Lecture 2: Hilbert function, dimension

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CIMAT Lectures

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Hilbert basis theorem

Theorem

Every ideal in $k[x_1, \ldots, x_n]$ is finitely generated.

Proof.

Let I be an ideal. Let $\langle LT(I) \rangle$ be its ideal of leading forms. By Dicskon's lemma, $\langle LT(I) \rangle$ has a finite set of generators $M_1, \ldots, M_s$. Let $G_1, \ldots, G_s$ be elements of I such that $LT(G_i) = M_i$. We claim that the G_i are a basis for I.

Let f be an element of I. Divide f by the G_i:

$$f = a_1 G_1 + \cdots + a_s G_s + r$$

The remainder r is in I. Therefore $LT(r) \in \langle LT(I) \rangle$. But monomials of r are not divisible by the M_i. Therefore $r = 0$. Q.E.D. □

Hilbert functions

Let I be an ideal of $R = k[x_1, \ldots, x_n]$. Let $I_{\leq s}$ be the space of elements of I of degree at most s The HIlbert function of I is

$$H_I(s) = \dim\left(R_{\leq s}/I_{\leq s}\right)$$

Example. Let $H_{0,n}(s) = \dim k[x_1, \ldots, x_n]_{\leq s}$ be the Hilbert function of the zero ideal in $k[x_1, \ldots, x_n]$.

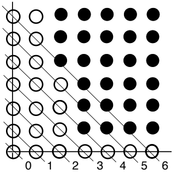
$$H_{0,n}(s) = \binom{n+s}{s} = \frac{s^n}{n!} + \cdots$$

In this case the Hilbert function is a polynomial of degree n which takes integer values for integer s.

What is the general form of the Hilbert function?

Hilbert function of a monomial ideal

Consider the ideal $I = \langle x^3 y, x^2 y^4 \rangle$.



The Hilbert function at s is the number of white dots below a diagonal with intercept $x = s$.

s	0	1	2	3	4	5	6	7	8	...
H(s)	1	3	6	10	14	18	21	24	27	...

$H_I(s) = 3s + \epsilon(s)$, where $\epsilon(s)$ is eventually constant: $\epsilon(s) = 3$ for $s \geq 6$.

What is general form of $H(s)$ for a monomial ideal?

Counting points "outside" a monomial ideal

For a monomial ideal, the problem of computing the Hilbert function is a problem of counting points outside $C(I)$, the cone of the ideal.

The region outside the $CI)$ is a union of translates of coordinate subspaces of dimension m, union a finite set.

The Hilbert function for the translate of an m-dimensional subspace by a vector α is

$$(*) \qquad H_{m,\alpha}(s) = \binom{m+s-|\alpha|}{s-|\alpha|}.$$

Why? The set of points in that translate of degree at most s is in one-to-one correspondence with points of the subspace of degree $s-|\alpha|$: map x to $x-\alpha$.

Using the inclusion-exclusion formula, we see that $H_I(s)$ is a sum of functions of the form $(*)$ for various m and α, plus a function which is eventually constant.

Conclusion

$H_I(s)$ is an eventually polynomial function.

Moreover:

(1) The degree d of $H_I(s)$ is the dimension of the largest linear subspaces in the exterior of $C(I)$.

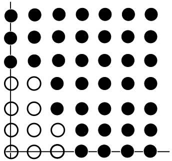
(2) The degree is the same as the dimension of the variety $V(I)$, which is a union of linear subspaces.

(3) The coefficient of $s^d/d!$ in $H_I(s)$ is the number of translates of linear subspaces outside of $C(I)$.

(4) The coefficient is the degree of $V(I)$: the sum of the multiplicities of the linear subspaces of largest dimension.

Example of zero-dimensional ideal

Consider the ideal $\langle x^3, x^2y^2, y^4 \rangle$



The values of the Hilbert function are 1, 3, 6, 9, 10, 10, 10, ... I is of dimension zero and $\dim_k R/I = 10$. $V(I)$ is the origin. But it is a "fat" origin with a 10-dimensional ring of functions.

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sage: R.<x,y> = PolynomialRing(QQ, order = 'lex')
sage: f = x^3; g = x^2*y^2; h = y^4
sage: I = ideal(f,g,h)
sage: I.dimension(), I.vector_space_dimension()
0, 10
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Hilbert function of a general ideal

Theorem

The Hilbert function of an ideal is the same as the Hilbert function of its ideal of leading terms.

Proof.

Let $f_1, \ldots, f_m$ be a basis for $I_{\leq s}$. Let $F_1, \ldots, F_m$ be the corresponding leading terms. Delete duplicates and put in order. Let $F_1, \ldots, F_k$ be the result. Let $f_1, \ldots, f_k$ be the corresponding elements of $I_{\leq s}$. We claim that the f_i are vector space basis of $I_{\leq s}$. Linear independence: easy.

Span? Let W be the span of the f_i. Let f be an element of $I_{\leq s} - W$ with $LM(f)$ minimal. $LT(f) = \lambda f_i$ for some i. $LM(f - \lambda f_i) < f_i$. Thus $f - \lambda f_i \in W$. Q.E.D.

□