now find square-triangular numbers by finding solutions to Pell's equation. One strategy is to inspect all pairs of integers x, y, with $0 \leq x \leq B$, $0 \leq y \leq B$. If a pair solves the equation, we write it down on our list of solutions. If it does not, we go on to the next pair. This is called the *brute-force method.*

Problem 10 *How many pairs of "candidate solutions" x, y for Pell's equation are there satisfying $0 \leq x \leq B$, $0 \leq y \leq B$?*

Problem 11 *Following the notation of the previous problem, find all nonnegative solutions of Pells equation with $B = 100$. What are the corresponding square-triangular numbers?*

Problem 12 *What is the probability p_B be that a randomly chosen lattice point (x, y) with $0 \leq x < B$, $0 \leq y < B$ is a solution of Pell's equation? Solve this problem for $B = 10$ and $B = 100$. Can you go further?*

Problem 13 *Discuss the feasibility of using brute-force search when $N = 1000$. Is there a way of improving our search?*

Let's talk about strategy. In the previous problem we could list all the points x, y with integer coordinates that lie in the box $0 \leq x, y < N$. There are N^2 such points. So when $N = 100$, there are $10,000$ points to inspect. Already a large number! Consider how much larger it is for $N = 1000$. The number of points to inspect grows quadratically in the magnitude of the input N. Can we do better? That is, can we speed things up?

Problem 14 *Suppose that a computer program can inspect all candidate solutions with $0 \leq x, y < 10^3$ in one second. How long do you expect it to take if $x, y < 10^6$? What if we try an upper bound of 10^9 or 10^{12}?*

Remarks

Computer scientists usually measure (or estimate) the time taken for a computation to produce its output in terms of the number of bits in the input. If the input is an integer, like 5, then the number of bits in the input is the number of binary digits needed to express the number. Since the binary representation of 5 is 101, the number of bits of input for an input "5" is three.

Problem 15 *Express 17 in binary form. What is the decimal form of the binary number 11011011?*

Problem 16 *How many bits does an n-digit number have? A good approximate answer is satisfactory.*

Problem 17 *Suppose that inspection of a lattice point takes t seconds. If the "inspection box" is defined by $0 < x < B$, $0 < y < B$. How many seconds T will it take to inspect all lattice points in the box? Express t as a function of B. The express it as a function of the number of bits in B. Comment on your formulas.*

5 Speeding things up

Two ways of speeding up the search for solutions to Pell's equation are: (a) use a computer, (b) be more clever. Of course, if we combine (a) and (b), we will do even better.

First let's discuss using a computer. Using the Python language, we will design a function `search(B)` that prints out all solutions to Pell's equation satisfying $0 \leq x < B$, $0 \leq y < B$. Thus we can do the following:

```
>>> search(100)
1 0
3 2
17 12
99 70
```

The function `search` listed all the solutions of Pell's equation of *size* at most B in the twinkling of an eye.

Here is how the `search` is defined:

```
def search(B):
   for x in range(0,B):
      for y in range(0,B):
          if x*x - 2*y*y == 1:
             print x, y
```

The code is almost self explanatory. Numbers x and y in the range $[0, B)$ are generated and for each pair (x, y) we test whether the equation holds. If it does, we print out the solution.

The function `search(B)` works quite well when $B = 1000$, but already when $B = 10000$ it requires considerable time. We could use a faster computer or a faster language, but the real difficulty is that the amount of work to be done grows quadratically in B.

Problem 18 *Use the Python program* `search` *with* $B = 100, 1000, 100000$. *Comment on what you find. Can you find a solution of Pell's equation with* $y > 10^6$ *using* `search`*?*

Now let's try to be more clever. To begin, rewrite Pell's equation as

$$x^2 = 1 + 2y^2.$$

Notice that we can systematically inspect values of y, accepting only those for which $1 + 2y^2$ is a perfect square. A table of squares helps to do this quickly, even if we are doing the inspection manually.

Problem 19 *Find all solutions of Pell's equation with $y < 100$ using "linear search on y."*

Problem 20 *Now that you have tried linear search by hand to find solutions of Pell's equation, design a computer program which uses the same method. Do you find more solutions? How big of a solution can you find?*

For the last problem it is useful to have a function that determines whether a number is a square or not. The following function works well if the numbers are not too big. For larger numbers one should use an integer version of Newton's method (xxx).

```
def issquare(n):
  isqrt = int(sqrt(n))
  if n == isqrt*isqrt:
    return 1
  else:
    return 0
```

The function `issquare` works by computing the integer part of $\sqrt{n}$. Thus, if $n = 5$, $isqrt = int(sqrt(5)) = int(2.23606...) = 2$. Then we check if the square of this "integer square root" is equal to n. If it is, we return 1 (true). Otherwise we return 0 (false). Since floating point arithmetic is innacurate for for very large numbers, this function will fail when n is too large. One really needs a true integer square root function — one that uses only integer operations. No floating point arithmetic allowed!

Problem 21 *It is natural to study the general Pell equation $x^2 - Ny^2 = 1$ where N is a fixed integer. What can you say about the cases $N = 1$ and $N = 3$?*

6 Solutions by algebraic numbers

In the last section you found that $x, y = 1, 0$ and $x, y = 3, 2$ are solutions to Pell's equation. These give the "trivial" square-triangular numbers 0 and 1. You also have found the solution $x, y = 17, 12$ which gives the square triangular number 36. Finally, you found larger solutions using a combination of brute force search and a somewhat more clever search, possibly aided by a computer. But there is a much better way of finding solutions which is based on a little theory that comes from a factored form of Pell's equation:

$$(x - \sqrt{2}y)(x + \sqrt{2}y) = 1. \tag{4}$$

The key idea is that every solution of Pell's equation gives a factorization of 1. Thus $x, y = 3, 2$ gives the factorization

$$(3 - 2\sqrt{2})(3 + 2\sqrt{2}) = 1.$$

We claim that from an "old" factorization one can produce a "new" one. To see this, square the preceding equation:

$$((3 - 2\sqrt{2})(3 + 2\sqrt{2}))^2 = 1.$$

It can be rewritten as

$$(3 - 2\sqrt{2})^2(3 + 2\sqrt{2})^2 = 1.$$

Doing the algebra, we find

$$(17 - 12\sqrt{2})(17 + 12\sqrt{2}) = 1$$

Thus $x, y = 17, 12$ should be a solution of Pell's equation — and it is, as you can easily check. It corresponds to the square-triangular number 36.

Problem 22 *What is the solution of Pell's equation corresponding to* $(3 + 2\sqrt{2})^3$*? What about* $(3 + 2\sqrt{2})^4$*? Check your results. What are the corresponding square-triangular numbers?*

Problem 23 *What is the solution of Pell's equation corresponding to* $(3 + 2\sqrt{2})^{16}$*? Check your result. What is the corresponding square-triangular number?*

Problem 24 *Are there finitely or infinitely many square-triangular numbers? Explain.*

Problem 25 *How many integer solutions are there to the equation* $x^2 - y^2 = 1$*?*

Problem 26 *Investigate integer solutions of quadratics equations of the form* $x^2 - Ny^2 = 1$ *for various values of* N*. What conclusions can you draw?*

Notes

It is worth reflecting on what we have done. Our problem is to find integer solutions to an equation with integer coefficients. But the best way so far to do this is to use irrational numbers. We solve a problem in one area of mathematics by using ideas from another area.

The numbers used are not random irrational numbers. First, they have the form $a + b\sqrt{2}$ where a and b are integers. Secondly these numbers form what is called a *ring*: the sum any two such numbers is again such a number, as is the product. (What about the quotient of two such numbers?)

We denote the ring of numbers $\{ a + b\sqrt{2} \mid a, b \in \mathbf{Z} \}$ by the symbol $\mathbf{Z}[\sqrt{2}]$ — "the integers $\mathbf{Z}$ with $\sqrt{2}$ adjoined." All numbers in $\mathbf{Z}[\sqrt{2}]$ have a remarkable property: each satisfies a quadratic equation with integer coefficients.

Problem 27 *Show that* $\mathbf{Z}[\sqrt{2}]$ *is a ring.*

Problem 28 *Describe the set of numbers obtained by taking quotients of elements of* $\mathbf{Z}[\sqrt{2}]$. *What properties does it have?*

Problem 29 *What is the quadratic equation with integer coefficients satisfied by* $5 + 7\sqrt{2}$*?*

Problem 30 *Snow that every number in* $\mathbf{Z}[\sqrt{2}]$ *satisfies a quadratic equation with integer coefficients.*

There is one final observation worth making. Some of the numbers U in $\mathbf{Z}[\sqrt{2}]$ have a *multiplicative inverse*. This means that there is a number V, also in $\mathbf{Z}[\sqrt{2}]$, such that $UV = 1$. Such numbers are called the *units* of $\mathbf{Z}[\sqrt{2}]$. The units form a *group* under multiplication: elements can be multiplied, and also divided.

Problem 31 *Describe the units of* $\mathbf{Z}[\sqrt{2}]$.

7 Recursion relations

In the last section we used powers of the "magic" irrational numbers $\mu = 3+2\sqrt{2}$ to find integer solutions to Pell's equation. (Mathematicians are fond of using Greek letters. The letter μ is called "mu" and is the Greek "m.")

We'll now show how to use μ to compute solutions even more efficiently. The idea is that the n-th solution is given by

$$\mu^n = x_n + y_n\sqrt{2}.$$

Our goal is to compute x_{n+1} and y_{n+1} from x_n and y_n. To this end notice that

$$x_{n+1} + y_{n+1} = (3 + 2\sqrt{2})(x_n + y_n\sqrt{2}).$$

If we multiply out the right hand side (RHS) and collect terms, we get

$$RHS = (3x_n + 4y_n) + (2x_n + 3y_n)\sqrt{2}.$$

Comparing with the left-hand side, we find that

$$x_{n+1} = (3x_n + 4y_n) \qquad (5)$$

$$y_{n+1} = (2x_n + 3y_n) \qquad (6)$$

These so-called *recursion relations* allow us to quickly computef large solutions to Pell's equation, as well as to solve many other problems.

Problem 32 *Let $U_n = (x_n, y_n)$ be the n-th solution of Pell's equation. Compute U_8, as well as the associated square-triangular number S_8.*

Problem 33 *Find a square-triangular number $S > 10^{20}$.*

Problem 34 *Let $U_n = (x_n, y_n)$ be the n-th solution of Pell's equation. How much bigger than x_n is x_{n+1} What can you say about the y coordinates?*

Problem 35 *Let S_n be the n-th square-triangular number. What can you say about its size as a function of n?*

Problem 36 *What is the (approximate) number of solutions to Pell's equation with $x < B$? What is the (approximate) number of square-triangular numbers with $S < B$?*

Problem 37 *Study the ratio $r_n = x_n/y_n$ as n becomes large. What can you conclude? Can you explain your conclusion?*

Notes

The recursion relation above can be written in terms of vectors and matrices:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

We can write this more compactly as

$$v_{n+1} = Av_n$$

where v_{n+1} is the vector on the right-hand side, A is the 2×2 matrix, etc. Investigate the relation between v_{n+1} and v_n when n is large.

8 Fundamental solutions, continued fractions

By following the pattern used to solve the equation $x^2 - 2y^2 = 1$ it is not hard to find infinitely many solutions to the equation $x^2 - Ny^2 = 1$ given a "nontrivial" solution (a, b) with $a > 0$ and $b > 0$. When N is small, we can do this by trial and error, if the nontrivial solution exists.

Problem 38 *Find a nontrivial solution of $x^2 - Ny^2 = 1$ for the prime numbers $N = 2, 3, 5, 7, 11, 13, 17$. For what nonprime coefficients N can we find nontrivial solutions? Any observations?*

For general N the trial and error (or systematic search) strategy may not be efficient enough to produce a solution in a reasonable amount of time. However, there is another method, based on the idea of *continued fractions*, that works quite well.

A continued fraction is an expression like

$$4 + \cfrac{1}{2 + \cfrac{1}{7 + \cfrac{1}{3}}}$$

Note that the numerators are all equal to 1. A more compact notation for the above continued fraction is [4; 2, 7, 3].

Problem 39 *What is the simple fraction corresponding to the contined fraction [4; 2, 7, 3]? Same question for [1; 2, 3, 4]?*

We can also convert any simple fraction into a continued fraction. For this we use three ideas — the integer part of a number, its remainder mod 1, and unit fractions:

1. Given a number α, let $[\alpha]$ be the "largest integer" in α. Thus $[3.1416] = 3$.
2. The "remainder mod 1" of a number , which we write as $\{\alpha\} = \alpha - [\alpha]$. Thus $\{3.1416\} = 0.1416$. The remainder of a number mod 1 is always less than 1 and greater than or equal to zero. Notice that
$$\alpha = [\alpha] + \{\alpha\}, \tag{7}$$
3. A *unit fraction* is a number of the form $1/x$. (These are also called "Egyptian fractions.")

Now let's find the continued fraction of 1281/401. The first step is to apply (7) to write the given number as an integer and a remainder mod 1:

$$1281/401 = 3 + 78/401.$$

Then express the last term as a unit fraction:

$$1281/401 = 3 + 1/(401/78)$$

Repeat this two-step process until the denominator of the unit fraction is an integer. Thus

$$1281/401 = 3 + 1/(5 + 11/78) = 3 + 1/(5 + 1/(78/11)).$$

and then

$$1281/401 = 3 + 1/(5 + 1/(7 + 1/11)).$$

The process stops at this point, since the denominator of the last unit fraction is an integer. The result is the continued fraction [3; 5, 7, 11].

Problem 40 *Find the continued fraction of* 1193/532*.*

Problem 41 *Find the continued fraction of* $\sqrt{2} \sim 1.4142135623730951$*. Comment on your results.*

The same process can be used to compute the continued fraction of $\sqrt{2}$. This can be done using the decimal expansion and a bit of guessing and a leap of faith, or exact algebra. Let's try it with the decimal expansion first. Step one yields

$$1.4142135623730951 = 1 + 0.4142135623730951,$$

Since

$$1/0.4142135623730951^{-1} = 1/(2.41421356454765235)$$

step two yields

$$1.414213562 = 1 + 1/(2.41421356454765235)$$

Applying steps one and two to 2.41421356454765235, we find

$$1.414213562 = 1 + 1/(2 + 1/(2.41421354969884616)$$

Notice that the denominators of the last two expressions are almost all the same — they agree through the seventh decimal place. If we used infinite precision arithmetic, there would be exact agreement of the remainders no matter how far out we carry the expansion. With exact agreement we would find the infinite continued fraction

$$\sqrt{2} = [1, 2, 2, 2, \dots].$$

Problem 42 *Find the continued fractions of* $\sqrt{3}$ *and* $\sqrt{7}$ *using a decimal approximation to each.*

There is, of course, a way of using full precision: just work with radicals. We illustrate the method for $\sqrt{2}$. First, write $\sqrt{2}$ as an integer part and a remainder mod 1:

$$\sqrt{2} = 1 + (\sqrt{2} - 1).$$

Next, write the remainder as a unit fraction:

$$\sqrt{2} - 1 = 1/(\sqrt{2} - 1)^{-1}$$

and rationalize the denominator to get

$$\sqrt{2} - 1 = 1/(1 + \sqrt{2}).$$

Thus

$$\sqrt{2} = 1 + 1/(1 + \sqrt{2}).$$

We could proceed as before (this is recommended for practice). However, we notice something remarkable: *the formula is self-similar*: we can substitute for the $\sqrt{2}$ on the right-hand side the entire right-hand side:

$$\sqrt{2} = 1 + 1/(2 + 1/(1 + \sqrt{2})) = 1 + 1/(2 + 1/(2 + 1/(1 + \sqrt{2}))) = \cdots$$

Repeating this argument, we *prove* that

$$\sqrt{2} = [1, 2, 2, 2, \ldots]$$

Problem 43 *Find the continued fraction expansions of $\sqrt{3}$ and $\sqrt{7}$ without using decimal approxmations — or approximations of any kind. Prove that your result is correct.*

We can now begin to see what the relation of continued fractions is to Pell's equation $x^2 - Ny^2 = 1$. Given a continued fraction $[a; b, c, d, \ldots]$ we can form its *convergents*. These are shortened continued fractions like $[a; b]$, $[a; b, c]$, etc.

Problem 44 *Compute the first three convergents of the continued fraction expansion of $\sqrt{2}$. How do they relate to the solutions of Pell's equation? Do you detect a general pattern?*

Problem 45 *Investigate continued fractions as a tool for solving other equations of Pell's type, $x^2 - Ny^2 = 1$. Be sure to try $N = 3$, $N = 7$. Look for patterns.*

A *fundamental* solution to Pell's equation is a solution (a, b), with both a and b positive, from which all other positive solutions can be found using the expression $(a + b\sqrt{N})^k$. A fundamental solution is smaller than all other positive solutions.

The *size* of a fundamental solution is $log(a)/log(N)$. It is roughly the number of digits of a divided by the number of digits in N.

Problem 46 *(Contest) Find the largest fundamental solution you can.*

9 More about continued fractions

Let's look more closely at our method for finding continued fractions. Define a *continued fraction with remainder* to be an expression of the form

$$F = [a_0, a_1, \dots a_n, \theta],$$

where $0 < \theta < 1$. We expand such a fraction by replacing θ by $[1/\theta], \{1/\theta\}$, where $[x]$ denotes the integer part of x and $\{x\}$ denotes its fractional part — the part between 0 and 1. Thus F becomes

$$F' = [a_0, a_1, \dots a_n, [1/\theta], \{1/\theta\}],$$

Let's try this rule on $\sqrt{7} \sim 3.6055512754639891$. We have

```
[3, 0.6055512754639891]  <== (a)
[3, 1, 0.65138781886599784]
[3, 1, 1, 0.53518375848799526]
[3, 1, 1, 1, 0.86851709182133385]
[3, 1, 1, 1, 1, 0.15138781886599184]
[3, 1, 1, 1, 1, 6, 0.60555127546422849]    <== (b)
[3, 1, 1, 1, 1, 6, 1, 0.65138781886534503]
```

The remainders at (a) and at (b) are 0.6055512754639891 and 0.60555127546422849 disagree in the 12th place. If you use a calculator that works to a higher precision, you will find that the agreement is higher. If you work with exact expressions you will find that the agreement is perfect. This suggests that the continued fraction is

$$[3, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, \dots]$$

with the pattern [1, 1, 1, 1, 6] repeating infintiely often.

Problem 47 *Compute the continued fraction expansion of $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$, $\sqrt{14}$. Do you notice a pattern? Can you prove that it is correct? What do you conclude about fundamental solutions to Pell's equation?*

The Python code below is useful in computing continued fractions:

```
def expand( C ):
  x = C[-1] # last element of C
  x = 1.0/x
  a = int(x)
  b = x - a
  return C[:-1]+[a,b]
```

For example, one finds that

```
expand([3, 0.6055512754639891]) =  [3, 1, 0.65138781886599784]
```

To work with exact expresions, we use numbers of the form $a + b\sqrt{7}$ where a and b are rational numbers. The set of all such expressions is closed under addition, subtraction, multiplication and division by nonzero numbers. It is a so-called *number field*, and is written $\mathbf{Q}(\sqrt{7})$. For some purposes (such as writing computer program), it is better to write elements of this field as $(a + b\sqrt{7})/c$. where a, b, and c are integers. Then the field operations are given by integer formulas involving triples of integers $[a, b, c]$ and $[a', b', c']$.

Problem 48 *Find integer formulas for the four field operations in* $\mathbf{Q}(\sqrt{D})$.

Problem 49 *Find an algorithm for computing the continued fraction expansion of* $\sqrt{D}$ *using only integer operations on lists* $[a, b, c]$

Problem 50 *Find the continued fraction expansion of* $\sqrt{61}$. *Find a fundamental solution to* $x^2 - 61y^2 = 1$. *What is the length of this solution?*

Problem 51 *Investigate the continued fraction expansion of* $2^{1/3}$.

10 Other Diophantine equations

Pell's equation is just one of many interesting Diophantine equations. Here are some others

1. $x^2 + y^2 = z^2$
2. $y^2 = x^3 + 1$
3. $x^3 + y^3 = z^3$
4. $x^3 + y^3 + z^3 = w^3$

(1) Solutions of the first Diophantine equation are called *Pythagorean triples*, and of course we know one: 3, 4, 5. These are the lengths of a right triangle with whole-number sides. Once we have one Pythagorean triple, it is easy to find others, such as 6, 8, 10. But these are not interesting, since they correspond to similar triangles. What we really seek are *primitive* Pythagorean triples. These are triples of numbers satisfying the equation which have no common factors.

A basic question about primitive Pythagorean triplets is: (a) how many of them are there? One can ask other questions. (b) Is there a formula for them? (c) Are there interesting patterns that we can find in the triples? By studying some "experimental data" — lists of triples — one can make a start on (c) and perhaps even say something about (a) and (b).

(2) The second equation defines what is called an "elliptic curve." There is a rich theory of these equations. One question is: given an elliptic curve $y^2 = x^3 + ax + b$ with rational coefficients a and b, is there a rational solution. No general method for answering this problem is known.

(3) This equation has no positive integer solutions. It plays a role in one of the most famous mathematical problems of all time is "Fermat's last theorem." Pierre Fermat (1601–1665) was a judge in Toulouse, France and also a very serious amateur mathematician. One evening reading a copy of Diophantus' *Arithmetica*, newly rediscovered and translated from Greek to Latin, he came on a theorem about Pythogorean triplets. In the margin of the book he wrote "I have found a marvelous proof that there are no positive integer solutions to the equation

$$x^n + y^n = z^n \tag{8}$$

for $n > 2$. Despite much progress in special cases, the problem remained unsolved until Andrew Wiles, a British mathematician working at Princeton University, announced his solution in 1994.

Wiles' proof, not only settled an old prolbems, but it also opened the doors to new areas of research thought the introduction of new ideas and techniques. It relies on a great deal of advanced mathematics that was developed since Fermat's time, but there is one part that we can mention here. Given a (hypothetical) solution a, b, c to Fermat's equation, one can consider the elliptic curve given by the cubic equation

$$y^2 = (x - a)(x - b)(x - c) \tag{9}$$

Cubic Diophantine equations are not completely understood, but they are much better understood than those of higher degree. They played a huge role in Wiles' proof. Indeed, the unsuspected connection between cubic equations and Fermat's equation, discovered in XXXX by the German mathematician Gerhard Frey, was one of the milestone events that led to the solution Fermat.

(4) We do not understand the set of integer solutions to equations of this general form (homogeneous cubics in four variables).

References

[1] H. Davenport, *The Higher Arithmetic: an Introduction to the Theory of Numbers*

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