Dynamical Systems

Jim Carlson

April 2003

Contents

1 Introduction

We will introduce the idea of discrete dynamical system through a series of examples. These include: two models of growth of bacterial population growth, random number generation, computing square roots, a simple model ecology, the conduction of heat, the game of billiards, and Conway's "Game of Life."

A dynamical system is characterized by a series of *states* that change through time. In a discrete dynamical system we consider the state only at equally spaced intervals, e.g., once per second, once per hour, etc. Thus the evolution of the system is given by a sequence of states

$$x_0,\ x_1,\ x_2,\ \ldots.$$

The evolution is determined by (1) the *initial state* x_0 and (2) a *next-state function* f, where $x_{n+1} = f(x_n)$. These words will have more meaning after having read the examples below.

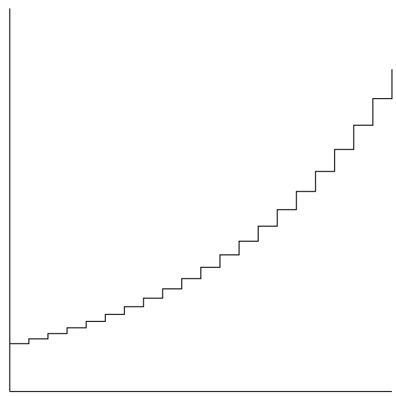
Historical note: The French mathematician Henri Poincaré introduced the notion of dynamical system around 1900 in order to study the stability of the solar system. The key question is to decide between two alternatives, one pleasant, the other less so: (a) the earth remains in an orbit close to its present one for a very long period of time, (b) a catastrophe occurs, e.g., the earth is captured by the sun while some other bodies (comets, ...) are ejected from the solar system, or the reverse: hte earth is ejected while other bodies fall into the sun.

1.1 Bacterial growth I

Consider the population of bacteria in a culture medium at one hour intervals. The population is just a sequence of numbers

$$P_0,\ P_1,\ P_2,\ \ldots$$

We can study these populations experimentally, and when we do, we find data like the following, at least as long as the concentration of bacteria in the medium is small.



Bacterial growth

We can also ask for a theoretical model of how the bacterial population grows. One model might be that a bacterium divides into two bacteria, on the average, in one hour's time. A still more simplified model is that all bacteria divide into two bacteria at time 1 hour, 2 hours, etc. The simplified model is very easy to compute. It obeys the rule

$$P_{n+1} = 2P_n \tag{1}$$

Thus if we start with one bacterium at time $T =$ one hour, then the sequence $\{ P_n \}$ is $1, 2, 4, 16, \ldots$. It can be described by the formula

$$P_n = 2^n \tag{2}$$

Using the formula we find that $P_{24} = 1.7 \times 10^7$ and $P_{48} = 2.6 \times 10^{14}$. Bacterial populations grow very rapidly, indeed "exponentially."

Exercise 1 *Show that $P_n = Ce^{kn}$ where C and k are constants and n is the time measured in hours. Find C and k.*

We will call the model just introduced "model I." It is given by a rule of the form

$$P_{n+1} = kP_n. \tag{3}$$

The population at all future times is determined by the population at time $n = 0$.

Exercise 2 *What is the general form for the population sequence $\{ P_n \}$ for model I given an initial population P_0?*

1.2 Discrete dynamical systems

The simple model of bacterial growth just discussed is an example of a *discrete dynamical system.* A dynamical system consists of

1. A set of states S.
2. A function $f : S \longrightarrow S$ which tells how the next state depends on the previous one.

In the dynamical system considered above, the set of states is the set of possible populations $\mathbf{N} = 0,\ 1,\ 2,\ \ldots$. The symbol $\mathbf{N}$ stands for the natural numbers. The next-state function f is the function $f(x) = 2x$. The system has an explicit solution for initial population $P_0 = 1$. It is given by the formula $P_n = 2^n$.

It is sometimes convenient to use for states the positive real numbers. Consider, for example, a model in which the population increases by a factor of 1.2 in one hour's time, and the initial population is 1.76 million.

Exercise 3 *For the model just discussed, with next-state function $f(x) = 1.2x$, compute the states P_n for $n = 1,\ 2,\ 3,\ 4,\ 5$.*

Exercise 4 *Susan sets up a savings account with an initial balance of \$100. Her bank gives an annual interest rate of 3%. At the end of each month, Susan deposits \$50 in her account. Describe her account balance as a dynamical system and compute its state for months $n = 0,\ 1,\ \ldots,\ 12$.*

Exercise 5 *SOMETHING RADIOACTIVE*

Exercise 6 *In 1202 the Italian mathematician Fibonacci (Leonardo of Pisa) posed the following puzzle:*

> *Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on. How many pairs will there be in one year?*

Let F_n denote the number of rabbits in the field at time n. At time $n = 0$ there are no rabbits: $F_0 = 0$. At time $n = 1$ there is one pair: $F_1 = 1$.

a) Discover the rule for calculating F_n in terms of previous values of the F's.

b) Consider the ratio F_n/F_{n-1}. How does it behave as n becomes larger and larger?

Exercise 7 *Consider the sequence of numbers below:*

```
0 1 6 14 3 16 13 15 8 7 2 11 5 9 12 10
```

It is generated by a discrete dynamical system. Do you see the pattern in these numbers? Can you discover the rule which produced the pattern?

1.3 Bacterial growth II

The model of bacterial growth considered above (call it model I) is quite realistic when the population is low and the time interval considered is not too large. (XX: experimental data) However, it is clear that the model cannot be accurate over long time intervals: the population growths ever larger and will eventually consume all available resources.

Exercise 8 *Assume that our bacteria are one micron in diameter. How long will it take an initial population to cover the surface of the earth according to model I?*

Can we find a more realistic model? Let us try this. We will replace the next-state function $f(x)$ by

$$f(x) = kx - \ell x^2 \tag{4}$$

We call this model II. In it we have added a quadratic correction term ℓx^2 to the linear term kx of model I. The constant k is assumed to be positive, so that growth occurs, and the constant ℓ is assumed to be negative. Consequently the growth rate will decline as the population rises. To make this statement precise, observe that the current state and the next state are related by the equation

$$Q = f(P). \tag{5}$$

The rate of growth from state to state is $Q/P = f(P)/P$. In our case,

$$\text{growth rate} = f(P)/P = k - \ell P. \tag{6}$$

If ℓP is very small in comparison with k, then the the model II growth rate is very nearly equal to the model I growth rate. To say that ℓP is small by comparison with k is to say that P is small compared to k/ℓ. Thus we expect model II to behave like model I when

$$P << \frac{k}{\ell}. \tag{7}$$

What happens when the population is not small compared to k/ℓ? As a first attempt to answer this question, we run a short simulation, computing 15 states for model II with parameters $k = 2$ and $\ell = 0.01$. This simulation (see below) is written in the Python language (see appendix). The code explains itself, except for two points. The first is that in defining the function `f` one uses the keyword `lambda` to signal the start of the list of independent variables. The colon signals the end of this list. Thus `length = lambda x,y:  sqrt(x*x + y*y)` defines the length of a vector with components `x` and `y`. The second is the form of the `print` statement. It is clear that the variables `n` and `x` are being printed. The part `'%4d %6.2f'` gives form in which the variables `n` and `x` are to be printed: `n` is a decimal integer printed in a slot four spaces wide; a space separates this integer from `x`, which is printed in decimal form in a slot six spaces wide, with to positions after the decimal point.

```
>>> K = 2
>>> L = .01
>>> f = lambda x: K*x - L*x*x
>>> for n in range(0,15):
...   print '%4d %6.2f' % (n, x)
...   x = f(x)
...
   0   1.00
   1   1.99
   2   3.94
   3   7.73
   4  14.85
   5  27.50
   6  47.44
   7  72.37
   8  92.37
   9  99.42
  10 100.00
  11 100.00
  12 100.00
  13 100.00
  14 100.00
```

The population rises rapidly at first, almost doubling at each stage. But the growth rate quickly slows down, and within ten generations has reached an equilibrium value of 100: the population is in a "steady state."

What happens if the initial population is larger than the equilibrium population? We can simulate this situation as well:

```
>>> x = 150
>>> for n in range(0,10):
...   print '%4d %6.2f' % (n, x)
...   x = f(x)
...
   0 150.00
   1  75.00
   2  93.75
   3  99.61
   4 100.00
   5 100.00
   6 100.00
   7 100.00
   8 100.00
   9 100.00
```

We see that the population rapidly decreases to its equilibrium value.

Can the fact that there is an equilibrium state be understood theoretically (as opposed to computationally)? Let us first try to understand the equilibrium value. It is a state x that satisfies

$$x = f(x). \tag{8}$$

Such states are called *fixed points* for f. In our case the fixed points are given by the equation

$$x = kx - \ell x^2. \tag{9}$$

One solution is $x = 0$. No bacteria, no growth. The population is zero at the beginning and forever afterwards. The other root of this equation is

$$x = \frac{k-1}{\ell}. \tag{10}$$

This formula gives a positive equilibrium solution. Substituting $k = 2$, $\ell = 0.01$ as in our example, we find $x = 100$, which is not a surprise.

Exercise 9 *Consider a population model with $f(x) = 3x - .0001x^2$. What is the equilibrium population level? How does the population behave over time? What happens when the initial population is slightly above or slightly below the equilibrium value? Now vary the parameters k and ℓ. How does the population behave?*

1. Note In the preceding problem it is useful to define a function `states(f, a, n)` which prints out n states of the dynamical system defined by f with initial state a. Here is an example:

```
>>> K = 3
>>> L = 0.002
>>> f = lambda x: K*x - L*x*x
>>> states(f, 1, 4)
    0     1.00
    1     3.00
    2     8.98
    3    26.77
```

The function `states` can be defined as follows:

```
def states(f, a, n):
  x = a
  for k in range(0,n):
    print "%5d %8.2f" % (k, x)
    x = f(x)
```

Exercise 10 *Consider a model II system with* $k = 2.45$ *and* $\ell = 0.03$. *What is the equilibrium population? Run the system with initial state* $x = 1$ *(one bacterium). How long does it take to get to equilibrium? Plot the population as a function of time and comment on the resulting graph.*

Exercise 11 *Experiment with other values of* k, ℓ, *and initial population. What do you observe?*

2 Other dynamical systems

If you have done the exercises in the previous section, you will have noticed that even our very simple model II can display a surprising range of behavior. We will explore this further — both computationally and theoretically — in what follows. We will also study other dynamical systems. To appreciate the range of possibilities, we consider XX examples: random number generators, Markov processes, and the conduction of heat.

2.1 Random number generators

Consider for a moment the sequence of numbers

```
0 1 6 14 3 16 13 15 8 7 2 11 5 9 12 10 ...
```

Is there a pattern? There does not seem to be an obvious one, and indeed the pattern is designed to be like a random sequence: one produced by flipping coins.

To make such "pseudorandom" sequences we use *modular arithmetic*. To this end, let

$$a \,\%\, N$$

denote the remainder of division of a by N. The numbers a and N are integers, with $a \geq 0$ and $N > 0$. The phrase "$a \,\%\, N$" is pronounced "a mod N," and N is called the *modulus*. Thus $7 \,\%\, 3 = 1$, $18 \,\%\, 5 = 3$, $12 \,\%\, 3 = 0$, etc.

Notice that for all $j \geq 0$, the number $j \,\%\, N$ is in the range $0,\ 1,\ \ldots,\ N-1$. We call this set of N numbers "the integers mod N, and we write it as $\mathbf{Z}/N$.

Now take $N = 17$ as the modulus, and take $f(x) = 5x + 1 \,\%\, 17$. Then $f(0) = 1$, $f(1) = 6$, $f(6) = 14$. Note that $5 \cdot 6 + 1 = 31$, and that 17 divides 31 once with a remainder of 14, so that $f(6)$ is indeed 31, as written. The rest of the values in the sequence $0,\ 1,\ 6,\ 14,\ 3,\ \ldots$ are computed in the same way.

Repeated application of the function f produces the longer sequence

```
0 1 6 14 3 16 13 15 8 7 2 11 5 9 12 10 0 1 6 14 ...
```

Notice that the 17th number is the same as the first, the 18th is the same and the second, and so on. Thus the sequence forms a *cycle* of 16 distinct numbers. (That is to say, they can be organized in a circle). Since the mod 17 number system has exactly 16 different numbers, this cycle is *maximal*: there is no larger cycle.

Exercise 12 *This exercise uses 16 as the modulus. Consider the function* $f(x) = (7x + 1) \,\%\, 16$. *What is the cycle of* 1, $f(1)$, $f(f(1))$, ...? *Can you organize all the numbers of* $\mathbf{Z}/16$ *into cycles for* f? *What are the lengths of these cycles?*

Exercise 13 *Consider other functions of the form* $f(x) = (ax+b) \,\%\, 16$. *What are their cycles? Can you find a function with a cycle of length 16? Such a cycle is called* maximal.

The function just given defines a 17-state dynamical system on the numbers 0, 1, ..., 16. We can define N-state dynamical systems using a function of the general form

$$f(x) = (ax + b) \,\%\, N$$

If the parameters a, b, and N are carefully chosen, the sequence of numbers generated by the dynamical system behaves much like a random sequence. These are used for everything from engineering studies to computer games.

2. Note The idea of modular arithmetic (congruences) was invented by Gauss and published in his *Disquisitiones Arithmetica* in 1801.

2.2 A strange system

Consider the dynamical system defined by the rule

1. if x is divisible by 3, then $f(x) = x/3$.
2. if x leaves 1 as remainder when divided by 3, then $f(x) = 2x$.
3. if x leaves 2 as a remainder when divided by 3, then $f(x) = 2x - 1$.

Here is what happens when we run the system with initial state $x = 4$:

```
4, 8, 15, 5, 9, 3, 1, 2, 3, 1, 2, ...
```

You should check that this is correct. Note that when we get to the sixth state a repeating pattern begins.

Exercise 14 *In the preceding example, how many states do we need to compute before we can* prove *that we get a repeating cycle?*

Exercise 15 *Experiment with different initial states. Is there a common pattern? Can you prove that any of your hunches are correct?*

2.3 Computing roots

Consider the problem of computing the square root of 3. This is the same as solving the equation $x^2 = 3$, which we can rewrite as

$$x = 3/x. \tag{11}$$

We can use this equation to find a square root by trial and error. Note that if $x < \sqrt{3}$, then $x > 3/x$, and that if $x > \sqrt{3}$, then $x < 3/x$. Thus we can determine whether a trial solution is too large or too small by doing one division.

Let us use this observation to efficiently find a sequence of successive approximations to $\sqrt{3}$. To begin, let $x_0 = 1$. Since $x_0 < 3/x_0$, the trial solution is too small. We compute a better trial solution x_1 by taking the average of x_0 and $3/x_0$. Thus $x_1 = 2$. Since $x_1 > 3/x_1$, the trial solution is too large. We compute a better trial solution x_2 by taking the average of x_1 and $3/x_1$. Thus $x_2 = 7/4$. In this way we obtain a sequence of trial solutions $x_0,\ x_1,\ x_2,\ x_3,\ \ldots$. The rule for computing the sequence is $x_{n+1} = f(x_n)$, where

$$f(x) = \frac{1}{2}\left(x + \frac{3}{x}\right) \tag{12}$$

Thus the square root of three can be computed using a dynamical system.

Exercise 16 *Run the dynamical system for the square root of three to find an approximate root which is accurate enough for you.*

Exercise 17 *What are the fixed points of the dynamical system with* $f(x) = (1/2)(x + a/x)$*?*

Exercise 18 *Find an approximate root of the equation* $x = x + x^3/100$

Exercise 19 *The equation* $x^3 = 27$ *has* $x = 3$ *as a root. Can we find a root of the equation* $x^3 + x = 27$*? Let us suppose that* u *is an approximate root. A good choice is* $u = 3$*. We would like to find a better approximation of the form* $u' = u + h$*, where* h *is "small." To find the correction term* h *substitute* u' *in the equation to be solved and expand. From*

$$(u + h)^3 + (u + h) = 27.$$

we obtain

$$u^3 + 3u^2h + 3uh^2 + h^3 + u + h = 27.$$

If h *is small, then* h^2 *and* h^3 *is smaller still. So let's ignore the smaller terms and solve the equation*

$$u^3 + 3u^2h + u + h = 27.$$

We find

$$h = -(u^3 + 27)/(3u^2 + 1).$$

Thus

$$u' = u - (u^3 + 27)/(3u^2 + 1).$$

If $u = 3.0$*, then* $u' = 2.89286$*. This computation can be summarized by saying that* $u' = f(u)$*, where*

$$f(u) = u - (u^3 + 27)/(3u^2 + 1).$$

Applying f *to an approximate root yields a better approximate root. Find a still better approximation by applying* f *again. Can you find a root that is accurate to six decimal places? What is the fixed point of the function* f*?*

Exercise 20 *Find a root of* $x^4 + 2.1x = 16$ *accurate to six decimal places using the method of the previous problem.*

2.4 A model ecology

Consider an island divided into two parts, A and B. A species of rodent lives on the island, but part A is more suitable to the rodents than part B. This reflected in the probability of a rodent staying put or migrating to the other part. The rules are as follows:

1. A rodent in part A will stay put with probability 0.7 and will move to part B with probability 0.3.
2. A rodent in part B will stay put with probability 0.4 and will move to part B with probability 0.6.

If $z = (x, y)$ gives the state of population at a given time, then the populations at one time unit later are given by $z' = (x', y')$, where

$$x' = 0.7x + 0.6y \qquad (13)$$

$$y' = 0.3x + 0.46y \qquad (14)$$

What will happen if the initial distribution of the population is $z = (10000, 0)$? This is something we can easily compute, by hand or with a machine:

0	10000	0
1	7000	3000
2	6700	3300
3	6670	3330
4	6667	3333
5	6667	3333

Notice the rapid convergence to what seems to be an equilibrium state, the population vector $z = (6667, 3333)$. Can we predict this equilibrium state in advance? Of course. We solve the equations "next state = current state," $x' = x$, $y' = y$:

$$x = 0.7x + 0.6y$$
$$y = 0.3x + 0.4y$$

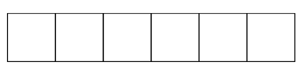
One finds that $x = 2y$. Thus, if the total population is $x + y = 10,000$, then $x = 6,666\frac{2}{3}$, $y = 3,333\frac{1}{3}$.

Exercise 21 *Run the model just discussed with different initial states. What happens in each case after a long time?*

Exercise 22 *Suppose the probabilities are as follows. An animal in part A stays put with probability 0.55 and moves with probability 0.45. An animal in part B stays put with probability 0.58 and moves with probability 0.42. What is the equilibrium state? Experiment with different initial states to see how rapidly the ecosystem approaches equilibrium.*

2.5 Conduction of heat I

Consider a thin, insulated rod of a conducting material, like iron, where the left and right ends are held at fixed temperatures, and where the temperature along the rod is known at a certain time. How can we determine the temperature of the rod at later times? One way to answer this question is to imagine the rod divided into short segments ("cells") of equal length, as in the figure below.



Thin rod divided into cells

Now consider an initial temperature distribution, as in the figure below. Each number represents the average temperature in the given cell. Thus the first cell has temperature eight degrees, the next one has temperature two degrees, etc.

8	2	4	6	4	0

Thin rod at $T = 0$

We can view this tempeature state numerically, as above, by plotting temperature versus position (cell number) or by filling in the cells with a shade of gray:

Thin rod at $T = 0$, gray scale

What is the temperature T seconds later? Well, the heat flows from hotter to colder, and so the temperature of a given cell tends towards the average temperature of nearby cells. Thus it is reasonable to try the *averaging rule*:

> The temperature of a given cell at a given time is the average of the temperatures of its left and right neighbors T seconds earlier.

This rule applies to all cells except those on the ends: their temperature is fixed. Thus the temperature T seconds later of the left-most cell is 8 degrees, and the temperature of the next cell to the right is $(8 + 2)/2 = 5$. This is the average of the temperatures of the left and right neighbor cells. In this way we compute the next temperature state:

8	6	4	4	3	0

Thin rod: state at $T = 1$

Below is the grayscale representation of this temperature state:

Thin rod at $T = 1$: gray scale

Exercise 23 *Compute the next two temperature states. What is the long-term behavior of the temperature?*

Exercise 24 *Graph temperature versus position for each of the four temperature states computed above. What do you notice about this sequence of pictures (motion picture)?*

Exercise 25 *Run the heat conduction model with six cells, with temperature 100 in the left-most cell and temperature 0 in the right-most cell. Fill in the other cells as you wish. What can you say about the temperature distribution after a long time? (Compare with other initial temperature distributions but with the same left-most and right-most temperatures).*

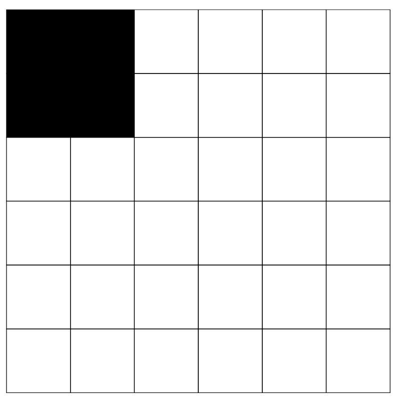
2.6 Conduction of heat II

We can also consider the conduction of heat in a thin insulated plate where the temperature around the edge is held fixed. The figure below illustrates an initial temperature distribution with a “hot spot” in the upper left corner: a group of four cells of temperature eight degrees. The other cells are at temperature zero. We should fill each such cell with a zero, but instead we leave it blank.

8	8				
8	8				

Thin plate at $T = 0$

As in the one-dimensional case, there is a gray-scale representation:



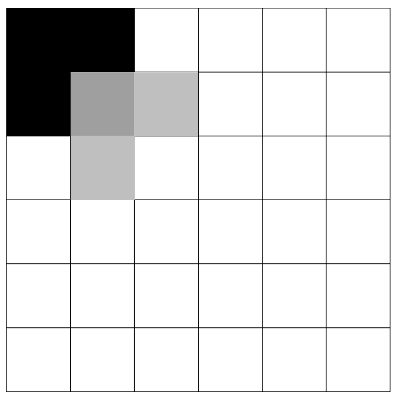
Thin rode at $T = 0$, gray scale

How do we compute the temperature T seconds later? We use a 2-dimensional averaging rule: the temperature of an interior cell is the average of its four nearest neighbors T seconds earlier. Four our purposes a cell and its neighbor must share an edge: it is across an edge that heat is transferred. Consider, for example the cell in the second row counting from the top and the second column from the left. We call this cell $(2, 2)$. It has four neighbors, $(1, 2)$, $(2, 1)$, $(3, 2)$, and $(2, 3)$. The temperatures of these cells are 8, 8, 0, 0. Their average is $4 = (8 + 8 + 0 + 0)/4 = 4$. The temperatures of the other cells are computed in the same way.

8	8				
8	4	2			
	2				

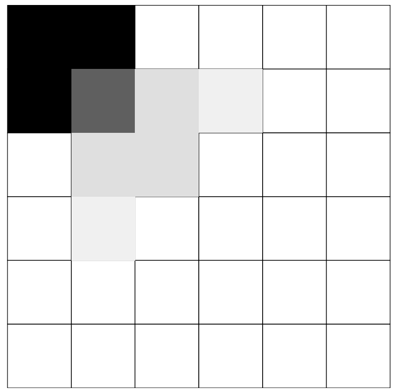
Thin rod at $T = 1$

The is a gray-scale representation looks like this:



Thin rod at $T = 1$, gray scale

Here is a gray-scale representation of the next state:



Thin rod at $T = 1$, gray scale

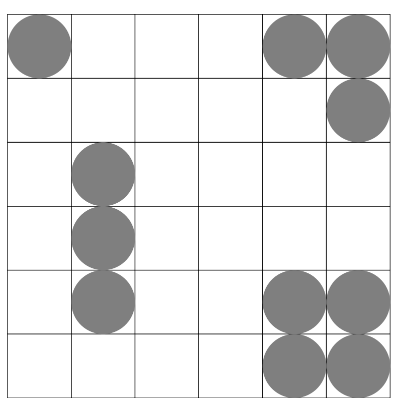
You can see the spread of heat from the hot spot in the upper left corner to the rest of the grid. This looks just like the spread of ink (especially if you use many cells). The same mathematics applies.

Exercise 26 *Find the next two temperature states. What is the long term behavior of the temperature?*

Exercise 27 *Choose an initial temperature distribution that you find interesting. Run the dynamical system and comment on the result after a long time. Compare with other initial distributions.*

2.7 The game of life

The game of life, invented by the mathematician John Conway in the late 1970's (see www.math.com/students/wonders/life/life.html), is played on an $n \times n$ board as in the figure below. Each square is either vacant or occupied. We code the states with colors: white for vacant, black for occupied.

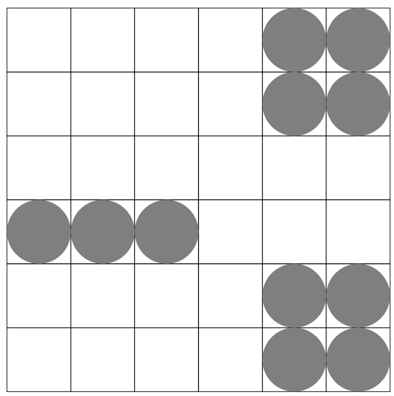


Game of life, initial state

A board with a particular arrangement of vacant and occupied squares is called a state. The next state is computed according to the following rules.

1. A dead cell with exactly three live neighbors becomes a live cell (birth).
2. A live cell with two or three live neighbors stays alive (survival).
3. In all other cases, a cell dies or remains dead (overcrowding or loneliness).

Thus the next state in our case is as follows.



Game of life, next state

Exercise 28 *Play the next move in the Game of Life. What conclusions can you draw?*

Exercise 29 *On a 6×6 board, choose some initial state for the Game of Life, then play a few moves. What conclusions can you draw? You may want to try this experiment more than once.*

The game of life is an example of a *celluar automaton*: a grid of cells that can take on various states, and a set of rules for computing the next state. One can consider one-dimensional automota, and also automota of three and more dimensions. The one and two-dimensional heat conduction models are examples of one and two-dimensional automata. Many models of physical phenomena can be discretized, formulated as celluar automota, and solved (or simulated) on a computer. Future states computed by the model deviate from physical reality after a certain interval of time, and the deviation grows larger as time increases. This is one of the difficulties inherent in predicting the weather, which use such models.

Cellular automata were considered by Turing and later by von Neumann. Stephen Wolfram has written a large book on the subject. His thesis, first enunciated by Fredkin, is that the universe is a cellular automaton. This idea is not generally accepted by physicists. For a discussion, see the New York Review of Books article by Steven Weinberg.

Exercise 30 *Experiment with small configurations of live cells in the Game of Life. What kind of behavior do you find? Reference for applet: www.bitstorm.org/gameoflife/*

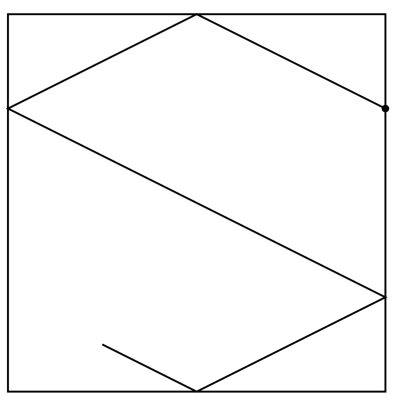
Exercise 31 *How many states does the* 8×8 *Game of Life have? How many of these states have five occupied cells?*

Exercise 32 *Consider the* $n \times n$ *Game of Life. What is the probability that a randomly chosen state is immortal? This is a research problem. Begin with* n *very small.*

.

2.8 Geometrical systems

Consider the square enclosure in in the figure below. A billiard ball first impacts the walls of the enclosure at the midpoint of the bottom wall. It bounces off according to the rule: *the angle of reflection equals the angle of incidence.* Subsequent impacts obey the same rule, which is the rule that billiard balls obey as do light rays striking a mirror. We have just described is a dynamical system. The states are the pairs (x, θ) that describe the position and angle of impact.



Billiard table, initial slope $= -1/2$

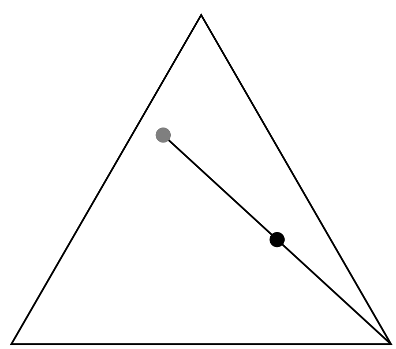
Exercise 33 *Under what conditions will the states of the dynamical system on the walls of a square form a closed cycle:* $(x_n, \theta_n) = (x_0, \theta_0)$ *for some* n*? Can one compute* n*, the length of the cycle, given the initial state* (x_0, θ_0)*?*

Exercise 34 *Study dynamical systems of the kind just described but with a shape different from a square.*

Consider the triangle in the figure below as well as a point in the interior of the triangle, indicated by the gray dot. Choose a vertex of the triangle at random. Draw a line from the given point to the vertex, and construct its midpoint, indicated by the solid dot. The rule

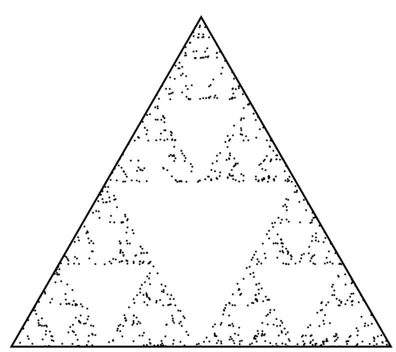
$$f(\text{gray dot}) = \text{solid dot}$$

given by this construction defines a "random dynamical system."



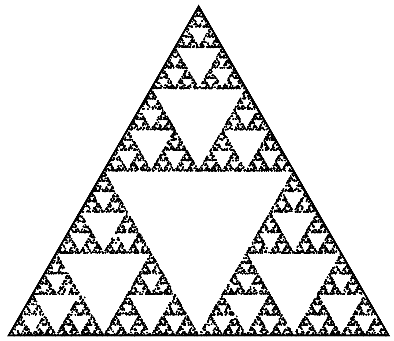
Random dynamical system on a triangle

The next figures shows a sequence of 1000 states computed using the rule f. Note the emerging pattern of nested triangles. (How would you describe it?)



Fractal triangle, 1000 iterations

The last figure shows a sequence of 10,000 states of the same system.

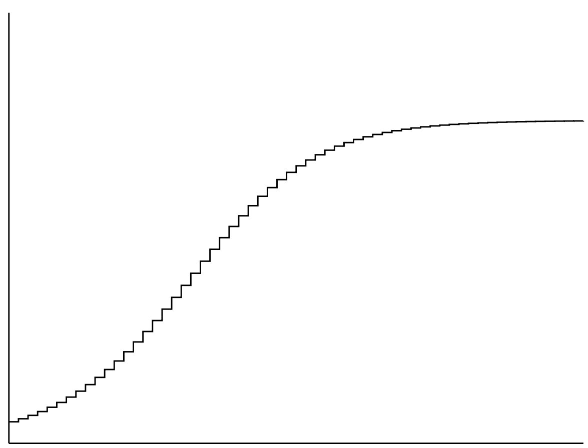


Fractal triangle, 10,000 iterations

Exercise 35 *Study the behavior of the random dynamical system just defined. How does the picture change when we change the starting point? Can one prove anything about these figures? Can one give a different description of them? What happens if we change the shape of the triangle? What happens if we replace the triangle by another figure?*

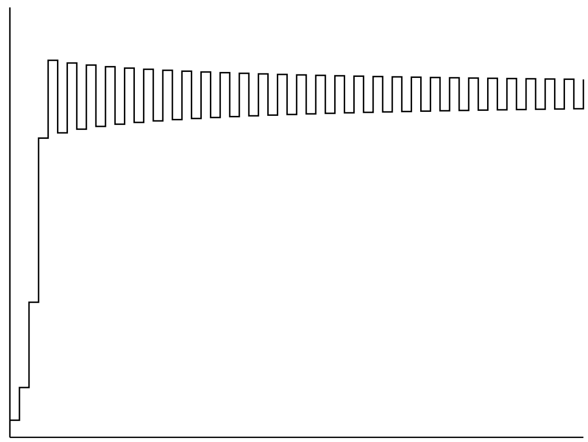
3 Understanding one-dimensional systems

We have seen the notion of a dynamical system has wide applicability: from population dynamics to heat conduction to finding roots of equations to generating random numbers. We have also seen that very simple systems can produce a wide range of behavior. For example, in the population model of section 1.3 with next-state function $f(x) = 1.15x - 0.01x^2$, we find an exponential rise in population followed by an exponential approach to equilibrium:



Exponential approach to equilibrium

Changing the parameters to $f(x) = 3x - 0.1x^2$ we find a much different behavior:



Decaying oscillations

If you experiment with other parameter values by playing around with different values of k and ℓ in the formula $f(x) = kx - \ell x^2$, you will find still other phenomena.

Experimentatation with other parameter values is both fun and enlightening. It also raises some fundamental questions. How can we be sure that we have found all the phenomena? Which parameter values produce which phenomena? Why do we have exponential approach to equilibrium in some cases and decaying oscillations in others? Etc.

To answer these questions we first make an important simplification. By a change of variables which replaces $x = au$ we can reduce the change-of-state function to the form

$$f(u) = Ku(1-u). \tag{15}$$

Thus to explore the phenomena we have only to vary one parameter, K.

To make this change of variables, replace x by au in XX to get

$$kx - \ell x^2 = aku - a^2\ell u^2$$

We can make the coefficients of u and u^2 equal if we set

$$ak = a^2\ell.$$

Thus, if we choose $a = k/\ell$, then

$$kx - \ell x^2 = Ku(1-u)$$

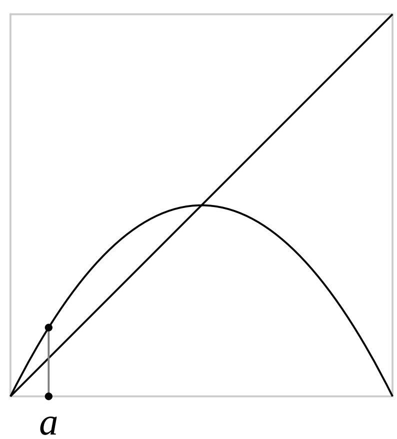
with $K = k^2/\ell$. Thus everything to be understood about the system with $f(x) = kx - \ell x^2$ can be understood for the system with $f(u) = Ku(1-u)$.

Exercise 36 *What is the equilibrium population for the dynamical system $f(x) = Kx(1-x)$? For what range of values of K does it make sense to consider the system? Examine the dynamics of the system for different values of K in this range. What phenomena do you observe?*

3.1 Graphical methods

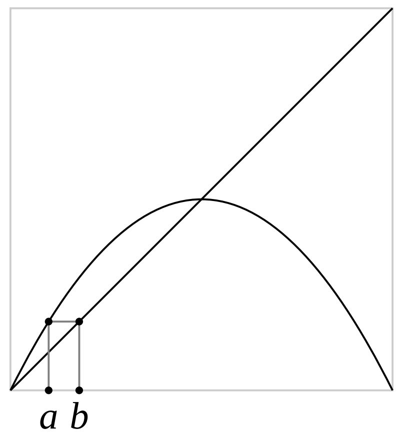
The results of the experiments in exercise 36 reveal a rich, even bewildering variety of phenomena. They illustrate the maximm "*simple systems can exhibit complex behavior.*" Nonetheless, this complexity can be understood, at least in part, by simple methods. One such method is *graphical analysis*. Consider the figure below. We have plotted the function $f(x) = 2x(1-x)$ on the interval $0 \leq xle1$. We have also drawn the box $0 \leq x \leq 1$, $0 \leq y \leq 1$, and in that box we have drawn the diagonal D, given by $y = x$. This drawing can be used to plot the sequence of states x_{n+1} beginning with an initial state x_0.

Here is the "graphical rule" for computing the next state b from a given state a. First, raise a vertical line a and find its point of intersection with the graph of f.



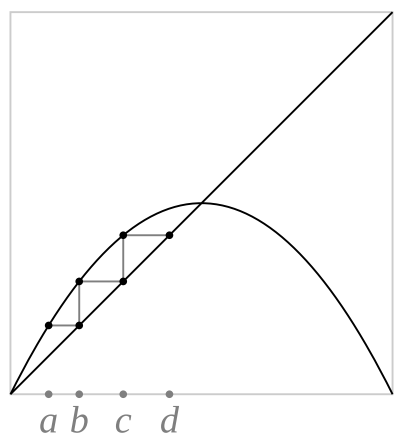
Graphical next-state rule, first step

Second, run a horizontal line from the graph to the diagonal line $y = x$. From the point of intersection with the diagonal, drop a vertical line to the x-axis. The point at which this vertical meets the x-axis is the next state b.



Graphical next-state rule, second step

With this graphical rule, we can quickly compute a sequence of states for f:



Sequence of states

It is pretty clear that if we continue this process the states a, b, c, etc. get closer and closer the state $z = 1/2$. This state is a fixed point of f, and it is also the limit value of the sequence.

Exercise 37 *Use the graphical rule to compute a sequence of states for the dynamical system with* $f(x) = 1.8x(1-x)$*, where the initial state is* $x_0 = 0.1$*. Then repeat your analysis with* $x_0 = 0.9$ *Can you form some general conclusions based on these analyses?*

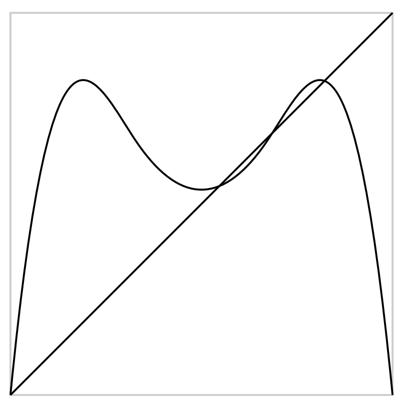
Exercise 38 *Analyze the system with* $f(x) = 3x(1-x)$*,* $x_0 = 0.1$*. Can you draw any general conclusions?*

Exercise 39 *Study the system with next-state function $f(x) = Kx(1-x)$ for various K. Can you draw any general conclusions?*

3.2 Fixed and periodic points

From the various examples we have studied (see exercise **??**) it is clear that there are different kinds of fixed points. An *attracting* fixed point p is one such that if x_0 is near p, then the states x_n get closer and closer to p. A *repelling* fixed point displays the opposite behavior: the states x_n get farther from p.

Another phenomenon we noticed is a sequence of states that seemed to approach alternation between a pair of states a and b. Is it possible to have perfect alternation between two states a and b, so that $f(a) = b$ and $f(b) = a$? If so, then $f(f(a)) = f(b) = a$, and $f(f(b)) = f(a) = b$. Therefore these two special points are roots of the equation $f(f(x)) = x$. A point of this kind is called *periodic*. More precisely, these are *points of period 2*. We can locate them by finding the places where the graph of $y = f(f(x))$ meets the graph of $y = x$. See the figure below.



Finding points of period two

Exercise 40 *Locate the points of period two for $f(x) = 3.25x(1-x)$ using the graphical method just described. Does $f(x) = 2.75x(1-x)$ have periodic points? For what value of K do there exist periodic points for $f(x) = Kx(1-x)$?*

4 Python

Notion of period.

References

[1] Freeman Dyson, Origin of Life

[2] Robert Devaney

[3] John Maynard Smith

[4] Steven Weinberg

[5] Steven Wolfram