On the Duflot filtration for equivariant cohomology rings

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Abstract

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We study the $\mathbb{F}_p$-cohomology rings of the classifying space of a compact Lie group G using methods from equivariant cohomology. Building on ideas of Duflot and Symonds we study a "rank filtration" on the p-toral equivariant cohomology of a smooth manifold. We analyze the structure induced by this filtration and construct a well behaved chain complex that controls the local cohomology of H^*BG.

We also refine the Duflot filtration to a filtration by a ranked poset, and from this get a detection result and restrictions on associated primes that generalize some of the work of Carlson and of Okuyama from finite groups to general compact Lie groups. We also use our methods to give new local cohomology computations for the cohomology of p-Sylow subgroups of S_{p^n}.

In the final chapter we show that the derived category of cochains on the Borel construction of a finite G-CW complex is stratified in the sense of Benson, Iyengar, and Krause by the equivariant cohomology ring.

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ACKNOWLEDGMENTS

This work owes a large debt to the work of Duflot in equivariant cohomology, especially [Duf83b] and [Duf83a]. Indeed this is largely an attempt to exploit to the fullest the techniques of Duflot. Of course, Quillen also deserves mention as the founder of studying the geometry of equivariant cohomology rings.

I was inspired to think about these ideas due to the work of Symonds [Sym10] on the regularity conjecture.

I was introduced to the subject by my thesis advisor, Steve Mitchell. Steve shared innumerable insights and I could not have done it without him. Steve was a great person and a great mathematician, and he is dearly missed.

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DEDICATION

to Steve Mitchell

Chapter 1
INTRODUCTION

1.1 Historical introduction

Group cohomology rings are a natural object of study in topology and in representation theory. For topologists, principal G bundles over a space X are classified by homotopy classes of maps from X into BG, so the cohomology of BG is the setting for characteristic classes of principal G bundles and the spaces BG are interesting homotopy types. Modular representation theorists study representations of a finite group G over fields of characteristic p, where p divides the order of G. In this setting Maschke's theorem does not hold, so there are interesting extensions of modules and not all modules are projective. The cohomology of G, $H^*(G, -)$, then appears as the derived functors of the G-fixed point functor. When G is a finite group, these coincide: $H^*_{\text{Sing}}(BG; k) = H^*_{\text{Group}}(G; k)$. This is true for k any field with trivial G-action, but there is higher cohomology only when k is of characteristic p and p divides the order of G. From now on, all cohomology will be with $\mathbb{F}_p$ coefficients, where p divides the order of G if G is finite.

So, understanding group cohomology rings is of interest to both topologists and to representation theorists, and group cohomology is an ongoing area of interaction between the two subjects. The first important theorem about group cohomology rings is a theorem of Venkov saying that group cohomology rings are reasonable rings from the point of view of commutative algebra (we have that H^*BG is graded commutative, as it is the cohomology of a space).

Theorem 1.1.1 (Venkov [Ven59])**.** *For G a compact Lie group, H^*BG is finitely generated as an $\mathbb{F}_p$-algebra.*

It is interesting to note that when G is a finite group this theorem is purely algebraic (and has a purely algebraic proof, due to Evens [Eve61]), but Venkov's proof crucially involves compact Lie groups. Here is a sketch of the proof: take a unitary embedding of G, and consider the fiber bundle $U(n)/G \to BG \to BU(n)$. The cohomology of $BU(n)$ is a polynomial algebra on n generators, so as $U(n)/G$ is a manifold its cohomology is a finite dimensional $\mathbb{F}_p$ vector space, and consequently the E_2 page of the Serre spectral sequence of this fibration is finitely generated over $H^*BU(n)$. The same is true for each successive page, and because the spectral sequence converges in finite time the E_∞ page is finitely generated over $H^*BU(n)$, from which we can extract the desired result.

Even though H^*BG is finitely generated, for most groups computing the cohomology ring is a daunting task, and knowing all the generators and relations in the ring is not necessarily illuminating. What is more desirable is structural or geometric information about these rings, relating the geometry of $\operatorname{Spec} H^*BG$ to the group theory of G. With a good structural result, we can understand H^*BG even without having an explicit presentation of this ring, and structural results can also help in computing the cohomology rings of specific groups.

The first structural result in the study of group cohomology rings is Quillen's stratification theorem. This result describes $\operatorname{Spec} H^*BG$ as a space, and also shows that H^*BG is detected modulo nilpotents on subgroups of the form $(\mathbb{Z}/p)^n$, which we follow Quillen in calling p-tori.

Before stating the theorem, we need to explain a bit of terminology. A uniform F-isomorphism is a map between $\mathbb{F}_p$-algebras such that the kernel is nilpotent, and where there is an n so that the p^{nth} power of every element of the codomain is in the image of the map. Given a compact Lie group G, the category $\mathcal{A}(G)$ is the category whose objects are p-tori of G, and where morphisms from E to E' are homotopy classes of G-equivariant maps from $G/E \to G/E'$. Quillen uses a slightly different category in the orginal paper.

Theorem 1.1.2 (Quillen [Qui71])**.** *Let G be a compact Lie group.*

1. *The map $H^*BG \to \varinjlim_{E\in\mathcal{A}(G)} H^*BE$ induced by the restriction maps is a uniform F-isomorphism.*

2. *The induced map on spectra $\varprojlim_{E\in\mathcal{A}(G)} \operatorname{Spec} H^*BE \to \operatorname{Spec} H^*BG$ is a homeomor-*

phism.

The second statement is a corollary of the first. Here are a few more corollaries.

Corollary 1.1.3 (Quillen [Qui71])**.** *Let G be a compact Lie group.*

1. *If $x \in H^*BG$ restricts to 0 on all p-tori in G, then x is nilpoent.*

2. *The Krull dimension of H^*BG is the largest rank of a p-torus of G.*

3. *The minimal primes of H^*BG are in bijection with conjugacy classes of the maximal (by inclusion) p-tori of G.*

Just as Venkov's proof of the finite generation of cohomology uses compact Lie groups, Quillen's stratification theorem crucially uses equivariant cohomology. If X is a G-space, then the equivariant cohomology of X, denoted by $H^*_G X$, is the cohomology of the Borel construction on X. Taking X equal to a point recovers the cohomology of BG, so we can write $H^*_G = H^*BG$.

In the proof of the stratification theorem, Quillen uses the G-space obtained by taking a faithful finite dimensional representation $G \hookrightarrow U(n)$ and quotienting by the right action of S, the diagonal matrices of order dividing p. The equivariant cohomology of this space is faithfully flat over H^*_G, and Quillen first proves his theorem for $H^*_G U(n)/S$ and then uses faithfully flat descent to deduce the result for H^*_G.

This is the first example of the use of equivariant cohomology to study group cohomology rings. There are several other mathematicians after Quillen who have used his ideas, but applying equivariant cohomology to group cohomology is not yet fully explored. Quillen also introduced techniques from commutative algebra and algebraic geometry into studying group cohomology rings, and Quillen's result suggests that group cohomology rings are interesting from a geometric point of view, and that there are rich connections between the geometry of Spec H^*_G and the group theory of G.

After Quillen's theorem on dimension and minimal primes, it is natural to ask about depth and associated primes. Duflot [Duf81, Duf83a] proved the following.

Theorem 1.1.4 (Duflot)**.** *Let G be a compact Lie group.*

1. *The associated primes of H^*_G come from restricting to p-tori, in the sense that for each associated prime $\mathfrak{p}$ there is a p-torus E so that $\mathfrak{p}$ is the kernel of the map $H^*_G \to H^*_E/\sqrt{0}$.*

2. *The depth of H^*_G is bounded below by the maximal rank of a central p-torus of G.*

3. *For G finite, the depth of H^*_G is bounded below by the maximal rank of a central p-torus in a p-Sylow subgroup of G. In particular,* $\operatorname{depth} H^*_G \geq 1$.

Duflot's results are very helpful in concluding that some groups have H^*_G Cohen-Macaulay and they provide some control over the associated primes of H^*_G, but they also raise the questions: which p-tori represent associated primes, and what is the depth of H^*_G? As we will see, these questions are closely related.

Duflot also uses equivariant cohomology and a technique of Quillen that reduces the study of G-equivariant cohomology to the study of S-equivariant cohomology, where S is a p-torus. In the special case of the S-equivariant cohomology of a smooth manifold M, Duflot [Duf83b] defined a filtration on $H^*_S M$ via a filtration defined on the manifold M by the ranks of isotropy groups of points in M. The key property of this filtration is that the subquotients are particularly nice: the i^{th} subquotient is free over the cohomology of a rank i p-torus.

The Duflot filtration puts a rich structure on the S-equivariant cohomology of smooth manifolds, and one of the main goals of this thesis is to describe and exploit this structure to the fullest extent possible. Before coming back to the Duflot filtration, we describe some further results in the geometry of group cohomology rings.

Carlson [Car83] applied Quillen's ideas to modular representation theory by defining for a finite dimensional modular representation M of a finite group G a subvariety of $\operatorname{Spec} H^*_G$

called the support variety of M. This subvariety controls much of the large scale structure of M, for example it reflects the indecomposability of M and the complexity of M. Using the theory of support varieties and techniques from algebraic topology, Benson, Carlson, and Rickard [BCR97] showed that the cohomology of a finite group controls the thick tensor ideals of the stable module category of that group, so that geometric facts about H^*_G have representation theoretic interpretations.

The idea of support theory has found broad applications, and the way in which H^*_G controls the structure of the stable module category is well understood by the work of many people, especially Benson, Iyengar, and Krause [BIK11c], but as of yet there are no applications of equivariant cohomology to the study of the stable module category. We hope to indicate in this thesis the relevance of equivariant cohomology to support theory, and some directions for future work in this area.

In [Ben04] Benson conjectured that group cohomology rings have Castelnuovo-Mumford regularity 0. Castelnuovo-Mumford regularity is an integer invariant of a local ring involving the local cohomology of that ring as a module over itself, and this conjecture has structural and computational consequences. The local cohomology modules of a ring encode much of the geometric information of the ring, including the depth and dimension, so if one is interested in the geometry of H^*_G it is natural to study its local cohomology. Moreover in the case of H^*_G, Greenlees [Gre95] has constructed a spectral sequence with E_2 page the local cohomology of H^*_G converging to $(H^*_G)^*$, the $\mathbb{F}_p$-linear graded dual of H^*_G, so the local cohomology modules are a natural approximation to the cohomology ring.

Symonds proved Benson's regularity conjecture in [Sym10], and he used equivariant cohomology and the Duflot filtration to reduce the conjecture to the p-torus case, where regularity can be computed via an explicit local cohomology computation. In this thesis we extend Symonds' techniques to get more information about the local cohomology modules of equivariant cohomology rings.

1.2 Overview of contents

Using some of the ideas in Symonds' proof of the regularity theorem, we construct a chain complex that essentially computes the local cohomology of a group cohomology ring, in a way in which structural information like Krull dimension, regularity, and bounds on depth can be read off immediately.

The construction of this chain complex necessitates the study of the algebraic structure present in equivariant cohomology provided by the Duflot filtration, and we define this structure purely algebraically and prove some geometric facts about rings having this structure. We also show how to enrich the structure of the Duflot filtration to a filtration by a poset, and we derive some computational consequences from this enrichment, including a cohomological detection result of Carlson and new restrictions on which p-tori can represent associated primes.

The general strategy is to study group cohomology rings by first studying related equivariant cohomology rings, and then descending down to group cohomology. These equivariant cohomology rings are the p-toral equivariant cohomology of homogeneous spaces arising from a unitary representation of a group, and in general the structural results have more content in the equivariant cohomology setting than they do only for group cohomology rings, suggesting that for families of groups where these homogeneous spaces can be well understood the techniques of this thesis will give even more cohomological information than the general theory developed here. For example, we compare the rich structure of the Duflot filtration with group theoretic properties of iterated wreath products to get new local cohomology computations for these itereated wreath products.

In the final chapter we begin an attempt to categorify the Duflot filtration, and to apply equivariant topology to the study of modular representation theory proper, rather than merely group cohomology. To this end, we show that just as group cohomology controls the structure of the stable module category of a group, the equivariant cohomology of a space controls the equivariant analogue. This leads to the classification of the localizing and thick subcategories in the equivariant version of the stable module category.

1.3 Outline

Now we review in more detail the contents of this document.

1.3.1 The Duflot chain complex

In this thesis, inspired by Symonds' proof of the regularity theorem, we show that the Duflot filtration of $H^*_S M$ for a smooth S-manifold M, with S a p-torus, gives rise to a cochain complex, DM, of graded modules we call the Duflot complex of M. The cohomology of DM is the local cohomology of $H^*_S M$, and DM enjoys the following properties, which we state here in the special case that G is a finite group and M is the manifold $G\backslash U(V)$ for V a faithful representation of G, with $S < U(V)$ the maximal p-torus of diagonal matrices of order p acting on the right.

Theorem 1.3.1 (2.1.9, 3.1.4, 3.2.1, 4.1.3)**.** *A finite dimensional representation $G \hookrightarrow U(V)$ can be chosen so that the Duflot complex of $G\backslash U(V)$ satisfies the following:*

1. $H^* D(G\backslash U(V)) = \mathcal{H}^* H^*_S(G\backslash U(V)) = \mathcal{H}^*(H^*_G) \otimes H^* U(V)/S$.

2. $D(G\backslash U(V))^i = 0$ *for i less than the p-rank of $Z(G)$ and i greater than the p-rank of G.*

3. *The Krull dimension of the i^{th} term of the dual chain complex $D(G\backslash U(V))^*$ is i.*

4. $D(G\backslash U(V))^{i,j}$ *is zero above the line $i = -j + \dim G\backslash U(V)$.*

From this theorem and basic commutative algebra we can conclude Quillen's theorem on the dimension of H^*_G, Duflot's lower bound for the depth of H^*_G, and Symonds' regularity theorem. This method of proof is different than the proofs of the theorems of Quillen and Duflot, but similar to Symonds' proof of the regularity theorem.

We also show that the layers of the Duflot filtration are good approximations to the local cohomology of $H^*_S M$ in the following sense.

Theorem 1.3.2 (2.5.1)**.** *Denote by F_i the i^{th} layer of the Duflot filtration for a smooth S-manifold M, where S is a p-torus. Then $F_i H^*_S M \to H^*_S M$ induces a surjection $\mathcal{H}^i F_i H^*_S M \to \mathcal{H}^i H^*_S M$, and an isomorphism $\mathcal{H}^j F_i H^*_S M \to \mathcal{H}^j H^*_S M$ for $j > i$.*

This lets us compute some of the local cohomology modules of $H^*_S M$ by studying higher layers of the Duflot filtration, which can be easier in a way that we will shortly describe.

1.3.2 Systematizing the Duflot filtration

This filtration puts a lot of structure on $H^*_S M$, and to clarify what structure is forced by the Duflot filtration we found it useful to systematize the Duflot filtration. We define a "free rank filtration", which is simply a filtration with subquotients of the form appearing in the Duflot filtration, and we show that any module with a free rank filtration has a Duflot chain complex and satisfies analogs of the dimension, depth, and regularity theorems, as well as Duflot's theorem on associated primes. The current application for this theory is to simplify the study of S-equivariant cohomology, but it would be very interesting to have more examples of algebras with free rank filtrations.

The Duflot filtration is a filtration by the natural numbers, but we show how it can be refined to be a filtration by a certain poset. This poset is the poset of connected components of M^A, as A ranges over all subtori of S, and it is equipped with a map to $\mathbb{N}^{op}$ by the rank of A. Precise definitions are given in sections 2.2 and 3.1. Studying this refinement of the Duflot filtration yields some restrictions on associated primes and a detection result for cohomology. It also lets us study some maps between S-manifolds in terms of the maps on their associated posets, which ultimately leads to new local cohomology computations.

1.3.3 Detection on subgroups and restrictions on associated primes

Our detection result is the following:

Theorem 1.3.3 (4.1.13)**.** *Let G be a compact Lie group, and let d be the depth of H^*_G. Then $H^*_G \to \prod_{E<G, \operatorname{rank} E = d} H^*_{C_G E}$ is injective.*

This result is due to Carlson [Car95] for finite groups. This result could also be derived using the techniques of Henn, Lannes, and Schwartz in [HLS95].

Our restriction on associated primes is the following. For finite groups Okuyama [Oku10] showed that one and three are equivalent, and the equivalence of two and three is a special case of a result of Kuhn [Kuh07] given below.

Theorem 1.3.4 (4.1.8)**.** *Let G be a compact Lie group and $E < G$ a p-torus. The following are equivalent:*

1. *E represents an associated prime in H^*_G.*
2. *E represents an associated prime in $H^*_{C_G E}$.*
3. *The depth of $H^*_{C_G E}$ is $\operatorname{rank} E$.*

This theorem is useful because it allows one to rule out, without computation, some p-tori from representing associated primes. For example if the Duflot bound for depth for $H^*_{C_G E}$ is bigger than $\operatorname{rank} E$, then E can't represent an associated prime in H^*_G. Our proof uses the Duflot filtration and goes through equivariant cohomology, and for particular groups where there is a good understanding of the relevant S-manifold, our methods could presumably give more restrictions on associated primes.

Carlson conjectured that for finite groups there is always an associated prime of dimension equal to the depth of H^*_G. Green [Gre03] and Kuhn [Kuh07, Kuh13] have shown that for p-groups and compact Lie groups respectively, this is true when the Duflot bound for depth is sharp. We give in 4.1.19 a different proof that also applies to compact Lie groups.

Theorem 1.3.5 (Kuhn)**.** *For G a compact Lie group,* $\operatorname{depth} H^*_G$ *is equal to the p-rank of the center of G if and only if the maximal central p-torus of G represents an associated prime.*

1.3.4 Local cohomology modules for p-Sylows of S_{p^n}

We also study the relationship between the posets mentioned above associated to manifolds M, N with $M \to N$ an equivariant principal K-bundle for K a finite group. In favorable situations there is an induced map between the posets that is an analog of a principal K-bundle in the category of posets. For certain group extensions that we call i-trivial (see section 4.2 for definitions), this leads to a spectral sequence:

Theorem 1.3.6 (4.2.11)**.** *If $1 \to H \to G \to K \to 1$ is i-trivial, then for any faithful representation $G \to U(V)$ there is a spectral sequence with $E_2^{p,q} \cong H^p(K, \mathcal{H}^q F_i H_S^* H \backslash U(V))$ converging to $\mathcal{H}^{p+q} F_i H_S^* G \backslash U(V)$.*

This leads to computations involving the top local cohomology modules of the group cohomology of the Sylow p-subgroups of S_{p^n}, which we denote by $W(n)$. In other word, if we let $W(1) = \mathbb{Z}/p$, then we can inductively define $W(n)$ by $W(n) = W(n-1) \wr \mathbb{Z}/p$. From the spectral sequence, we can compute the local cohomology of $W(n)$ in terms of the Tate cohomology of $W(n-1)$.

Theorem 1.3.7 (4.3.10)**.** *Let k be the dimension of a faithful representation $W(n) \to U(V)$, and let d be $-p^{n-1} + k$. For $0 < i < p-3$, we have that $\mathcal{H}^{p^{n-1}-(p-3)+i} H_S^*(W(n) \backslash U(V))$ is isomorphic as an $H_{W(n)}^*$-module to*

$$\widehat{H}^{i-(p-2)}(W(n-1), \Sigma^d (H_S^* E(n) \backslash U(V))^*).$$

For the top local cohomology, we have that $\mathcal{H}^{p^{n-1}}(H_S^* W(n) \backslash U(V))$ is isomorphic as an $\mathbb{F}_p$-vector space to

$$\widehat{H}^{-1}(W(n-1), \Sigma^d (H_S^* E(n) \backslash U(V))^*) \oplus N(\Sigma^d (H_S^* E(n) \backslash U(V))^*).$$

Here N is the norm map, and $\widehat{H}$ is Tate cohomology.

Local cohomology is known to vanish in degrees below the depth and above the dimension, and it is nonvanishing at degrees equal to the depth and dimension, and at any degree equal to the dimension of an associated prime.

The regularity theorem concerns the vanishing of $\mathcal{H}^{i,j}H_G^*$ for $j > -i$, but little is known about the vanishing and nonvanishing of the entire $\mathcal{H}^i(H_G^*)$ beyond what is known for general graded rings. In particular, it is not known if there is any finite group G with depth $H_G^* = d$ and $\dim H_G^* = r$ and some i with $d < i < r$ with $\mathcal{H}^i H_G^* = 0$. In other words, in the range where local cohomology can be either zero or nonzero, it is not known if there is a group where local cohomology is ever nonzero. Perhaps no such group exists. Using 1.3.7, we can show that:

Proposition 1.3.8 (4.3.11)**.** *For* $0 \leq i < p-3$, $\mathcal{H}^{p^{n-1}-i}(H_{W(n)}^*) \neq 0$.

In other words, there are arbitrarily long intervals where local cohomology is nonzero and where there is no a priori reason (in light of the fact noted above about associated primes) for local cohomology to be nonzero (the embedded primes for $H_{W(n)}^*$ are all of rank less than $p^{n-1} - (p-3)$).

Proposition 1.3.9 (4.3.12)**.** *For each* $p \geq 5$*, for each* n *there exists a* p*-group* G *and an* i *so that* $\mathcal{H}^{i+j}(H_G^*) \neq 0$ *for all* $0 < j < n$ *, and so that* $i+j$ *is not the dimension of an associated prime.*

Additionally, the strong form of Benson's regularity conjecture is still open. This conjecture states that $\mathcal{H}^{i,-i}(H_G^*) = 0$ for $i \neq \dim H_G^*$. It is known by the work of Symonds that this can only fail for i equal to the rank of a maximal p-torus in G, so it is already known that this conjecture is true for $W(n)$ in the range which we compute in 1.3.7. Nevertheless, it is interesting to observe that for the i in our range the largest j for which $\mathcal{H}^{i,j}H_{W(n)}^* \neq 0$ is smaller than predicted by the strong form of the regularity conjecture.

In fact, we have the following:

Corollary 1.3.10 (4.3.13)**.** *For* $0 < i < p-3$, $\mathcal{H}^{p^{n-1}-i,j}H_{W(n)}^* = 0$ *for* $j > -p^{n-1}$.

In other words, the strong regularity conjecture, which is confirmed for these groups in this range, tells us that $\mathcal{H}^{i,j}H_{W(n)}^*$ should be 0 for $j > -(i+1)$, but in fact we can show in this range the bound can be improved to $j > -p^{n-1}$, independent of i.

1.3.5 Stratification of derived categories of cochains on Borel constructions

One of the exciting connections between cohomology and representation theory is that, in the language of Benson, Iyengar, and Krause [BIK11b] for G a finite group H_G^* *stratifies* $\mathbf{Stmod}\,\mathbb{F}_pG$. We will define stratificaiton in Chapter 5. The connection to topology is that $\mathbf{Stmod}\,\mathbb{F}_pG$ is closely related to the derived category of C^*BG-modules. The latter is also stratified by H_G^* by the work of Benson and Greenlees [BG14] and Barthel, Castellana, Heard, and Valenzuela [BCHV17], even in the case when G is a compact Lie group and there is no stable module category.

The proof of Benson and Greenlees uses ideas from the Quillen stratification theorem and equivariant cohomology, and we show that their proof and some of the results of [BCHV17] together extend this stratification result to the derived category of cochains on the Borel construction of a G-space.

Theorem 1.3.11 (5.4.1)**.** *For X a finite G-CW complex, $D(C^*(EG \times_G X))$ is stratified by H_G^*X.*

1.4 Notation and conventions

Throughout, we fix a prime p.

We will denote by P_W the polynomial algebra $S(W^*)$, where W is a fixed i-dimensional $\mathbb{F}_p$-vector space. The grading on P_W is inherited from that on W, which is concentrated in degree -1 when $p = 2$ and in degree -2 otherwise. So, after choosing a basis for W, $P_W = \mathbb{F}_p[y_1, \ldots, y_i]$, where $|y_i|$ is either 1 or 2. For $V \subset W$, we denote by P_V the symmetric algebra $S(V^*)$, graded in the same manner as P_W. We have a P_W module structure on P_V induced by the inclusion $V \to W$.

All rings are graded commutative and all modules are graded, and Σ^d denotes the suspension functor. All homological algebra is to be done in the graded sense. See Appendix B for a discussion on the difference between commutative graded rings and graded commutative rings.

We use $\mathcal{H}^*_{\mathfrak{m}}$ to denote local cohomology with respect to $\mathfrak{m}$. All our local cohomology modules will be taken with respect to the maximal ideal of positive degree elements in an obvious algebra, so we will often omit $\mathfrak{m}$. For a treatment of local cohomology that includes the graded case, see [BS13]. We collect the results on local cohomology we need in Appendix A.

A p-torus is a group isomorphic to $(\mathbb{Z}/p)^r$ for some r, and if a p-torus E is isomorphic to $(\mathbb{Z}/p)^r$ we say that the rank of E is r.

For X a G-space, we write $H^*_G X$ for $H^*(EG \times_G X; \mathbb{F}_p)$, and write H^*_G for $H^*_G pt$, which is $H^*(BG; \mathbb{F}_p)$.

For V a complex unitary representation of G, we will use $U(V)$ to denote the group of unitary isomorphims of V, which is of course equipped with a map $G \to U(V)$. If V is equipped with a direct sum decomposition, i.e. $V \cong V_1 \oplus V_2$ as G-vector spaces, we will require that $U(-)$ respects this direction sum decomposition, so $U(V_1 \oplus V_2) = U(V_1) \times U(V_2)$.

This is so we can refer to a map $G \hookrightarrow U(V)$, where we more properly mean a map $G \hookrightarrow \prod_i U(n_i)$. Generally any faithful representation of G suffices for our theory, but there are a few points where we want to specify a representation V of G so that the center of G maps to the center of $U(V)$, which is only possible in general if $U(V)$ is allowed to denote a product of unitary groups.

Chapter 2

AN ABSTRACT TREATMENT OF MODULES WITH FREE RANK FILTRATIONS

In this chapter we describe the algebraic structure that is present on the S-equivariant cohomology of a smooth manifold, and explore this structure in depth. We then show that whenever this structure is present there are analogs of Quillen's theorem on dimension, Duflot's theorem on depth, Carlson's detection theorem, and Symonds' regularity theorem. In section 2.5 we describe how the structures studied earlier in the chapter behave with respect to certain classes of maps, which leads to a spectral sequence for local cohomology modules.

2.1 Modules with free rank filtrations

2.1.1 Structure on $H^*_S M$

Our initial goal is to develop a framework for studying the algebraic consequences of the Duflot filtration. To motivate the definitions used in this framework, we first recall some of the structure apparent in the Duflot filtration. We will return to this in much more detail in section 3.1.

Let M be a smooth S-manifold, where S is a p-torus. Then Duflot in [Duf83b] defines a filtration on $H^*_S M$ so that F_i/F_{i+1} is a sum of modules of the form $\Sigma^d(H^*_A \otimes H^*N)$, where N is a manifold and A is a rank i p-torus. This $H^*_A \otimes H^*N$ arises as the S-equivariant cohomology of a submanifold L of M on which A acts trivially and S/A acts freely, so it is an $H^*_S M$-module via the restriction map $H^*_S M \to H^*_S L \cong H^*_A \otimes H^*N$.

Because A acts trivially on L, there are S-equivariant maps:

$$\begin{array}{ccc} & & pt \\ & \nearrow & \uparrow \\ M & \longleftarrow & L \\ & \nwarrow & \uparrow \\ & & S/A \end{array}$$

. Taking S-equivariant cohomology gives:

$$\begin{array}{ccc} & & H_S^* \\ & \swarrow & \downarrow \\ H_S^*M & \longrightarrow & H_S^*L \\ & \searrow & \downarrow \\ & & H_A^* \end{array} \cdot$$

When $p = 2$, the top and bottom objects of this diagram are already polynomial rings, and when p is odd there is an inclusion of a polynomial ring P_S on rank S variable of degree two, and a map from H_A^* onto a polynomial ring on rank A variables in degree 2. For p odd, we can write H_S^*L as a polynomial ring on rank A variables tensored with the tensor product of H^*N and an exterior algebra on rank A variables, so as $P_A \otimes B$, where B is bounded as a graded ring. So, for both p odd and for p even we have a diagram:

$$\begin{array}{ccc} & & P_S \\ & \swarrow & \downarrow \\ H_S^*M & \longrightarrow & P_A \otimes B \\ & \searrow & \downarrow \\ & & P_A \end{array} \cdot$$

The map $P_S \to P_A$ is induced by the linear inclusion $A \to S$. For our description of these polynomial rings, recall that for $p = 2$ the cohomology of a 2-torus A is naturally $S(A^*)$ where A^* is concentrated in degree 1, and for p odd the cohomology of a p-torus B is naturally $S(B^*) \otimes S(\Sigma B^*)$, where B^* is in degree 1 and therefore ΣB^* is in degree 2 (and symmetric algebras are to be taken in the graded commutative sense). Note that this is compatible with our grading conventions.

Suspensions of modules of this form we will call "r-free" where r is the rank of A. We will now define this in a purely algebraic setting.

2.1.2 *Free rank filtrations*

Fix an $\mathbb{F}_p$ vector space W.

We are interested in Noetherian $\mathbb{F}_p$-algebras R that are also finite algebras over P_W, i.e. finitely generated $\mathbb{F}_p$-algebras R with a (graded) $\mathbb{F}_p$-algebra map $P_W \to R$ making R into a finitely generated P_W-module.

Throughout this section, R will be a fixed finite P_W-algebra.

Definition 2.1.1. An R-module M is called *j-free* if it is a suspension of a P_W-module which is a P_W-algebra of the form $P_V \otimes N$, where V is a j-dimensional subspace of W, N is a bounded connected $\mathbb{F}_p$-algebra, and M is an R-module via an algebra map $R \to P_V \otimes N$.

Moreover, in the commuting diagram of algebra maps:

$$\begin{array}{ccc} & & P_W \\ & \swarrow & \downarrow \\ R & \longrightarrow & P_V \otimes N \\ & \searrow & \downarrow \\ & & P_V \end{array}$$

we require that the map $P_W \to P_V$ is induced by the inclusion $V \to W$ (the map $P_V \otimes N \to P_V$ is the obvious one, induced by the identity $P_V \to P_V$ and the augmentation $N \to \mathbb{F}_p \to P_V$).

The name is indicating that as P_W-modules, j-free modules are in particular the pullback of a free P_V-module under the map $P_W \to P_V$.

Definition 2.1.2. A descending R-module filtration $0 = F_{i+1} \subset F_i \subset F_{i-1} \subset \cdots \subset F_0 = L$ of an R-module L is called a *free rank filtration* if F_j/F_{j+1} is a finite sum of j-free modules.

We will write F_j/F_{j+1} as $\bigoplus_{V\subset W, \dim V = j} \bigoplus_{k=0}^{l} \Sigma^{d_{V,k}}(P_{V,k} \otimes N_{V,k})$, and we denote $\Sigma^{d_{V,k}} P_{V,k} \otimes N_{V,k}$ as $M_{V,k}$. We will denote the map $R \to P_{V,k}$ as $\phi_{V,k}$, but whenever it is possible to do so without confusion we will omit the k in the subscript. We regard the direct sum decomposition of F_j/F_{j+1} as part of the data of a free rank filtration; the same j-dimensional subspace of W can occur more than once in this direct sum decomposition.

Definition 2.1.3. In the future we will want to add the condition that the kernels of the $\phi_V : R \to P_V$ are distinct; we will call a free rank filtration with this extra property *minimal*.

Lemma 2.1.4. *If an R-module L has a free rank filtration, then the Krull dimension of L/F_s is less than s. If R is in addition connected, and $F_{s+1} \neq F_s$, then* $\operatorname{depth} F_s \geq s$.

Remark 2.1.5. This is an analog of Quillen's theorem on dimension and the Duflot bound for depth, and we will later use this theorem to derive these theorems in group cohomology.

Proof. The proofs of both statements follow from examining induced filtrations on L/F_s and F_s.

For the first statement, consider the filtration $F_s/F_s \subset F_{s-1}/F_s \subset \dots F_0/F_s = L/F_s$. The subquotients in this filtration are $F_{s-1}/F_s, F_{s-2}/F_{s-1}$,
$\dots, F_0/F_1$. The Krull dimension of F_s/F_{s+1} is less than or equal to s (and in fact equal to i unless $F_s = F_{s+1}$ and $F_s/F_{s+1} = 0$), so as $\dim L/F_s = \max_{k \leq s} \dim F_{k-1}/F_k$, the result for L/F_s follows.

For the second statement, we consider the filtration of F_s: $F_i \subset F_{i-1} \subset \cdots \subset F_s$. The subquotients of the filtration are $F_{i-1}/F_i, \dots, F_s/F_{s+1}$. The subquotient F_i/F_{i+1} has depth i, so as the depth of a filtered module must be greater than or equal to the minimum depth of the subquotients, the depth must be greater than or equal to s. The assumption that R is connected is just because it is traditional to only define depth in the context of modules over local rings.

□

Corollary 2.1.6. *If R is connected and L has a free rank filtration, then the depth of L is greater than or equal to the smallest k such that $F_{k+1} \neq F_k$.*

Proof. This follows from Lemma 2.1.4, because for such a k, $F_k = F_0 = R$. □

Definition 2.1.7. A prime $\mathfrak{p}$ of an algebra R with a free rank filtration is called *toral* if it is the kernel of one of the maps $\phi : R \to P_V$ appearing in the free rank filtration.

Theorem 2.1.8. *If L has a free rank filtration, then every element of* $\operatorname{Ass}_R L$ *is toral.*

Proof. We have that $\operatorname{Ass}_R L \subset \bigcup_{V,k} \operatorname{Ass}_R M_{V,k}$, so it suffices to prove that each element of $\operatorname{Ass}_R M_V$ is toral. The map $\operatorname{Ass}_{M_V} M_V \to \operatorname{Ass}_R M_V$ is surjective.

But M_V is a domain tensored with a finite dimensional (as an $\mathbb{F}_p$-vector space) algebra, and any such algebra has a unique associated prime. So the only associated prime of M_V as a module over itself is the kernel of the map $M_V \to P_V$, and we have our result. □

Theorem 2.1.9. *If R is in addition a connected P_W algebra and L has a free rank filtration, then there is a cochain complex DL where*

$$DL^j = \bigoplus_{V \subset W, \dim V = j} \bigoplus_{k=0}^{l} \Sigma^{-\sigma_j}(\Sigma^{d_{V,k}}(P^*_{V,k} \otimes N_{V,k})),$$

where $\sigma_j = j$ when the characteristic is 2 and $\sigma_j = 2_j$ when the characteristic is odd, and $(-)^*$ denotes the graded linear dual, and this cochain complex has the property that $H^i(DL) = \mathcal{H}^i(L)$, the i^{th} local cohomology of L as an R module with respect to the maximal homogeneous ideal $\mathfrak{m}$ of positive degree elements.

We call this chain complex the Duflot complex of L, and this complex is functorial for module maps preserving the filtration.

Remark 2.1.10. In other words, the i^{th} term of the Duflot complex is a sum of shifts of modules of the form of the dual of polynomial ring in i variables, tensored with a bounded module.

Proof. We have a diagram:

$$
\begin{array}{ccc}
 & & L \\
 & & \downarrow \\
F_{i-1}/F_i & \longrightarrow & L/F_i \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
F_1/F_2 & \longrightarrow & L/F_2 \\
 & & \downarrow \\
F_0/F_1 & \longrightarrow & L/F_1 \\
 & & \downarrow \\
0 & \longrightarrow & L/F_0 = 0
\end{array}
$$

Here, going to the right and down is a short exact sequence. All maps are maps of R-modules, so we can apply the local cohomology functor to get an exact couple:

$$
\begin{array}{ccc}
 & & \mathcal{H}^* L \\
 & & \downarrow \\
\mathcal{H}^* F_{i-1}/F_i & \longrightarrow & \mathcal{H}^* L/F_i \\
 & \nwarrow & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
\mathcal{H}^* F_1/F_2 & \longrightarrow & \mathcal{H}^* L/F_2 \\
 & \nwarrow & \downarrow \\
\mathcal{H}^* F_0/F_1 & \longrightarrow & \mathcal{H}^* L/F_1 \\
 & \nwarrow & \downarrow \\
0 & \longrightarrow & \mathcal{H}^* L/F_0 = 0
\end{array}
$$

Here the dotted maps have cohomological degree 1, and are the boundary maps in the local cohomology long exact sequence. This exact couple gives a spectral sequence with $E_1^{p,q} = \mathcal{H}^{p+q} F_p/F_{p+1}$ converging to $\mathcal{H}^* L$.

However, $F_p/F_{p+1} = \bigoplus_{V \subset W, \dim V = p} \bigoplus_{k=0}^{l} \Sigma^{d_{V,k}} (P_{V,k} \otimes N_{V,k})$, so $\mathcal{H}^{p+q} F_p/F_{p+1}$ is a sum of modules of the form $\mathcal{H}^{p+q}(\Sigma^{d_{V,k}} P_V \otimes N_V)$ where $\dim V = p$, which we now compute.

We recall that the independence theorem for local cohomology, A.0.4, tells us that we can compute the local cohomology of $P_V \otimes N_V$ either over R or over P_V (this uses the fact that $R \to P_V$ is finite, that these are connected rings, and that local cohomology is radical invariant).

But by A.0.5 $\mathcal{H}^*(P_V \otimes N_V) = \mathcal{H}^* P_V \otimes \mathcal{H}^* N_V$.

So by A.0.6, our spectral sequence collapses to the bottom row at the E_1 page, and we have our result.

Functoriality follows from a map of filtered modules inducing a map on the exact couples giving the spectral sequence.

□

Definition 2.1.11. For R a local graded ring and L a module over R, let $a_i L = \sup_j\{\mathcal{H}^{i,j} L \neq 0\}$. Then recall that the Castelnuovo-Mumford regularity of L, or $\operatorname{reg} L$, is $\sup_i\{a_i L + i\}$.

Corollary 2.1.12. *Let $t(N)$ denote the top nonzero degree of a bounded module N. If L satisfies the hypotheses of the previous theorem, then $\operatorname{reg} L \leq \max\{t(N_{V,k}) + d_{V,k}\}$ when $p = 2$, and $\operatorname{reg} L \leq \max\{t(N_V) + d_{V,k} - \operatorname{rank}(V)\}$ when $p \neq 2$.*

Proof. This comes from examining the Duflot chain complex. Under the hypotheses of the theorem $DL^{i,j}$ is zero for $j > -\sigma_i + \max\{t(N_{V,k}) + d_{V,k}\}$, giving the result. □

Finally, we remark that this cochain complex is perhaps better understood via its dual. For a connected P_W algebra R, let $\mathfrak{d}$ denote the Matlis duality functor. In our setting this coincides with $\operatorname{Hom}_k(-, k)$, see [BS13] exercise 14.4.2.

Theorem 2.1.13. *For R a connected P_W algebra and L a module with a free rank filtration, taking the Matlis dual of DL gives a chain complex with*

$$\mathfrak{d}(DL^j) = \bigoplus_{V \subset W, \dim V = j} \bigoplus_{k=0}^{l} \Sigma^{\sigma_j}(\Sigma^{-d_{V,k}} P_{V,k} \otimes N^*_{V,k}),$$

and the homology of this chain complex is the Matlis dual of $\mathcal{H}^* L$.

In other words, the i^{th} term of the dual chain complex is a sum of shifts of polynomial rings tensored with bounded modules.

2.2 Modules with a stratified free rank filtration

2.2.1 Preliminary definitions and motivation from topology

In the application we are focused on, the filtration on $H^*_S M$ will be related to the fixed points of subgroups of S. In fact, these submanifolds refine the filtration and clarify how we can understand $H^*_S M$ by studying the different $H^*_S Y$, for Y a component of the fixed points of a subgroup of S. In this section we give an algebraic description of how a free rank filtration can be refined to a filtration by a poset.

First, we describe some of the algebraic structure present when considering embedded manifolds along with their Gysin and restriction maps.

Definition 2.2.1. We say that a P_W-algebra T is *embedded* in R with codimension d_T if there is a map of P_W algebras $R \to T$ and a map of R-modules $\Sigma^{d_T} T \to R$, so that the composition $\Sigma^{d_T} \to R \to T$ is a map of T modules. We call the map $i_* : \Sigma^{d_T} T \to R$ the *pushforward*, and the map $i^* : R \to T$ *restriction*. We define the *Euler class* of T, or e_T, to be $i^* i_* 1$. We will also refer to i_* as tr or *transfer* or also as the Gysin map, and to i^* as *restriciton*, res.

Note that if T is embedded in R, the composition of pushforward followed by restriction is multiplication by e_T.

The terminology comes from topology, if $X \to Y$ is an embedding of smooth manifolds where X has codimension d, then H^*X is embedded in H^*Y with codimension d. This is also true equivariantly.

We are only interested in the case of embedded P_W-algebra in the case that the Euler class is not a zero divisor; we give a name to this class of algebras.

Definition 2.2.2. If T is embedded in R and e_T is a nonzero divisor, then we say that T is *fixed* in R.

The terminology comes from equivariant topology, if M is a smooth S-manifold, for S

a p-torus, and Y is a connected component of the fixed point set of a sub-torus of S, then $H^*_S Y$ is fixed in $H^*_S M$.

Note that if T is fixed in R, then the pushforward $\Sigma^{d_T} T \to R$ is injective.

2.2.2 Filtrations by posets

The free rank filtrations that occur in topology have refinements to filtrations by posets. Here we discuss some features of the posets we are interested in.

Let P be a poset weakly coranked by the natural numbers, i.e. equipped with a map of posets $r : P \to \mathbb{N}^{op}$.

Given such a poset, there are a few related posets: $P_{\geq j}$ is the subposet of elements of rank greater than or equal to j, and $P_{<Y}$ for $Y \in P$ is the subposet of elements less than Y, and similary $P_{\leq Y}$, $P_{>j}$, etc. We'll use P_j for the set of elements of rank j.

A filtration of a module L by such a poset P is a submodule $F_X(L)$ (or just F_X if L is understood) for each $X \in P$, and where if $X \leq Y$, $F_X \subset F_Y$, and so that $\sum_{X \in P} F_X = L$.

We want to think of such a filtration as a functor from P to the category of R-modules N equipped with an injection $N \to L$ (where L is the module being filtered), so from now on we write $F(X)$ for $F_X L$.

We will continually be referring to the lowest subquotient of $F(X)$ in the filtration on $F(X)$ by $P_{\leq X}$, so if $r(X) = j$, let $\mathrm{gr}_j F(X) = F(X)/\sum_{Y \in P_{<X}} F(Y)$.

Given a module filtered by P, there is an associated filtration by $\mathbb{N}$, where $F_j = \sum_{X \in P_{\geq j}} F(X)$.

Definition 2.2.3. The filtration of M by P is called *good* if for all j the map $\oplus_{Y \in P_j} \mathrm{gr}_j F(Y) \to F_j/F_{j+1}$ is an isomorphism.

All the filtrations we are interested in are good, so the associated graded can be computed one element of the poset at a time. The point is that a filtration by a weakly coranked poset gives a filtration by $\mathbb{N}$, so there are two associated graded: one indexed by $\mathbb{N}$ and one indexed by the poset. When the filtration is good, these associated gradeds agree.

Remark 2.2.4. For a trivial example where the filtration is not good, consider a filtration on $\mathbb{Z}$ by the poset $a \leftarrow b \rightarrow c$, where b has rank 1 and a, c have rank 0, and filter by $f(b) = 0$ and $f(a) = f(c) = \mathbb{Z}$. Then the associated filtration by $\mathbb{N}$ just has $F_0 = \mathbb{Z}$ and $F_1 = 0$, and we see that the filtration is not good, since $f(a)/f(b) \oplus f(c)/f(b) = \mathbb{Z} \oplus \mathbb{Z}$, which of course is not F_0/F_1.

There is a potential for confusion between the filtration by P and the induced filtration by $\mathbb{N}$, so we will use letters like X, Y for elements of P and i, j, l for natural numbers. We will use capital letters for elements of P and natural numbers will always be lower case.

Note that if the filtration of L by P is good, then for each $X \in P$, $F(X)$ has a good filtration by $P_{\leq X}$. The fact that there is a filtration is automatic; to see that it is good consider the following commutative diagram.

$$\begin{array}{ccc} \bigoplus_{Y \in P_j} \mathrm{gr}_j F(Y) & \longleftarrow & \bigoplus_{Z \in (P_{\leq X})_j} \mathrm{gr}_j F(Z) \\ \downarrow & & \downarrow \\ F_j/F_{j+1} L & \longleftarrow & F_j/F_{j+1} F(X) \end{array}$$

The right hand arrow is always surjective (the map that is required to be an isomorphism in a good filtration is always surjective), so because the left hand arrow is injective, the righthand arrow must be injective also, and therefore an isomorphism.

2.2.3 Stratified rank filtrations

Definition 2.2.5. An R-module L has a *free rank filtration stratified by a weakly coranked finite poset* P if there is a good filtration of L by P such that for all j each $\mathrm{gr}_j F(Y)$ is j-free.

Note that the induced filtration by $\mathbb{N}$ of such a stratified free rank filtration is a free rank filtration, where the decomposition of F_j/F_{j+1} into a sum of j-free modules is induced by $F_j/F_{j+1} = \oplus_{Y \in P_j} \mathrm{gr}_j F(Y)$. Also note that for each $Y \in P$, $F(Y)$ has a stratified free rank filtration by $P_{\leq Y}$. The condition that our poset is finite ensures that our associated filtration by $\mathbb{N}$ is finite.

Definition 2.2.6. If L is an ideal of R, a free rank filtration stratified by a finite poset P is called a *topological* filtration stratified by P if the following are satisfied:

1. For each $Y \in P$, $F(Y) = i_*(\Sigma^{d_T} T)$ for an embedded, fixed T. We use the same symbol for $T \in P$ and the embedded T corresponding to $F(T)$.

2. For $T \in P$, the filtration on R induces on $F(T) = \Sigma^{d_T} T$ a free rank filtration (of T-modules) stratified by $P_{\leq T}$ on T, and the structure maps $T \to P_V$ for the various j-free modules occurring as subquotients give us a commuting diagram:

$$\begin{array}{ccc} R & & \\ \downarrow & \searrow & \\ T & \longrightarrow & P_V \end{array} .$$

3. For all $U < T$ in P with corresponding embedded U and T in R, U is also embedded in T, and the composition of the two transfers $\Sigma^{d_{T,R}}(\Sigma^{d_{U,T}} U \to T) \to R$ is the transfer of $\Sigma^{d_{U,R}} U \to R$.

In the third point, the notation $d_{U,T}$ refers to the codimension of U in T. Condition three implies that the codimension of U in T plus the codimension of T in R is the codimension of U in R. Note that if $L \subset R$ has a free rank filtration that is topologically stratified by P, then each embedded T appearing in the stratification for L has a free rank filtration topologically stratified by $P_{\leq T}$.

Here we record some useful properties of such filtrations.

Proposition 2.2.7. *If L has a topological filtration stratified by P, then the following hold for all $T \in P$. We use the same symbol F for the functor appearing in the filtration of L by P and for the functor appearing in the filtration of the embedded T by $P_{\leq T}$.*

1. *For $j = r(T)$ we have that $F_j T = F_0 T = T$, so $F_l/F_{l+1} T = 0$ for $l < j$.*

2. *Each $F_l/F_{l+1}\Sigma^{d_T} T \to F_l/F_{l+1} L$ is the inclusion of direct summands.*

3. *For each M_V appearing in the decomposition of $F_j/F_{j+1}L$ as j-free modules (so $\dim V = j$), there is a unique $T \in P_j$ such that $\mathrm{gr}_j F(T)$ maps isomorphically onto M_V.*

Proof. 1. Recall that $F_l T = \sum_{X \in (P_{\leq T})_{\geq l}} F(X)$. So if $l \leq j$, then T occurs in this sum and $F_l T = T$.

2. For $T \in P$, we consider $F_l/F_{l+1}T$, which because the original filtration is good, is $\oplus_{U \in (P_{\leq y})_l} \mathrm{gr}_l F(U)$. But each such U is also embedded in R, so on each component of $F_l/F_{l+1}T$ our map is part of the composition $\Sigma^{d_Y}(F_l/F_{l+1}\Sigma^{d_{U,Y}}U \to F_l/F_{l+1}Y) \to F_l/F_{l+1}L$, giving our result.

3. This follows from the fact that the filtration is good.

□

Note that the stratification condition implies that the coproduct of all the $\Sigma^{d_T}T$ surjects onto L, because this map is surjective on associated gradeds. Actually, in many examples $L = R$, and R itself will be one of the Ts appearing in the topological stratification, corresponding to the maximal element of P.

However even in this situation when R is one of the Ts, we can restrict to those elements of the poset that have rank equal to l, then by the definition of the associated filtration on $\mathbb{N}$ the coproduct of all of these (with the appropriate shifts according to the codimension) surjects onto $F_l(R)$. In fact, we can look at the subposet $P_{\geq j}$, and we can compute F_j as a colimit over this poset.

Lemma 2.2.8. *Suppose L has a topological filtration stratified by P and that Y, Z are embedded rings appearing in the stratification. Then, if x is in the image of the transfer from Y and the transfer from Z, there are W_ts embedded in R, Y and Z as part of each stratification, and $w_t \in W_t$ so that $x = \Sigma_t tr_{W_t} w_t$, and so that for each t there is a commuting diagram:*

$$\begin{array}{ccccc} \Sigma^{d_{W_t}} W_t & \longrightarrow & \Sigma^{d_Y} Y & \longrightarrow & L \\ & \searrow & & \nearrow & \\ & & \Sigma^{d_Z} Z & & \end{array} .$$

Note: The subscript on W_t is just an indexing, it has nothing to do with the various filtrations.

Proof. The proof is by downward induction on the filtration degree. First, consider the highest degree filtration that is potentially nonzero, F_iL, so suppose that $x \in F_iL$ is in the image of transfer from some Y and from some Z, so there exist $y \in \Sigma^{d_Y}Y, z \in \Sigma^{d_Z}Z$ with $tr_Y(y) = tr_Z(z) = x$. Then $x \in F_i/F_{i+1}L = F_iL$ is equal to $x_1 + x_2 + \ldots x_k$, where $x_j \in \mathrm{gr}_i F(W_j)$,. Similarly, $y = y_1 + \cdots + y_k$, $z = z_1 + \cdots + z_k$, and each y_j, z_j maps to x_j.

Then, because Y and Z are stratified, for each j there is a $w_j \in \Sigma^{d_{W_j}}W_j$ so that W_j is embedded in Y and Z and so that under the transfer from W_j to Y, $w_j \mapsto y_j$, and under the transfer from W_j to Z, $w_j \mapsto z_j$. That there is some W_j with this property for Y and Z separately follows because Y and Z are stratified, but there is a common W_j with this property because there is a unique W_j appearing in the stratification for R mapping to the summand supporting x_j. This also explains why the following diagram commutes:

$$\begin{array}{ccccc} \Sigma^{d_{W_j}}W_j & \longrightarrow & \Sigma^{d_Y}Y & \longrightarrow & L \\ & \searrow & & \nearrow & \\ & & \Sigma^{d_Z}Z & & \end{array} .$$

Now, suppose the result is true for $x \in F_lL$ with $l > j$, and consider $tr_Y(y) = tr_Z(z) = x$, where $x \in F_jL - F_{j+1}L$. Then $\overline{x} \in F_jL/F_{j+1}L$ is equal to $x_1 + \cdots + x_k$, where $x_l \in \mathrm{gr}_j Y_{V_{j,l}}$. By similar logic as in the preceding paragraphs, the images of y, z in $F_jY/F_{j+1}Y$, $F_jZ/F_{j+1}Z$ can be written as a sum $\overline{y} = y_1 + \cdots + y_k$, $\overline{z} = z_1 + \cdots + z_k$, where $y_l, z_l \mapsto x_l$. As X, Y, Z are compatibly stratified, for each l there is some $w_l \in \Sigma^{d_{W_l}}W_l$ where W_l is embedded in Y, Z, R so that the transfer of w_l hits y_l, z_l, x_l and so that the following diagram commutes:

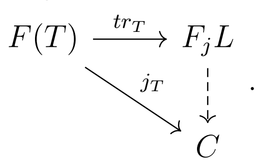
$$\begin{array}{ccccc} \Sigma^{d_{W_l}}W_l & \longrightarrow & \Sigma^{d_Y}Y & \longrightarrow & L \\ & \searrow & & \nearrow & \\ & & \Sigma^{d_Z}Z & & \end{array} .$$

But then $tr_Y(y - \sum_l tr_{W_l}w_l) = tr_Z(y - \sum_l tr_{W_l}w_l) = x - \sum_l tr_{W_l}$. Additionally, $x - \sum_l tr_{W_l} \in F_{j+1}L$, where our result is already assumed to be true, so we are done by induction.

□

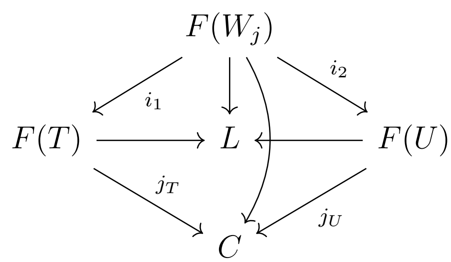
Proposition 2.2.9. *If L has a topological filtration stratified by P, then* $\varinjlim_{T \in P_{\geq j}} F(T) = F_j L$.

Proof. We show that $F_j L$ satisfies the required universal property. First, there are compatible maps $F(T) \to F_j L$ for $T \in P_{\geq j}$. Now, suppose that there are compatible maps $F(T) \to C$, where C is some other R-module. We need to fill in the dashed arrow in the diagram:



Because each map $F(T) \to F_j L$ is injective and because $\bigoplus_{T \in P_{\geq j}} F(T) \to F_j L$ is surjective by the definition of our filtration by $\mathbb{N}$ on L, there is at most one map from $F_j L \to C$ making the diagram commute. We need to show that this map is well defined.

In other words, if $tr_T(t) = tr_U(u)$, we need to show that $j_T(t) = j_U(u)$. But this follows from Lemma 2.2.8, because we will have a commuting diagram



and $w_j \in F(W_j)$ so that $t = \sum i_1(w_j)$, $u = \sum i_2(u_j)$. Then we see that $j_T(t) = j_U(u) = \sum j_{W_j}(w_j)$, which completes the proof. □

There is one final very strong condition we can add to a free rank filtration topologically stratified by P.

Definition 2.2.10. An R-module with a topological filtration stratified by P is called *Duflot* if for all $T \in P$, the corresponding embedded T is isomorphic as P_W algebras to $P_V \otimes T'$ where T' is a P_W algebra, and where $\dim V = r(T)$, and $T/\sum_{X \in P_{<T}} X$ is $r(T)$-free and a suspension of a module of the form $P_V \otimes N$, and $T \to T/\Sigma_{X \in P_{<T}} X \cong \Sigma^d(P_V \otimes N)$ is induced by a linear map $T' \to \Sigma^d N$.

For $X \in P$ as mentioned in 2.2.6 we just write X for $F(X)$, so $T/\sum_{X \in P_{<T}} X$ means $F(T)/\sum_{X \in P_{<T}} F(X)$.

We will refer to "Duflot algebras" and "Duflot modules" in the sequel: R is a *Duflot algebra* when R itself has the structure of a Duflot module.

2.3 An example

We will see how to extract examples of Duflot algebras from any compact Lie group G and representation $G \hookrightarrow U(n)$ in Chapter 3. Here we work out one example that is not completely trivial, but is simple enough that everything can be written down very explicitly.

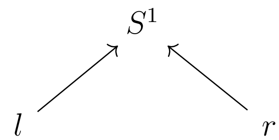
The example comes from the C_2-equivariant cohomology of S^1 with the reflection action. This C_2-manifold is described as the unit sphere in the sum of the trivial representation and the sign representation, or it can be given an explicit C_2-CW complex structure where there are two zero cells both with isotropy C_2, which we call l and r, and a single one cell with isotropy e. The attaching map $C_2 \times S^0 \to \{l, r\}$ is given by $(e, -1) \mapsto l$ and $(e, 1) \mapsto r$. Here we are thinking of S^0 as $\{\pm 1\}$.

We have that $H^*_{C_2} S^1 = \mathbb{F}_2[x, y]/(xy)$. We define a filtration on $H^*_{C_2} S^1$ by $F_2 = 0$, $F_1 = H^{>0}_{C_2} S_1$ and $F_0 = H^*_{C_2} S^1$. We can compute directly that this is a free rank filtration: $F_1/F_2 = F_1 \cong \Sigma^1 \mathbb{F}_2[t_1] \oplus \Sigma^1 \mathbb{F}_2[t_2]$ where in the first summand we map 1 to x and the second we map 1 to y, and F_0/F_1 is just $\mathbb{F}_2$.

In the Duflot complex DR for $R = H^*_{C_2} S^1$, we have that DR^1 is $\mathbb{F}_2$ concentrated in degree 0, and DR_2 by the computation A.0.6 is $(\mathbb{F}_2[t_1])^* \oplus (\mathbb{F}_2[t_2])^*$ (the shift down from A.0.6 and the shift up in F_1 cancel out). If we label the generators in degree 0 for the two summands of DR^1 e_1 and e_2, we have that the single differential in the Duflot complex takes the generator for DR^0 to $e_1 + e_2$. Consequently, upon taking cohomology we get that $\mathcal{H}^*(H^*_{C_2} S^1) = (H_{C_2} S^1)^*$.

We can read off that the regularity, depth, and dimension are all one. Note that this is consistent with Theorem 2.1.12: for each layer in the free rank filtration the bounded module N_V is just $\mathbb{F}_2$, and the shift d_V is 0 for the rank 0 part and 1 for each other the rank 1 pieces.

We can also observe that this filtration is topologically stratified by the following poset.



The rank of S^1 is 0 and the rank of l and r are 1.

We define $F(S^1) = H^*_{C_2}S^1$, $F(l) = (x)$, and $F(r) = (y)$. Note that $F(l)$ and $F(r)$ are the image of the Gysin map induced by $l \hookrightarrow S^1$ and $r \hookrightarrow S^1$. Note that $F(l)$ and $F(r)$ are each embedded; we have two different embeddings $\Sigma^1\mathbb{F}_2[t] \to H^*_{C_2}S^1 \to \mathbb{F}_2[t]$

This gives $H^*_{C_2}S^1$ the structure of a Duflot algebra.

2.4 Results on associated primes and detection for Duflot modules

Theorem 2.4.1. *If L has a minimal topological filtration stratified by P, then $\phi_{V,k}$ represents an associated prime in $\operatorname{Ass}_R L$ if and only if $\phi_{V,k}$ represents an associated prime in the embedded T corresponding to ϕ_V, as a module over itself.*

Proof. Let us unpack what this means. Here we suppress the subscripts on our vector spaces for clarity. Let $\phi_V : R \to P_V$ be one of the structure maps appearing the stratified free rank filtration for L. Associated to this ϕ_V, there is a rank V-free M_V appearing as a direct summand of $F_{\operatorname{rank} V}/F_{\operatorname{rank} V+1}L$. By the stratification hypothesis there is a unique $T \in P$ whose highest filtered subquotient hits M_V. So, we have a triangle $\begin{array}{ccc} R & \xrightarrow{\phi_V} & P_V \\ \big\downarrow & \nearrow_{\phi_V} & \\ T & & \end{array}$. The vertical map is the restriction map, and the maps going to the right are the two different structure maps, both of which we denote by ϕ_V.

First, note that if ϕ_V represents an associated prime in $\operatorname{Ass}_T T$, then considering the surjective map $\operatorname{Ass}_T T \to \operatorname{Ass}_R T$, $\ker \phi_V : T \to P_V$ is also an associated prime of $\operatorname{Ass}_R T$. But then it is also an associated prime of $\operatorname{Ass}_R \Sigma^{d_T} T$, and because there is an injective map $\Sigma^{d_T} T \to L$, it is an associated prime of $\operatorname{Ass}_R L$.

Conversely, we consider the short exact sequence $\Sigma^{d_T} T \to L \to \text{coker}$. Any associated prime of L as an R module must be in $\text{Ass}_R \Sigma^{d_T} T$ or $\text{Ass}_R \text{coker}$. We see that coker has a free rank filtration induced from those on T and L, and moreover ϕ_V doesn't appear as one of the structure maps in the free rank filtration for coker, as the filtered subquotients for coker are just those for L modulo those for $\Sigma^{d_T} T$, and our filtration is minimal. So, if ϕ_V represents an associated prime in R, it must represent one in T as well.

□

In the above proof we crucially use the minimality of the free rank filtration, which tells us that we can check every potential associated prime on a unique T. However, we don't use the full strength of our embedding hypothesis: we just use that the transfer is injective, not that the composition of transfer and restriction is a map of T-modules.

Also note that under the hypotheses of the above theorem, by Lemma 2.1.4 the depth of T is greater than or equal to $\dim V = i$, the lowest nonzero filtration degree. But $\ker \phi_V : T \to P_V$ is i dimensional, so if $\ker \phi_V$ is associated in T, it is the smallest dimensional associated prime in T and $\operatorname{depth} T = i$, as the dimension of an associated prime cannot be less than the depth.

Under the additional assumption that the filtration is Duflot, we can conclude the converse of this statement.

Theorem 2.4.2. *If L is a Duflot module, then a ϕ_V occurring in the filtration represents an associated prime in R if and only if the embedded T corresponding to ϕ_V has depth equal to $\dim V$.*

Proof. We saw above that ϕ_V represents an associated prime in L if and only if ϕ_V represents an associated prime in T, and that if ϕ_V represented an associated prime in T, then the depth of T was $\dim V$.

For the converse, if $\operatorname{depth} T = \dim V$, then as $T = P_V \otimes T'$, by the Künneth theorem for local cohomology the depth of T' as an module over itself must be zero, so there is some element $x \in T'$ annihilated by all of T, therefor $\text{ann}_T\, 1 \otimes x = \ker(T \to P_V)$ is in $\text{Ass}_T T$. □

Proposition 2.4.3. *Suppose R is Duflot algebra (so it has the required filtration as a module over itself). If* $\operatorname{depth} R = d$*, then* $R \xrightarrow{\prod res_T} \prod_{T \in P_d} T$ *is injective.*

We say that R is *detected* on such T

Proof. Suppose for a contradiction that x is a nonzero element in the kernel of $R \xrightarrow{\prod res_T} \prod_{T \in P_d} T$, so x restricts to zero on each T. We claim that x is annihilated by F_d. To see this, it is enough to show that x is annihilated by the image of each $\Sigma^{d_T} T$ with $r(T) = d$. So, suppose $i_*(t) \in \text{image}(\Sigma^{d_T} T \to R)$, and consider $x i_*(t)$. But, because i_* is a map of modules, this is $i_* i^* x t$, and therefore 0 since $i^*(x) = 0$.

Therefore x is annihilated by F_d, so F_d consists entirely of zero divisors, and is consequently contained in the union of all the associated primes of R, and therefore in one of the associated primes of R by prime avoidance, so we have $F_d \subset \mathfrak{p}$, where $\mathfrak{p}$ is associated. But by 2.1.4 $\dim R/F_d < d$. Therefore $\dim R/\mathfrak{p} < d$, and the depth of R is less than d, a contradiction.

Here we are using the fact that the depth of a ring gives a lower bound for the dimension of an associated prime. □

2.5 K Duflot modules and maps of Duflot modules

Sometimes, given an extension $1 \to H \to G \to K \to 1$ and a representation $G \hookrightarrow U(n)$, the associated bundle $K \to H\backslash U(n) \to G\backslash U(n)$ gives a close relationship between the poset controlling $H^*_S H\backslash U(n)$ and $H^*_S G\backslash U(n)$, perhaps after restricting to some higher level of the filtrations. Here we describe the algebraic structure this puts on the associated Duflot filtrations.

The main point of this section is Theorem 2.5.16. In order to make sense of this theorem we have to study how the map $F_i H^*_S G\backslash U(n) \to F_i H^*_S H\backslash U(n)$ mentioned above is controlled by a map between the stratifying posets and a natural transformation between the functors appearing in the filtration, and we need to similarly understand how the K-module structure on $H_S^*G\backslash U(n)$ is determined by an action of K on the stratifying poset that is compatible with the filtering functor.

Before we do this, we note that most of the pleasant algebraic properties of a Duflot algebra R are shared by each F_iR.

Lemma 2.5.1. *If R is a P_W-algebra that is Duflot as a module over itself with filtration stratified by P, then for all i:*

1. *F_iR is a Duflot module, with filtering poset $P_{\geq i}$.*

2. $\varinjlim_{X \in P_{\geq i}} F(X) = F_iR$

3. *The map $F_iR \to R$ induces an isomorphism on Duflot complexes in degree greater than or equal to i, and consequently an isomorphism on local cohomology in degree greater than i and a surjection in degree i.*

In this section, we will be focusing on these F_iR.

Definition 2.5.2. We will write a Duflot module as (L, P, F), where L is the module, P is the filtering poset, and F is the functor $P \to P_W$-mod realizing the filtration.

Note: The functor F actually takes values in R-mod. However, we will want to consider maps between Duflot modules that come from different Duflot algebras, so we forget to P_W-mod.

2.5.1 Morphisms of Duflot modules

Definition 2.5.3. A map $P \xrightarrow{\pi} Q$ of posets is a covering map if over every chain $C \subset Q$, there is a commutative diagram of posets
$$\begin{array}{ccc} \pi^{-1}C & \xrightarrow{\sim} & C \times n \\ \downarrow & \swarrow & \\ C & & \end{array}.$$

By n we mean the poset with n objects and only identity arrows, in other words $C \times n$ is just n copies of C.

Definition 2.5.4. A morphism of Duflot modules $(L, P, F) \xrightarrow{\pi} (N, Q, G)$ consists of the following data:

1. A covering map of posets $\pi : P \to Q$
2. A natural transformation $G\pi \to F$.

Lemma 2.5.5. *A morphism $\pi : (L, P, F) \to (N, Q, G)$ induces a P_W-module map $\pi^* : N \to L$ which is uniquely characterized by: for all $Y \in Q$, the diagram:*

$$\begin{array}{ccc} G(Y) & \longrightarrow & N \\ \downarrow & & \downarrow \\ \prod_{X \in \pi^{-1}Y} F(X) & \longrightarrow & L \end{array}$$

commutes.

Proof. We show how this information defines a map, and then uniqueness follows from the universal property of colimits.

Recall that $L = \varinjlim_{X \in P} F(X)$ and that $N = \varinjlim_{Y \in Q} G(Y)$. We define compatible maps $G(Y) \to L$ by $G(Y) \to \prod_{X \in \pi^{-1}Y} F(X) \to L$, where the left hand arrow is the product of the maps $G(\pi(X)) \to F(X)$ appearing in the natural transformation, and the right hand arrow is the sum of the inclusions.

To see that this gives a map from N we must show that if we have a map $Y \to Y'$ in Q, then the diagram:

$$\begin{array}{ccc} G(Y) & \longrightarrow & G(Y') \\ \downarrow & & \downarrow \\ \prod_{X \in \pi^{-1}Y} F(X) & \dashrightarrow & \prod_{X' \in \pi^{-1}Y'} F(X') \\ \downarrow & \swarrow & \\ N & & \end{array}$$

commutes, and this follows because local triviality ensures the existence of the dotted arrow. □

Proposition 2.5.6. *If $\pi : (L, P, F) \to (N, Q, G)$ be a morphism of Duflot modules, then the map on associated gradeds is a follows. Recall that $F_j/F_{j+1}L = \oplus_{X \in P_j} \mathrm{gr}_j F(X)$, and that $F_j/F_{j+1}N = \oplus_{W \in Q_j} \mathrm{gr}_j G(W)$. The map $\mathrm{gr}_j G(W) \to \mathrm{gr}_j F(X)$ is the one induced by the map $G(W) \to F(X)$ if $\pi(X) = W$ (the map appearing in the natural transformation), and zero if $W \neq \pi(X)$*

Proof. To determine $\mathrm{gr}_j G(W) \to L$, we must study the composition:
$$\begin{array}{ccc} & & L \\ & & \uparrow \\ G(W) & \longrightarrow & N \end{array}.$$

This map factors as:
$$\begin{array}{ccc} \prod_{X\in\pi^{-1}W} F(X) & \longrightarrow & L \\ \uparrow & & \uparrow \\ G(W) & \longrightarrow & N \end{array}$$
, so the map is as claimed when $\pi(X) = W$.

For the other case, because the map factors as in the above diagram, if $x \in \mathrm{image}(G(W)) \cap F(Y)$ with $\pi_*(Y) \neq W$, then x is in filtration degree greater than j, so we are done. □

2.5.2 K-Duflot modules

Definition 2.5.7. For K a finite group, a *K-Duflot module* is a Duflot module L with a left action of K on P by poset maps along with for all $k \in K$ natural isomorphisms $Fk \Rightarrow F$ so that for all $X \in P$, the induced map $Fk(X) \to F(X)$ is an isomorphism satisfying the following axioms:

1. $F \circ id \Rightarrow F$ is the identify

2. The induced natural isomorphisms $Fhk \Rightarrow Fk$ satisfy associativity, in the sense that for all $l, h, k \in K$,
$$\begin{array}{ccc} Fk(hl) & \Longrightarrow & Fhl \\ \Downarrow & & \Downarrow \\ F(kh)l & \Longrightarrow & Fl \end{array}$$
commutes.

Here the left vertical arrow is the identity, the functors $k(hl)$ and $(kh)l$ are equal as functors on P.

Note: In addition to the natural isomorphism $Fk \Rightarrow F$, we have a natural isomorphism $F \Rightarrow Fk$, obtained from k^{-1}, which we will use.

To put this more explicitly, we have an action of K on P, and for a morphism $X \to Y$ in P, we have a diagram:
$$\begin{array}{ccc} F(X) & \xleftarrow[k^*]{} & F(kX) \\ \downarrow & & \downarrow \\ F(Y) & \xleftarrow[k^*]{} & F(kY) \end{array}$$

Lemma 2.5.8. *A K-Duflot module L has a P_W-linear right K action, uniquely characterized by: for all $k \in K$, the diagram:*

$$\begin{array}{ccc} F(X) & \longrightarrow & L \\ \downarrow{\scriptstyle k^*} & & \downarrow{\scriptstyle k^*} \\ F(k^{-1}X) & \longrightarrow & L \end{array}$$

commutes.

Proof. We show how this data defines an action on L, and then uniqueness follows from the universal property of colimits.

For $k \in K$, we define a map $k^* : L \to L$ as follows. For all $X \in P$, we need a map $F(X) \to L$ commuting with the maps $F(X) \to F(Y)$ defining our colimit. For $X \in P$, we define $F(X) \to L$ by the map $F(X) \xrightarrow{k^*} F(k^{-1}X) \to L$, where the second map is the natural map $F(k^{-1}X) \to L$. That this defines a system of compatible maps follows from the commutativty of the diagram:

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow{\scriptstyle k*} & & \downarrow{\scriptstyle k^*} \\ F(k^{-1}X) & \longrightarrow & F(k^{-1}Y) \end{array}$$

. That it defines a group action follows from the conditions on the natural transformations, namely that multiplication by the identity acts as the identify and that the composition $F(hkx) \xrightarrow{h^*} F(kx) \xrightarrow{k^*} F(x)$ is the composition $F((hk)x) \xrightarrow{hk^*} F(x)$, i.e. that $(hk)^* = k^*h^*$. □

Proposition 2.5.9. *The action of K on L induces an action of K on the associated graded: on each summand the map $k^* : \mathrm{gr}_j F(kX) \to \mathrm{gr}_j F(X)$ is the isomorphism induced by the isomorphism $F(kX) \to F(X)$, and the image of $\mathrm{gr}_j F(kX)$ in the other summands is zero.*

Proof. This follows because the action by k on $F(kX)$ fits into the diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & L \\ {\scriptstyle k^*}\uparrow & & {\scriptstyle k*}\uparrow \\ F(kX) & \longrightarrow & L \end{array}$$

. This shows that the map is as claimed on these summands, and if $k^*(F(kX))$ intersects any other $F(Y)$ it is in higher filtration degree, so is in $\sum_{Z<Y} F(Z)$. □

Definition 2.5.10. For K-Duflot modules L, N, a morphism $\pi : (L, P, F) \to (N, Q, G)$ is *equivariant* if:

1. $\pi_* : P \to Q$ is equivariant

2. For all $k \in K$, there is a commuting diagram $\begin{array}{ccc} G\pi & \Longrightarrow & F \\ \Uparrow & & \Uparrow \\ Gk\pi & \Longrightarrow & Fk \end{array}$. In this diagram, the vertical natural transformation and the upper horizontal one are stipulated in the definitions of a K-Duflot module and a morphism, and the bottom one is induced from these.

Lemma 2.5.11. *The unique map of 2.5.5 induced by an equivariant morphism is an equivariant map* $\pi^* : N \to M$*.*

Proof. This follows from the commutativity of diagrams of the form:

$$\begin{array}{ccc} G(Y) & \longrightarrow & \prod_{Y\in\pi^{-1}Y} F(X) \\ \big\downarrow k^* & & \big\downarrow k^* \\ G(k^{-1}Y) & \longrightarrow & \prod_{Z\in\pi^{-1}k^{-1}Y} F(Z) \end{array}$$

, because the lower right hand object is also $\prod_{X\in\pi^{-1}Y} F(k^{-1}X)$.

□

Definition 2.5.12. A covering map of posets $P \to Q$ is a *principal K-bundle* if $K = \mathrm{Aut}(P \to Q)$ acts transitively on all fibers, or equivalently if $P \to Q$ fits into a triangle

$$\begin{array}{ccc} P & & \\ \downarrow & \searrow & \\ P/K & \longrightarrow & Q \end{array}$$

where the bottom map is an isomorphism.

By Aut$P \to Q$ we mean the group of poset isomorphisms $P \to P$ fitting into triangles:

$$\begin{array}{ccc} P & & P \\ \downarrow & \swarrow & \\ Q & & \end{array} .$$

Definition 2.5.13. If $\pi : (L, P, F) \to (N, Q, G)$ is a morphism where all the maps $Q(\pi(X)) \to F(X)$ are isomorphims, $\pi : P \to Q$ is a principal K-bundle, L is K-Duflot with respect to the K action on P, N is K-equivariant with trivial action, and π is an equivariant map with respect to these actions, then we say that $\pi : (L, P, F) \to (N, Q, G)$ is a K-bundle.

Proposition 2.5.14. *If* $\pi : (L, P, F) \to (N, Q, G)$ *is a K-bundle, then* $\pi^* : N \to L^K$ *is an injection.*

Proof. First, because the map is equivariant and because the action on N is trivial, π^* must map into the invariants. Then to check injectivity it is enough to check on associated gradeds, where the result is clear because the maps $Q(\pi(X)) \to F(X)$ are all isomorphisms. □

Proposition 2.5.15. *If $\pi : (L, P, F) \to (N, Q, G)$ is a K-bundle, then $DN^* \to (DL^*)^K$ is an isomorphism, and each DL^i is a sum of free K-modules.*

Proof. That each DL^i is a sum of free modules comes from the computation of the action of K on the subquotients of the filtration, because the action on the poset P is free, and because the summands of DL^i are indexed on P_i.

For the map $DN^* \to (DL^*)^K$, this follows because on associated gradeds, $F_j/F_{j+1}N \to F_j/F_{j+1}L^K$ is an isomorphism. This is so because over one summand of $F_j/F_{j+1}N$, the map is the diagonal map $\mathrm{gr}_j G(X) \to \sum_{Y \in \pi^{-1}X}(\mathrm{gr}_j F(Y)) \cong \mathrm{ind}_1^K \mathrm{gr}_j F(X)$.

□

Theorem 2.5.16. *If $\pi : (L, P, F) \to (N, Q, G)$ is a K-bundle, then there is a spectral sequence $H^p(K, \mathcal{H}^q(L)) \Rightarrow \mathcal{H}^{p+q}N$.*

Proof. This is the one of the two hypercohomology spectral sequences associated to the complex of K modules DL^*. One spectral sequence has the E_2 term listed in the statement of the theorem, and the other spectral sequence has E_2-term $H^{p+q}(H^p(K, DL^*))$. But $H^p(K, DL^*)$ is zero above degree 1 and is DN^* in degree 0, so the hypercohomology is $\mathcal{H}^*N$ as claimed. □

Chapter 3

CONNECTIONS WITH EQUIVARIANT COHOMOLOGY

In this chapter we show that the algebraic structures studied in the previous chapter are present in the S-equivariant cohomology rings of smooth manifolds, and we apply the theorems of the previous chapter to deduce structural results in S-equivariant cohomology. Recall that all cohomology with with $\mathbb{F}_p$ coefficients.

3.1 $H^*_S M$ has a Duflot free rank filtration

Now we show that this theory isn't vacuous: $H^*_S M$ has a Duflot algebra structure, where S is a p-torus and M is a smooth S-manifold. For applications to group cohomology, the most interesting S-manifold is $M = G \backslash U(V)$, where we have a representation $G \to U(V)$ and S is the maximal diagonal p-torus of $U(V)$ acting on the right of $G \backslash U(V)$. This manifold has two useful geometric properties:

1. For each subgroup $A \subset S$, each component Y of $G \backslash U(V)^A$ is S-invariant.

2. For each subgroup $A \subset S$ and component Y of $G \backslash U(V)^A$, the open submanifold of Y consisting of those points that have isotropy exactly equal to A is connected.

We call S-manifolds satisfying these two properties S-connected. There is a straightforward modification of the definition of a free rank filtration that applies to smooth S-manifolds that aren't S-connected, but it is more complicated to state so we don't do so here.

In this section, fix a p-torus S and a smooth S-connected manifold M.

Lemma 3.1.1. *For $A < S$ and Y a component of M^A, we have that Y is also S-connected.*

Proof. This follows from the fact that a component of Y^B is also a component of M^B. □

Definition 3.1.2. Let $M_i^j = \{x \in M : i \leq \operatorname{rank} S_x \leq j\}$.

Proposition 3.1.3 (Duflot)**.** *Define a filtration of H_S^*M-modules on H_S^*M by $F_j = \ker H_S^*M \to H_S^*M_0^{j-1}$. Then $F_j/F_{j+1} = \oplus_{\{V \subset S : \operatorname{rank} V = j\}} \oplus_{[Y] \in \pi_0(M^V)} \Sigma^{d_Y} H_S^* Y_j^j$, where d_Y is the codimension of Y in M. A shorter way to write this is as $F_j/F_{j+1} = \oplus_{[M'] \in \pi_0 M_j^j} \Sigma^{d_Y} H_S^* M'$, where Y is the component of the fixed points of a rank j p-torus containing M'.*

Proof. Duflot's original proof from [Duf83b] shows this, although it is not stated there in this generality. Duflot states the result for $p \neq 2$ and only gives a filtration as vector spaces, however her proof works verbatim to give the result here. A treatment is found on the section on Symonds' theorem in [Tot14]. □

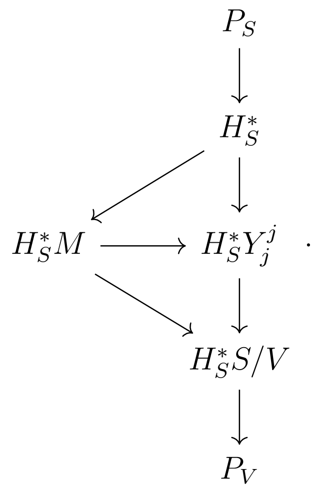
Theorem 3.1.4. *We have that H_S^*M has a free rank filtration.*

Proof. We use the notation as in Proposition 3.1.3.

This is essentially just a computation of each H_S^*M'. First, the points of M' have isotropy of rank exactly equal to j, so there is a unique rank j p-torus V with $M'^V = M'$. So, V acts trivially on M', so $H_S^*M' = H_V^* \otimes H_{S/V}^* M'$. But S/V acts freely on M', so this is $H_V^* \otimes H^*M'/(S/V)$. But $H_V^* = S(V^*)$ when $p = 2$ and is $S(\beta(V^*)) \otimes \Lambda(V^*)$ when p is odd.

Finally, we see that we have the diagram of S-spaces:
$$\begin{array}{ccc} & \nearrow & pt \\ & & \uparrow \\ M & \longleftarrow & Y_j^j \\ & \nwarrow & \uparrow \\ & & S/V \end{array} \quad .$$

Applying H_S^* gives:



In this diagram, when $p = 2$ the top and bottom vertical arrows are just the identity. This shows that each $\Sigma^{d_Y} H_S^* Y_j^j$ is j-free, so we are done. □

To show how to refine this free rank filtration to get a Duflot free rank filtration, we first observe that all the associated primes of H_S^*M are toral in the sense of definition 2.1.7, and also toral in the sense of being pulled back from the map $H_S^*M \to H_S^*S/V/\sqrt{0}$ induced by a map $S/V \to M$.

Lemma 3.1.5. *Every associated prime of H_S^*M is toral in the sense of 2.1.7, and toral primes are exactly the primes that come from restricting to a p-torus.*

Proof. That the associated primes are toral in the sense of 2.1.7 follows immediately from 2.1.8 once we know that H_S^*M has a free rank filtration.

To see that this definition of toral coincides with restriction to a p-torus, in the diagram above that shows that the Duflot filtration is a free rank filtration, observe that the map $H_S^*M \to P_V$ is induced by $S/V \to M$. □

Lemma 3.1.6. *For Y a component of M^A and for $p \neq 2$, the normal bundle of Y in M is S-equivariantly orientable.*

Proof. First we show that the normal bundle is orientable. Denote the total space of the normal bundle by N, and consider the bundle $N \to Y$ as an A-vector bundle, i.e. restrict the action to A. Now A acts trivially on Y, and because Y is a component of M^A, no trivial representations of A appear in the isotypical decomposition of $N \to Y$. Therefore, since every irreducible real representation of A has a complex structure, $N \to Y$ is in fact a complex vector bundle, and therefore orientable.

Now, in order to show that the normal bundle is S-equivariantly orientable, we need to show that the S-action preserves a given orientation. For this, consider the bundle of orientations $O \to Y$ of the bundle $N \to Y$; by previous discussion this is a trivial $\mathbb{Z}/2$-bundle. Now the action of S on $O \to Y$ gives a map $S \to \mathbb{Z}/2$ which is trivial if and only if S preserves the orientation. But S is a p-group, so the result follows. □

Proposition 3.1.7. *Let Y be a component of M^A. Then H_S^*Y is fixed in H_S^*M.*

Proof. To see that H_S^*Y is fixed in H_S^*M, we need to show that the composition of pushfoward and restriction $\Sigma^{d_Y} H_S^*Y \to H_S^*M \to H_S^*Y$ is injective, and to show that this composition is injective we just need to show that the equivariant Euler class of the normal bundle of Y is a non-zerodivisor. We are using 3.1.6 to guarantee the existence of a pushforward. So, suppose that the Euler class e is a zero divisor. Then it is contained in one of the associated primes of H_S^*Y. These associated primes are all toral, so e must restrict to 0 under a map $S/B \to Y$. But Y is fixed by A, so the map $S/B \to Y$ fits in the diagram $S/A \to S/B \to Y$, so e must restrict to zero on S/A.

However, when the normal bundle is restricted to S/A, we have an A representation with no trivial summands, so its Euler class is nonzero. □

Proposition 3.1.8. *For Y a component of M^A, the map $i_* : \Sigma^{d_Y} H_S^*Y \to H_S^*M$ induces a map from the Duflot filtration of Y to the Duflot filtration of M, which induces an inclusion from a suspension of the Duflot complex of Y to the Duflot complex of M.*

Proof. To see that we have the induced map on Duflot filtrations, we use the naturality and functoriality of the Gysin map. We have the following diagram, where each square is a pullback square:

$$\begin{array}{ccccc} Y_i^i & \xrightarrow{i_1} & Y_0^i & \longleftarrow & Y_0^{i-1} \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \downarrow \\ M_i^i & \xrightarrow{i_2} & M_0^i & \longleftarrow & M_0^{i-1} \end{array}$$

This induces the following commutative diagram, which is one piece of the Duflot filtration. In the following diagram, the two left horizontal arrows and the three slanted arrows are pushforwards, and the other maps are restrictions. We omit the suspensions for clarity.

$$\begin{array}{ccccc} & & H_S^*M_i^i & \longrightarrow & H_S^*M_0^i \\ & \nearrow & & \nearrow & \downarrow \\ H_S^*Y_i^i & \longrightarrow & H_S^*Y_0^i & & H_S^*M_0^{i-1} \\ & & \downarrow & \nearrow & \\ & & H_S^*Y_0^{i-1} & & \end{array}$$

Note that $H_S^*Y_i^i \to H_S^*M_i^i$ is the inclusion of a summand, since the map $Y_i^i \to M_i^i$ is the inclusion of a component. After applying local cohomology we get the following ladder:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{H}^i(H_S^*Y_i^i) & \xrightarrow{d_Y^i} & \mathcal{H}^{i+1}H_S^*(Y_{i+1}^{i+1}) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathcal{H}^i(H_S^*M_i^i) & \xrightarrow{d_M^i} & \mathcal{H}^{i+1}H_S^*(M_{i+1}^{i+1}) & \longrightarrow & \cdots \end{array}$$

Here the vertical arrows are inclusions of summands, since local cohomology is an additive functor.

□

Definition 3.1.9. Let $\mathrm{Fix}(M)$ be the weakly coranked poset whose elements are components Y of M^V, where V is some subtorus of S, and where $Y_{\mathrm{rank}\,V}^{\mathrm{rank}\,V}$ is non empty, and where the morphisms are given by inclusions. The rank of Y is the rank of the largest p-torus that fixes it.

There is a functor $F : \mathrm{Fix}(M) \to H_S^*M - mod$ sending Y to $\Sigma^{d_Y} H_S^*Y$. We now show that this refines the filtration from 3.1.4.

Lemma 3.1.10. *The filtration on H_S^*M by $\mathrm{Fix}(M)$ refines the filtration we have already defined, i.e.* $\sum_{Y \in \mathrm{Fix}(M)_{\geq s}} F(Y) = F_s H_S^*M$.

Proof. Let $J_s = \sum_{Y \in \mathrm{Fix}(M)_{\geq s}} F(Y)$. First, we observe that $J_s \subset F_s$: this is because for each Y $i_* : H^*_S Y \to H^*_S M$ respects the Duflot filtration, and because s is the lowest possibly nonzero term in the Duflot complex for Y, where Y is a connected component in the fixed point set for a rank s p-torus.

The other inclusion is by downward induction on s. First, when $s = \dim H^*_S M$, then $F_s H^*_S M = F_s/F_{s+1} H^*_S M$, which by is J_s.

So, suppose $F_{i+1} = J_{i+1}$, and consider $x \in F_i$. By our description of F_i/F_{i+1} from the first part of Proposition 3.1.3, there is some $y \in J_{i+1}$ so that x and y are equal modulo F_{i+1}, so we are done by induction. □

Lemma 3.1.11. *The filtration on $H^*_S M$ induced by $\mathrm{Fix}(M)$ is good.*

Proof. We need to show that the map $\oplus_{Y \in \mathrm{Fix}(M)_j} \mathrm{gr}_j F(Y) \to F_j/F_{j+1} H^*_S M$ is an isomorphism. This follows from 3.1.8: F_j/F_{j+1} has a direct sum decomposition indexed on $\mathrm{Fix}(M)_j$, and the map $F(Y) \to H^*_S M$ for $Y \in \mathrm{Fix}(M)_j$ induces an inclusion of the direct summand corresponding to Y on $F_j/F_{j+1} Y = \mathrm{gr}_j F(Y) \to F_j/F_{j+1} H^*_S M$. □

Proposition 3.1.12. *The poset $\mathrm{Fix}(M)_{\geq i}$ puts a Duflot module structure on $F_i H^*_S M$.*

Proof. We have seen in 3.1.11 that the filtration by $\mathrm{Fix}(M)$ is a stratified free rank filtration.

To see that it is topological, we use 3.1.7 and 3.1.8.

To see that it is minimal, we need to check that for each $Y \in \mathrm{Fix}(M)$ each $H^*_S M \to H^*_S Y$ has a distinct kernel. For $Y \in M^A$, we have the diagram:

$$\begin{array}{ccccc} H^*_S & & & & \\ \downarrow & \searrow & & \searrow & \\ H^*_S M & \longrightarrow & H^*_S Y & \longrightarrow & H^*_S S/A \end{array}$$

The maps $H^*_S \to H^*_S S/A = H^*_A$ have distinct kernel for distinct A, so we only need to show that if Y, Y' are different components of M^A, then the restrictions to Y and to Y' have different kernels. However, this is immediate because Y and Y' are disjoints, so the compostion $\Sigma^{d_Y} H^*_S Y \to H^*_S M \to H^*_S Y'$ is zero, while $\Sigma^{d_Y} H^*_S Y \to H^*_S M \to H^*_S Y$ is nonzero.

Finally, to see that it is Duflot, we use the computation that if Y is a component of M^A, then $H^*_S Y \cong H^*_A \otimes H^*_{S/A} Y$.

□

3.2 Structural properties of H_S^*M

Here we collect various algebraic properties of H_S^*M that follow from H_S^*M having a Duflot algebra structure. We will use these in 4.1.

As in the previous section, here M is a smooth S-connected manifold.

Corollary 3.2.1 (Duflot [Duf81], Quillen [Qui71]). *The depth of H_S^*M is no less than the largest rank of a subtorus of S that acts trivially on M, and the dimension is equal to the largest rank of a subtorus of S that acts with fixed points on M.*

Proof. This follows from 2.1.4. If we let d be the largest rank of a subtorus of S that acts trivially on M and r the largest rank of a subtorus that acts with fixed points on M, then $F_d H_S^*M = H_S^*M$, and $F_r H_S^*M$ is the smallest nonzero level of the filtration. Note that 2.1.4 only gives that the dimension of H_S^*M is less than or equal to r, but by studying the dual of the Duflot chain complex for H_S^*M we conclude that $\mathcal{H}^r H_S^*M$ is nonzero. □

Corollary 3.2.2 (Duflot [Duf83a]). *Each associated prime of H_S^*M is toral.*

Proof. This is just Lemma 3.1.5. □

Note: The proof we have given here is essentially the same as Duflot's proof.

Corollary 3.2.3. *There is a cochain complex DM with $H^i(DM) = \mathcal{H}^i(H_S^*M)$, and $DM^i = \oplus_{Y \in \mathrm{Fix}(M)_i} \Sigma^{d_Y} \mathcal{H}^i(H_S^* Y_i^i)$.*

Proof. This follows from 2.1.9. □

Remark 3.2.4. This result should be compared with the compuation of the Atiyah-Bredon complex in [AFP14]. There, Allday, Franz, and Puppe study a chain complex that occurs in the rational torus equivariant cohomology of a manifold, which is analagous to the dual of the Duflot chain complex. In unpublished work the authors extend some of their results to the setting of p-torus equivariant cohomology.

Corollary 3.2.5 (Symonds [Sym10]). *The regularity of H_S^*M is less than or equal to the dimension of M.*

Proof. This follows from 2.1.12. Each j-free summand appearing in the free rank filtration has the form $\Sigma^{d_Y} H_V \otimes H^*(Y_j^j/S)$, and $d_Y + \dim Y_j^j/S = \dim M$, so the result is as claimed.

The distinction between $p = 2$ and $p \neq 2$ appearing in 2.1.12 does not change the result because of the exterior terms in H_V^* cancel out the extra j term. □

Corollary 3.2.6 (Symonds [Sym10]). *If $a_i(H_S^*M) = -i + \dim M$, then there is a maximal element of* $\mathrm{Fix}(M)$ *of rank i.*

Proof. Consider the Duflot chain complex for H_S^*M. The i^{th} degree term of this chain complex is a sum of modules of the form $\mathcal{H}^i(\Sigma^{d_Y} H_V^*) \otimes H^*(Y_i^i/S)$, where V is a rank i p-torus. The top nonzero degree of this will be $d_Y + -i + H^{\mathrm{top}} H^*(Y_i^i/S)$. We have that Y_i^i/S is a manifold of dimension $\dim M - d_Y$, so $a_i(H_S^*M) = -i + \dim M$ only if there is some Y so that $H^{\dim Y} Y_i^i/S$ is nonzero, which happens if and only if Y_i^i/S is compact, and consequently if and only if Y_i^i is compact.

However, $Y_i^i = Y - \cup_{W \in \mathrm{Fix}(M)_{>Y}} W$ (recall that every point of Y has isotropy of rank greater than or equal to i), so Y_i^i is compact if and only if $Y_i^i = Y$ and Y is maximal in $\mathrm{Fix}(M)$. □

Remark 3.2.7. Symonds shows this without explicitly stating it in his proof of some special cases of the strong regularity theorem in [Sym10].

Corollary 3.2.8. *Take $(Y, A) \in \mathrm{Fix}(M)$, and let $\mathfrak{p}$ be the kernel of $H_S^*M \to H_S^*Y \to H_S^*S/A$. The following are equivalent.*

1. $\mathfrak{p} \in \mathrm{Ass}_{H_S^*M} H_S^*M$.

2. $\mathfrak{p} \in \mathrm{Ass}_{H_S^*Y} H_S^*Y$.

3. *The depth of H_S^*Y is* $\mathrm{rank}\, Y$.

Proof. This follows from 2.4.2. □

Corollary 3.2.9 (Quillen [Qui71]). *The minimal primes of H_S^*M are in bijection with the minimal elements of* Fix(M)*, i.e. the pairs A, Y where A is a p-torus of Y and Y is a component of Y^A, and S/A acts freely on Y. The primes are obtained via the kernel of $H_S^*M \to H_S^*Y \to H_S^*S/A$.*

Proof. It is clear by our description of the associated primes that if the primes arising in this manner are associated, then they are minimal among the associated primes and therefore minimal primes. That there are distinct primes for each maximal element was noted in the proof of 3.1.12.

To see that these primes are associated we use 3.2.8. Since S/A acts freely on H_S^*Y, H_S^*Y is isomorphic to $H_A^* \otimes H^*Y/S$, so the Duflot bound is sharp for H_S^*Y and the result follows. □

This proof is different than Quillen's proof. In [Qui71] he obtains this as a consequence of the F-isomorphism theorem.

The following is the S-equivariant cohomology version of a detection result of Carlson [Car95]. We will use this later to derive Carlson's detection result.

Corollary 3.2.10. *If H_S^*M has depth d, then H_S^*M is detected by restricting to the H_S^*Y for $Y \in Fix(M)_d$.*

Proof. This follows from 2.4.3. □

Recall Carlson's conjecture:

Conjecture 3.2.11 (Carlson). *If H_G^* has depth d, then H_G^* has a d-dimensional associated prime.*

We will use the following to prove Carlson's conjecture in the special case where G is a compact Lie group with the Duflot bound sharp.

Theorem 3.2.12. *If the Duflot bound for $H^*_S M$ is sharp, then there is associated prime of dimension the depth of G given by $H^*_S M \to H^*_S S/A$, where A is the maximal p-torus that acts trivially on M.*

Proof. This follows from the third part of 3.2.8: $M^A = M$, so M is a minimal element of $\mathrm{Fix}(M)$ of rank $\dim A$, so if $\operatorname{depth} H^*_S M = \dim A$, then $H^*_S M \to H^*_S S/A$ represents an associated prime.

□

Remark 3.2.13. It is worth stating that this proof does not essentially use the machinery of a Duflot filtration, and this can be converted to an elementary proof of Carlson's conjecture when the Duflot bound for depth is sharp.

3.3 Exchange between S and G equivariant cohomology

The S-manifolds we are primarily interested in are those of the form $G\backslash U(V)$, where G is a compact Lie group and $G \to U(V)$ is faithful. Recall that by convention $U(V)$ respects direct sum decompositions of V. In other words, if we have a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_n$ of a faithful unitary representation V of G, then $U(V) = \prod_{i=1}^n U(V_i)$.

The space $U(V)$ has a maximal p-torus S of diagonal matrices of order dividing p. We take G to act on the left of $U(V)$ and S to act on the right, so we write $G\backslash U(V)$ instead of $U(V)/G$. Our first main goal is to show is that $G\backslash U(V)$ is an S-connected manifold. This will show that $H^*_S G\backslash U(V)$ has a Duflot algebra structure, and then we will see that $H^*_G U(V)/S$ does as well.

We need to explain the relationships between three closely related manifolds. First we have $U(V)$. The left action of G and the right action of S gives an action of $G \times S$ on $U(V)$ (if this is to be a left action, we must modify the right action of S so that s acts by s^{-1}, and if we want this to be a right action we similarly modify the G action). We also have the left action of G on $U(V)/S$, and the right action of S on $G\backslash U(V)$.

Note that there are maps $G\backslash U(V) \leftarrow U(V) \rightarrow U(V)/S$. If $A < S$ and $Gu \in G\backslash U(V)^A$, then ${}^uA < G$, and $uS \in (U(V)/S)^{({}^uA)}$. Here we mean the action of ${}^uA$ as a subgroup of G, so it is acting on the right.

Let $D < {}^uA \times A < G \times S$ be the image of A under $a \mapsto (uau^{-1}, a)$. We give D the action on $U(V)$ that is restricted from $G \times S$ acting on $U(V)$ on the left, so $(uau^{-1}, a) \cdot v = uau^{-1}va^{-1}$.

Parts of the following lemma are used in [Sym10]. These results could also be derived from [Duf83a], but for completeness we include a proof here.

Lemma 3.3.1. *1. For $A < S$ and $x = Gu \in (G\backslash U(V))^A$, the connected component Y of x in $(G\backslash U(V))^A$ is isomorphic as S-spaces to $C_{G^u}A\backslash C_{U(V)}A$.*

2. Moreover, the connected component X of $xS \in U(V)/S^{{}^uA}$ is $C_G({}^uA)$-invariant and isomorphic as $C_G({}^uA)$-spaces to $C^u_{U(V)}A/{}^uS$, and the G orbit of the connected component is $G \times_{C_G({}^uA)} C_{U(V)}({}^uA)/{}^uS$.

3. Given $Gu \in (G\backslash U(V))^A$, the connected component Z of $u \in U(V)^D$ (where the D is as in the discussion preceding the lemma) is the left $C_{U(V)}({}^uA)$ orbit of u, and the right $C_{U(V)}A$-orbit of u (in fact, $U(V)^D$ has one connected component). Under the maps $G\backslash U(V) \leftarrow U(V) \rightarrow U(V)/S$, Z maps onto to Y and onto X. We have that Z is invariant under the right S action, but not necessarily under the left G action. However, the G-orbit of Z is $G \times_{C_G({}^uA)} C_{U(V)}A$ and it maps onto the G-orbit of X, and the G orbit of Z also maps onto Y.

Proof. 1. First, observe that if Gu is fixed by A, then ${}^uA < G$, and $A < G^u$. Also, it is immediate that $C_{U(V)}A$ acts on the right on $(G\backslash U(V))^A$, and because $C_{U(V)}A$ is a product of unitary groups it is connected, and consquently the $C_{U(V)}A$-orbit of x is connected.

We will first show that Y is the $C_{U(V)}A$-orbit of x. It is enough to show that there are only finitely many orbits of the $C_{U(V)}A$ action on $(G\backslash U)^A$, from this it follows that each orbit is a component of $(G\backslash U)^A$. To see that there are only finitely many

$C_{U(V)}A$-orbits is it enough to show that there are only finitely many $N_{U(V)}A$-orbits on $(G\backslash U(V))^A$, because $C_{U(V)}A$ has finite index in $N_{U(V)}A$.

If $Gu, Gv \in (G\backslash U)^A$ have the property that ${}^uA$ and ${}^vA$ are conjugate p-tori in G, so ${}^{gv}A = {}^uA$ for some $g \in G$, then $u^{-1}gv \in N_{U(V)}A$, and Gu and Gv are in the same $N_{U(V)}A$-orbit. So, since G has only finitely many conjugacy classes of p-tori, there are only finitely many $N_{U(V)}A$-orbits, and we have shown that Y is the $C_{U(V)}A$-orbit of x.

To see that Y is $C_{G^u}A\backslash C_{U(V)}A$, observe that the stabilizer of Gu under the $C_{U(V)}A$ action is $C_{G^u}A$.

2. We can follow the same strategy as above to identify X. First note that $C_{U(V)}({}^uA)$ acts on $U(V)/S^{{}^uA}$, then observe that the $C_{U(V)}({}^uA)$-orbits are each a component of $U(V)/S^{{}^uA}$, and then that the stabilizer is as claimed. Once we have identified X this also shows that X is $C_G^{{}^uA}$-invariant, as $C_G({}^uA) < C_{U(V)}({}^uA)$.

   To see that the G-orbit of X is isomorphic to $G \times_{C_G({}^uA)} X$ as G-spaces, note that G acts on the components of the G-orbit of X, and there is an obvious equivariant homeomorphism from $G \times_{\operatorname{stab} X} X$ to the G-orbit of X. We have already observed that $C_G({}^uA)$ is contained in $\operatorname{stab} X$. To see the other containment, suppose that $gcuS = c'uS$, where $c, c' \in C_{U(V)}({}^uA)$ and $g \in G$. Then $c'^{-1}gc \in^u S < C_{U(V)}({}^uA)$, so g must centralize ${}^uA$.

3. If we can show that connected component of $u \in U(V)^D$ is as claimed, then the other claims follow. This is just a computation, it is easy to check that the left action of $C_{U(V)}({}^uA)$ and the right action of $C_{U(V)}A$ preserve Z, and if $y \in U(V)^D$, then we can write $y = cu = uc'$, where $c \in C_{U(V)}({}^uA)$ and $c' \in C_{U(V)}A$.

□

This is very nice, because if we start with a right S space of the form $G\backslash U(V)$, then each component of $(G\backslash U(V))^A$ has the same form for some conjugate subgroup of G: the components are of the form $C_{G^u}A\backslash U(V')$, where V' is a direct sum of representations of $C_{G^u}A$, giving a faithful representation of $C_{G^u}A$.

Similarly, if we start with the right G-space $U(V)/S$, then each component of $U(V)/S^B$ has the same form for some subgroup of G: the components are of the form $U(V')^uS$.

We'll explore the implications of this for group cohomology in the next section. For now, we show that $G\backslash U(V)$ is S-connected.

Proposition 3.3.2. *Let $M = G\backslash U(V)$. Then M is S-connected.*

Proof. That S preserves each component of M^A is clear from our description of each component of M^A. So, we need to show that, for Y a component of M^A, if $Y^{\operatorname{rank} A}_{\operatorname{rank} A}$ is nonempty it is connected.

Via the above correspondence between connected components of the fixed point sets of $U(V)/S$ and $G\backslash U(V)$, it is enough to show that $Z^{\operatorname{rank} D}_{\operatorname{rank} D}$ is connected, where the notation is as in 3.3.1.

For this, note that for each E a sub p-torus of D, $U(V)^E$ is the total space of a torus bundle over a (product of) flag manifolds, and for $U(V)^E \subset U(V)^D$, we have the following diagram of these T-bundles:

$$\begin{array}{ccc} T & \longrightarrow & T \\ \downarrow & & \downarrow \\ U(V)^E & \longrightarrow & U(V)^D \\ \downarrow & & \downarrow \\ F'' & \longrightarrow & F', \end{array}$$

where $F'' \to F'$ is a proper sub (product of) flag manifolds. But F'' therefore has even codimension in F', so $U(V)^E$ has even codimension in $U(V)^D$.

However, $Z^{\operatorname{rank} D}_{\operatorname{rank} D} = Z - (\cup_{E<D} Z^E)$. Each Z^E therefore has even codimension in Z, so $Z^{\operatorname{rank} D}_{\operatorname{rank} D}$ is connected and we are done.

Instead of looking at these flag manifolds, we could get this same result by comparing the dimension of $C_{U(V)}A$ and $C_{U(V)}B$ for $A < B$, each is a product of unitary groups corresponding to the isotypical decomposition of the representation for the respective p-torus, and if $A < B$ the isotypical decomposition for B refines that for A. $\square$

So, this tells us that all the theorems from 3.1 and 3.2 apply to $H^*_S G\backslash U(V)$.

Moreover, we now show that these theorems also apply to $H^*_G U(V)/S$.

Proposition 3.3.3. *Under the maps $G\backslash U(V) \leftarrow U(V) \rightarrow U(V)/S$, and for Y, Z, X components of the fixed point sets of A, D and ${}^uA$ respectively, as above, we have this diagram:*

$$\begin{array}{ccccc}
H^*_S Y & \longrightarrow & H^*_{G\times S} GZ & \longleftarrow & H^*_G GX \\
\downarrow & & \downarrow & & \downarrow \\
H^*_S G\backslash U(V) & \longrightarrow & H^*_{G\times S} U(V) & \longleftarrow & H^*_G U(V)/S \\
\downarrow & & \downarrow & & \downarrow \\
H^*_S Y & \longrightarrow & H^*_{G\times S} GZ & \longleftarrow & H^*_G GX
\end{array}\ .$$

The notation GZ and GX denote the G-orbits of Z and X. In this diagram, all the horizontal arrow are isomorphisms.

Proof. The diagram of the theorem comes from the diagram of spaces:

$$\begin{array}{ccccc}
ES \times_S Th(Y) & \longleftarrow & EG \times ES_{G\times S} Th(GZ) & \longrightarrow & EG \times_G Th(GX) \\
\uparrow & & \uparrow & & \uparrow \\
ES \times_S G\backslash U(V) & \longleftarrow & EG \times ES_{G\times S} U(V) & \longrightarrow & EG \times U(V)/S \\
\uparrow & & \uparrow & & \uparrow \\
ES \times_S Y & \longleftarrow & EG \times ES_{G\times S} GZ & \longrightarrow & EG \times_G GX
\end{array}\ .$$

The horizontal arrows in the bottom two rows are homotopy equivalences by our description of Y, GZ, and GX. For the top row, we first apply the equivariant Thom isomorphism to the cohomology of the top row, and then our description of Y, GZ and GX gives us the result.

□

Remark 3.3.4. It is worth noting that in order for this proof to be correct, we must verify that in the case that p is odd all the bundles in question are actually orientable, so that the claimed Thom isomorphisms exist. However, this follows in a straightforward way from the normal bundle for Y in $G\backslash U(V)$ being S-equivariantly orientable.

Definition 3.3.5. Let $\mathrm{Fix}(U(V)/S)$ be the weakly coranked poset whose elements are G-orbits of connected components of $U(V)/S^A$, for A a p-torus of G.

Lemma 3.3.6. *Under the correspondence between connected components of* $G\backslash U(V)^A$ *and connected components of* $U(V)/S^{uA}$, $\mathrm{Fix}(G\backslash U(V)) \xrightarrow{\sim} \mathrm{Fix}(U(V)/S)$.

Proof. This is immediate from the discussion of these connected components above. □

Note that there is a functor $F : \mathrm{Fix}(U(V)/S) \to H^*_G U(V)/S - mod$ where X is mapped to the module $(\Sigma^{d_X} H^*_G(G \times_{\mathrm{stab}\, X} X)$, which via the Gyisn map is equipped with an inclusion into $H^*_G(U(V)/S))$.

Theorem 3.3.7. $H^*_G U(V)/S$ *is a Duflot algebra, stratified by* $\mathrm{Fix}(U(V)/S)$.

Proof. This follows immediately from 3.3.3: we know that $H^*_S G\backslash U(V)$ has a Duflot algebra structure stratified by $\mathrm{Fix}(G\backslash U)$, and this puts a Duflot algebra structure on $H^*_G U(V)/S$ via the isomorphisms in 3.3.3. □

Chapter 4

APPLICATIONS TO THE COHOMOLOGY OF BG

In this chapter we apply the Duflot algebra structure on $H^*_S U(n)/G$ to study H^*BG. In section 4.1 we give structural results that apply to H^*_G for any compact Lie group G, and in sections 4.2 and 4.3 we show how certain extensions of finite groups lead to the structures studied in 2.5, from which we derive some local cohomology computations for the p-Sylow subgroups of S_{p^n} and for S_{p^n} itself.

4.1 Structural results in group cohomology

Here we demonstrate how the results of this paper recover several classical results in group cohomology. All of the results listed in this section were previously known at least for finite groups, but except where indicated otherwise with different proofs from our methods. There are several new results for compact Lie groups.

Throughout this section, G is a compact Lie group and V a faithful finite dimensional unitary representation of G.

Crucial to applying the techniques of this paper to get results in group cohomology is a theorem of Quillen [Qui71] showing to what extent H^*_G can be recovered from $H^*_G U(V)/S$.

Proposition 4.1.1. *The map $H^*_G \to H^*_G U(V)/S$ is faithfully flat, and as H^*_G-modules $H^*_G U(V)/S$ is non-canonically isomorphic to $H^*_G \otimes H^*U(V)/S$.*

Corollary 4.1.2. *The map $H^*_G \to H^*_G U(V)/S$ induces a surjection* $\operatorname{Ass} H^*_G U(V)/S \to \operatorname{Ass} H^*_G$.

This is a consequence of the map being faithfully flat.

Theorem 4.1.3 (Duflot [Duf81], Quillen [Qui71]). *The depth of H^*_G is greater than or equal to the p-rank of the center of G, and the dimension of G is equal to the maximal rank of a p-torus of G.*

The result on depth is due to Duflot, and the result on dimension is due to Quillen. Duflot's original proof does not use the Duflot filtration in the way that we do here, but it has the advantage that it explicitly constructs a regular sequence of length the lower bound for depth.

The result on dimension follows from the Quillen stratification theorem. Here we give a proof of this theorem using the Duflot filtration.

Proof. By 4.1.1, it is enough to show the analogous result for $H^*_G U(V)/S = H^*_S G\backslash U(V)$. However, we note that we can choose for V a sum of representations for G so that $Z(G) \to Z(U(V))$. With this choice of V, the result follow from 3.2.1. □

Remark 4.1.4. Duflot's bound can be made stronger in the case that G is a finite group: the depth is in fact no less than the maximal rank of a central p-torus in a p-Sylow of G. This stronger statement follows immediately from the statement we have given, by using the fact that for P a p-Sylow of G, H^*_G is a summand of H^*_P as H^*_G-modules: if $\mathcal{H}^i H^*_G$ is nonzero, so is $\mathcal{H}^i H^*_P$.

Quillen also showed that the minimal primes of H^*_G are given by restricting to a maximal (by inclusion) p-torus. We can also get this result using our techniques.

Theorem 4.1.5 (Quillen [Qui71]). *The minimal primes of H^*_G are obtained by restricting to maximal p-tori.*

Proof. Because $H^*_G \to H^*_G U(V)/S$ is faithfully flat, all the minimal primes of $H^*_G U(V)/S$ pull back to minimal primes of H^*_G. But 3.2.9 describes the minimal primes of $H^*_S U/S$, which are given by restricting to minimal elements of $\mathrm{Fix}(U/S)$. These minimal elements correspond to a maximal p-torus A of G and a component of U/S^A, so it is clear that all minimal primes of H^*_G have the claimed description.

□

Remark 4.1.6. Quillen in fact shows that there is a bijection between the conjugacy classes of maximal p-tori and the minimal primes of H^*_G. In order to conclude this stronger result from what we have done so far, it is only necessary to show that if A and B are non-conjugate p-tori in G then $H^*_G \to H^*_G G/A/\sqrt{0}$ and $H^*_G \to H^*_G G/B/\sqrt{0}$ have different kernels. Note that we already have shown that $H^*_G U(V)/S \to H^*_G G/A/\sqrt{0}$ and $H^*_G U(V)/S \to H^*_G G/B/\sqrt{0}$ have different kernels.

We haven't yet completed this final step using the methods of this thesis, but there are various ways to conclude the desired result, for example Quillen's original proof or using the Even's norm.

Duflot also showed that the associated primes of H^*_G are given by restricting to p-tori, but there is no known group theoretic description of what p-tori give associated primes. Our proof of 3.1.5 is essentially a reformulation of Duflot's original proof.

Theorem 4.1.7 (Duflot [Duf83a]). *The associated primes of H^*_G all come from restricting to p-tori of G.*

Proof. This follows from 4.1.1 and 3.1.5. □

In the following, Okuyama [Oku10] shows the equivalence of 1 and 3 for finite groups, and the equivalence of 2 and 3 is due to Kuhn [Kuh13].

Theorem 4.1.8. *Let A be a p-torus of G. The following are equivalent.*

1. *A represents an associated prime in H^*_G.*

2. *A represents an associated prime in $H^*_{C_G A}$.*

3. *We have that* $\operatorname{depth} H^*_{C_G A} = \dim A$.

Proof. This follows from 3.2.8, 4.1.1, and our description of the fixed points of the A-action on $U(V)/S$. □

Remark 4.1.9. Note that this gives some restrictions on which p-tori can represent associated primes: such a p-torus must be the maximal central p-torus in its centralizer. This for example rules out all the p-tori of $\mathbb{Z}/p \wr \mathbb{Z}/p$ of rank from 3 to $p-1$.

Theorem 4.1.10 (Symonds [Sym10]). *For G a compact Lie group with orientable adjoint representation,* $\operatorname{reg} H_G^* = -\dim G$.

Proof. Since $H_S^* G\backslash U(V) \cong H_G^* U(V)/S$, by 3.2.5, $\operatorname{reg} H_S^* G\backslash U(V) = \operatorname{reg} H_G^* U(V)/S$ and this quantity is less than or equal to $\dim U(V) - \dim G$. Hwoever as H_G^*-modules, $H_G^* U(V)/S \cong H_G^* \otimes H^* U(V)/S$. Now, because $\operatorname{reg} H^* U(V)/S = \dim U(V)$ and regularity is additive under tensor products, we have that $\operatorname{reg} H_G^* \leq -\dim G$.

That $-\dim G \leq \operatorname{reg} H_G^*$ follows from the Greenlees spectral sequence of [Gre95]; it is this direction that requires the adjoint representation to be orientable. □

Theorem 4.1.11 (Symonds [Sym10]). *For G a compact Lie group, if $a_i(H_G^*) = -i - \dim G$ for some i, then i must be the rank of a maximal p-torus in G.*

Proof. This follows from 3.2.6 and the fact that the minimal elements of $\operatorname{Fix}(G\backslash U(V))$ are in bijection with the maximal p-tori of G. □

Remark 4.1.12. Symonds uses this without explicitly stating it in his proof of a special case of the Strong Regularity conjecture.

Theorem 4.1.13. *If* $\operatorname{depth} H_G^* = d$, *then* $H_G^* \to \prod_{\operatorname{rank} E = i} H_{C_G E}^*$ *is injective.*

Proof. By 3.2.10 and our analysis of the elements of $\operatorname{Fix}(G\backslash U(V))$ and $\operatorname{Fix}(U(V)/S$, we have an injective map $H_G^* U(V)/S \to \prod_{\operatorname{rank} E=i} H_{C_G E}^* C_{U(V)} E/^u S$. This fits into a diagram:

$$\begin{array}{ccc} H_G^* U(V)/S & \longrightarrow & \prod_{\operatorname{rank} E=i} H_{C_G E}^* C_{U(V)} E/^u S \\ \uparrow & & \uparrow \\ H_G^* & \longrightarrow & \prod_{\operatorname{rank} E=i} H_{C_G E}^* \end{array}.$$

Both vertical arrows are injections by 4.1.1, so because the top arrow is an injection the bottom arrows is too.

□

Remark 4.1.14. Carlson in [Car95] proves this theorem when G is a finite group. The techniques of Henn, Lannes, and Schwartz [HLS95] could also prove this theorem, presumably including the case of compact Lie groups.

Theorems 4.1.13 and 4.1.8 can be combined to show that H_G^* is detected on the family of centralizers of p-tori representing associated primes, and this family is minimal.

First recall the definition of a family of subgroups:

Definition 4.1.15. For G a compact Lie group a collection $\mathcal{F}$ of subgroups of G is called a family if $\mathcal{F}$ is closed under subconjugacy, in other words if $H \in \mathcal{F}$ and $K^g < H$ for some $g \in G$ implies that $K \in \mathcal{F}$ as well.

Definition 4.1.16. Let $\mathcal{C}_{\text{Ass}}$ denote the family generated by the set of subgroups $C_G E$, where E represents an associated prime in H_G^*.

Theorem 4.1.17. *We have that H_G^* is detected on $\mathcal{C}_{\text{Ass}}$, and if $\mathcal{F}$ is any other family generated by centralizers of p-tori, then $\mathcal{C}_{\text{Ass}} \subset \mathcal{F}$.*

Proof. First, we show that G is detected on $\mathcal{C}_{\text{Ass}}$. Suppose that $\operatorname{depth} H_G^* = d$, so H_G^* is detected on the centralizers of rank d p-tori. If E is a rank d p-torus, if E doesn't represent an associated prime in H_G^*, then $\operatorname{depth} H_{C_G E}^* > d$, so $H_{C_G E}^*$ is itself detected on centralizers of higher rank p-tori. So, because $C_{C_G E} A = C_G A$ for a p-torus A in $C_G E$ in any detecting family of centralizers of p-tori we can always replace $C_G E$ by a collection of centralizers of higher rank p-tori, unless E represents an associated prime.

To show that this family is the minimal family of centralizers of p-tori that detected cohomology, suppose that $\mathcal{F}$ is another such family.

Then $\operatorname{Ass}_{H_G^*} H_G^* \subset \operatorname{Ass}_{H_G^*} \prod_{H \in \mathcal{F}} H^* H = \cup_{H \in \mathcal{F}} \operatorname{Ass}_{H_G^*} H_H^*$, and $\operatorname{Ass}_{H_G^*} H^* H$ is mapped onto by $\operatorname{Ass}_{H_H^*} H_H^*$. So, if E represents an associated prime in H_G^*, then this associated prime must be in some $\operatorname{Ass}_{H_G^*} H_H^*$, so there is some $H \in \mathcal{F}$ containing E. But by assumption $H = C_G A$ for some p-torus A, so $C_G E \subset H$, which shows that $\mathcal{C}_{\text{Ass}} \subset \mathcal{F}$. $\square$

Remark 4.1.18. Note that $\mathcal{C}_{\text{Ass}}$ is not the minimal detecting family, but merely the minimal detecting family generated by centralizers of p-tori. Indeed, if the Duflot bound for depth is sharp (and G is a p-group), then as a central p-torus will represent an associated prime in H^*_G, $\mathcal{C}_{\text{Ass}}$ is the family of all subgroups. The family of all subgroups is the minimal detecting family if and only if there is essential cohomology, i.e. cohomology classes that restrict to zero on all subgroups.

However, there are examples of groups where the Duflot bound is sharp and that do not have essential cohomology. The semidihedral group of order 16 is one such example, it is detected on the family of all proper subgroups (i.e. it has no essential cohomology), but the depth is one and $\mathcal{C}_{\text{Ass}}$ is the family of all subgroups. If there is essential cohomology, then by 4.1.13 the Duflot bound must be sharp and therefore $\mathcal{C}_{\text{Ass}}$ is the minimal detecting family.

It would be interesting to have an understanding of those groups that have no essential cohomology and where the Duflot bound for depth is sharp. In the special case where H^*_G is Cohen-Macaulay, Adem and Karagueuzian [AK97] give a group theoretic criterion for when there is essential cohomology and prove that under the assumption that H^*_G is Cohen-Macaulay having essential cohomology is equivalent to the Duflot bound being sharp.

We now give a proof of the result due to Kuhn that Carlson's conjecture is true in a special case.

Theorem 4.1.19. *If the Duflot bound is sharp for the depth of H^*_G, then the maximal central p-torus of G represents an associated prime in H^*_G.*

In the case when G is finite and the strengthened Duflot bound is sharp (see 4.1.4), then the maximal central p-torus of a p-Sylow of G represents an associated prime.

Proof. The first sentence of the theorem follows immediately from 4.1.8, where we let A be the maximal central p-torus.

For the case when the strengthened Duflot bound is sharp, let A be the maximal central p-torus of a p-Sylow, let $\dim A = d$, and suppose that the depth of H^*_G is d. Then C_GA has index prime to p in G, so H^*_G is a direct summand of $H^*_{C_GA}$. Therefore, if $\mathcal{H}^dH^*_G$ is nonzero, so is $\mathcal{H}^d H^*_{C_G A}$. Therefore the Duflot bound is sharp for $H^*_{C_G A}$, so A represents an associated prime. $\square$

Remark 4.1.20. For finite groups, Green proved this for p-groups in [Gre03] and Kuhn extended the result to all finite groups in [Kuh07], and to compact Lie groups in [Kuh13].

4.2 i-trivial bundles

Here we explore the relationship on cohomology and local cohomology when we have an S-equivariant principal K-bundle $M \to N$, where M, N are smooth S-connected manifolds. Our goal is to describe a manifestation of the structures studied in Section 2.5 in equivariant topolgy.

Recall that S denotes a p-torus.

Definition 4.2.1. For K a finite group an S-equivariant principal K-bundle $\pi : M \to N$ is *i-trivial* if for all $Y \in \mathrm{Fix}(N)_{\geq i}$ $\pi^{-1}Y \to Y$ is a trivial K-bundle.

Remark 4.2.2. If M is a left S-space and K any finite group, then the trivial K-bundle $M \times K \to M$ is i-trivial for any i. Of course the interesting cases are when the bundle is not trivial. We will construct examples of i-trivial bundles by group extensions that we also call i-trivial momentarily.

Let $K \to M \to N$ be an i-trivial S-equivariant bundle.

Lemma 4.2.3. *If an equivariant submanifold $W \subset N$ is fixed by $A < S$ with $\operatorname{rank} A \geq i$, so is $\pi^{-1}W$, and if Y is a component of M^A, with $\operatorname{rank} A \geq i$, then π restricted to Y is a diffeomorphism onto a component of N^A.*

Proof. For the first part, if $x \in N$ is fixed by A, then $x \in W$ where W is a component of N^A. Then $\pi^{-1}W \to W$ is an S-equivariant trivial principal K-bundle, so the fiber over x is also fixed by A.

For the second part, if Y is a component of M^A, then $\pi(Y)$ is connected and fixed by A, so $\pi(Y) \subset W$, for some $W \in \mathrm{Fix}(N)_{\geq i}$. Then there is a pullback square:

$$\begin{array}{ccc} \pi^{-1}\pi(Y) & \longrightarrow & \pi^{-1}W \\ \downarrow & & \downarrow \\ \pi(Y) & \longrightarrow & W \end{array}$$

which is isomorphic to

$$\begin{array}{ccc} \pi(Y) \times K & \longrightarrow & W \times K \\ \downarrow & & \downarrow \\ \pi(Y) & \longrightarrow & W \end{array}$$

We see that $Y \subset \pi^{-1}\pi(Y)$, and as Y is connected Y is contained in a unique component $W \times k \subset W \times K$. But then as Y is a component of M^A and Y is fixed by A, we must have that that Y is all of $W \times k$ □

Corollary 4.2.4. *An i-trivial map $\pi : M \to N$ induces a map $\pi_* : \mathrm{Fix}(M)_{\geq i} \to \mathrm{Fix}(N)_{\geq i}$ by $Y \mapsto \pi(Y)$, and this map gives $\mathrm{Fix}(M)_{\geq i} \to \mathrm{Fix}(N)_{\geq i}$ the structure of a principal K-bundle, as defined in section 2.5.*

Proof. The previous lemma shows that π_* defines a map $\mathrm{Fix}(M)_{\geq i} \to \mathrm{Fix}(N)_{\geq i}$ compatible with the ranking. To see that it is a map of posets we must show that if $Y \subset Y'$ then $\pi(Y) \subset \pi(Y')$, which is completely obvious.

The previous lemma also shows that over each $W \in \mathrm{Fix}(N)_{\geq i}$, $\pi^{-1}W \cong W \times K$, but we also need to see that this is true over each chain in $\mathrm{Fix}(N)_{\geq i}$. So, given a chain $W_1 \to \ldots \to W_k$ is a chain in $\mathrm{Fix}(N)_{\geq i}$, consider $\pi^{-1}W_1$. Choosing an isomorphism $\pi^{-1}W_1 \cong W_1 \times K$ determines an isomorphism $\pi^{-1}W_i \times K$ for all i because $W_1 \times e$ lies in a unique component of $\pi^{-1}W_i$, so we have shown that $\pi_* : \mathrm{Fix}(M)_{\geq i} \to \mathrm{Fix}(N)_{\geq i}$ is a covering map.

To see that it is a principal K-bundle, we note that the K-action on M induces a K action on $\mathrm{Fix}(M)_{\geq i}$, and that $\mathrm{Fix}(M)_{\geq i}/K = \mathrm{Fix}(N)_{\geq i}$. □

Denote the functors filtering $H^*_S M$ and $H^*_S N$ by F and G respectively.

Lemma 4.2.5. *The K-action on M makes $(F_i H^*_S M, \mathrm{Fix}(M)_{\geq i}, F)$ a K-Duflot module.*

Proof. We only need to define the natural transformations $Fk \mapsto F$, and these comes from the restriction maps $H_S^*Y \leftarrow H_S^*kY$ induced by the K action, that they satisfy the axioms from 2.5.10 is immediate from the fact that the maps are coming from a group action. □

Theorem 4.2.6. *There is an equivariant morphism*

$$\pi : (F_iH_S^*M, \mathrm{Fix}(M)_{\geq i}, F) \to (F_iH_S^*N, \mathrm{Fix}(N)_{\geq i}, G)$$

making

$$\pi : (F_iH_S^*M, \mathrm{Fix}(M)_{\geq i}, F) \to (F_iH_S^*N, \mathrm{Fix}(N)_{\geq i}, G)$$

into a K-bundle, and the induced map $F_iH_S^*N \to F_iH_S^*M$ is the map coming from restriction $H_S^*N \to H_S^*M$.

Proof. The only piece of data we haven't defined yet is the natural transformation $G\pi_* \Rightarrow F$, and this also comes from the restriction maps $H_S^*(\pi(Y)) \to H_S^*Y$. That all the given data satisfies the requirements of 2.5.10 is immediate.

To see that the map $F_iH_S^*N \to F_iH_S^*M$ induced by the equivariant morphism agrees with the map induced by the map of spaces $M \to N$, we appeal to the uniqueness result of 2.5.5 □

Theorem 4.2.7. *There is a spectral sequence with $E_2 = H^p(K, \mathcal{H}^q(F_iH_S^*N))$ converging to $\mathcal{H}^{p+q}F_iH_S^*M$.*

Proof. This follows immediately from 2.5.16. □

Now, we can connect this to the group theory.

Definition 4.2.8. For i less than or equal to the p-rank of G, we say that an extension $1 \to H \to G \to K \to 1$ with K finite is *i-trivial* if for all $j \geq i$, if E is a rank j p-torus of G, then $C_GE < H$.

Remark 4.2.9. The main example we will be concerned with are iterated wreath products. For example, the extension $1 \to (\mathbb{Z}/p)^p \to \mathbb{Z}/p \wr \mathbb{Z}/p \to \mathbb{Z}/p \to 1$ for $p \geq 3$ is 3-trivial.

Proposition 4.2.10. *If $H \to G \to K$ is i-trivial, and V is a faithful representation of G, then $K \to H\backslash U(V) \to G\backslash U(V)$ is i-trivial.*

Proof. To show this, we use our description of the fixed points of $G\backslash U(V)^A$, where $A < S$. Recall that the connected component of $Gu \in G\backslash U(V)^A$ is isomorphic to $C_G^u A\backslash C_U(V)^u A$. By the i-triviality assumption, this is isomorphic to $C_H^u A\backslash C_{U(V)}^u A$.

Now the points of $H\backslash U(V)$ lying over Gu are $\{Hku : k \in K\}$. Under the i-triviality assumption each Hku is also fixed by A, and the connected component of each Hku is $C_H^{ku} A\backslash C_{U(V)}^{ku}$. Recall how S acts on one of the $C_H^u A\backslash C_{U(V)}^u A$: it is via the natural action of S^u on $C_H^u A\backslash C_{U(V)}^u A$, and the twist map $S \to S^u$. Therefore these are all isomorphic as S-manifolds.

In order to complete the proof, we just need to show that these components are all disjoint. So, suppose that $Hgu = Hg'uc$, where $c \in C_{U(V)}A$. We wish to show that g and g' have the same image in K. We have that $Hg = Hg'ucu^{-1}$. So, $ucu^{-1} \in G$. We also have that $ucu^{-1} \in C_{U(V)}^{uA}$, so $ucu^{-1} \in C_G^u A$. Therefore by i-triviality, $ucu^{-1} \in H$ as well, so $Hg = Hg'h$, so g and g' have the same image in H and we are done. □

Putting it all together, 4.2.7 gives us a spectral sequence computing the local cohomology of $H_S^* G\backslash U(V)$ when we have an i-trivial extension.

Theorem 4.2.11. *If $1 \to H \to G \to K \to 1$ is i-trivial, and V a faithful representation of G, then there is a spectral sequence starting at the E_2 page: $H^p(K, \mathcal{H}^q F_i H_S^* H\backslash U(V)) \Rightarrow \mathcal{H}^{p+q} F_i H_S^* G\backslash U(V)$.*

4.3 The top $p-2$ local cohomology modules of $H_S^* W(n)\backslash U(V)$

Here, we give an application of the theory we have developed so far to do some computations in local cohomology. Our goal will be to exploit i-triviality to get at the top local cohomology modules of $H_S^* W(n)\backslash U(V)$, where $W(1)$ is $\mathbb{Z}/p$ and for $n > 1$ $W(n)$ is $W(n-1) \wr \mathbb{Z}/p$, and V is any faithful representation of $W(n)$.

Recall that $W(n)$ is the p-Sylow of S_{p^n}. The depth of $H^*_{W(n)}$ is known by [CH95], it is n. The Krull dimension is p^{n-1}, so these groups have a large difference between their depth and dimension and consequently a lot of room for nontrivial local cohomology. However, other than the regularity theorem, which was proved for these groups by Benson in [Ben08], nothing is known about the structure of their local cohomology modules.

For a p-group H, let $r(H)$ (the rank of H) be the maximal rank of a p-torus of H, i.e. the dimension of H^*_H. Let $G = H \wr \mathbb{Z}/p$ be the split extension $H^p \to G \to \mathbb{Z}/p$, and denote the generator of $\mathbb{Z}/p$ by σ.

Theorem 4.3.1. *We have that $H^p \to G \to \mathbb{Z}/p$ is $r(H) + 2$ trivial.*

We'll prove this in a series of lemmas.

Lemma 4.3.2. *Any subgroup of $H \wr \mathbb{Z}/p$ not contained in H^p is generated by a subgroup K of H^p and an element of $H \wr \mathbb{Z}/p - H^p$.*

Proof. Suppose that g', h' are elements of $H \wr \mathbb{Z}/p - H^p$. We will show that the subgroup generated by g' and h' is equal to the subgroup generated by g' and k, where $k \in H^p$.

First, write $g' = g''\sigma^j$, and $h' = h''\sigma^l$, where $g'', h'' \in H^p$ and $j, l \not\equiv 0 \pmod p$. Then there are $m, n \not\equiv 0 \pmod p$ so that $(g')^m = g\sigma$, $(h')^n = h\sigma$. But since H and therefore $H \wr \mathbb{Z}/p$ are p-groups, $(g')^m$ and $(h')^n$ generate the same subgroup as g' and h', so we can reduce to the case where we have two elements of the form $g\sigma$ and $h\sigma$. But then $h\sigma(g\sigma)^{-1} \in H^p$, and we are done. ∎

Lemma 4.3.3. *Given an element $g\sigma^j$ of $H \wr \mathbb{Z}/p - H^p$, the maximal rank of a p-torus of H^p centralized by $g\sigma^j$ is $r(H)$.*

Proof. We want to classify the elements of H^p that commute with $g\sigma^j$. As above, by raising $g\sigma^j$ to the appropriate power we can reduce to the case that $j = 1$. Then if $h = (h_1, \ldots, h_p)$ commutes with $g\sigma = (g_1, \ldots, g_p)\sigma$, we have that $hg\sigma h^{-1} = g\sigma$, so $hg(\sigma \cdot h^{-1})\sigma = g\sigma$, so $hg(\sigma \cdot h)^{-1} = g$.

Therefore we have the equations:

$$
\begin{aligned}
h_1 g_1 h_p^{-1} &= g_1 \\
h_2 g_2 h_1^{-1} &= g_2 \\
&\dots \\
h_i g_i h_{i-1}^{-1} &= g_i \\
&\dots \\
h_p g_p h_{p-1}^{-1} &= g_p
\end{aligned}
\tag{4.3.1}
$$

Therefore, h_1 determines at most one h that commutes with $g\sigma$. But if $g\sigma$ is to centralize a p-torus of H^p, all the choices for the first coordinate of h must commute with one another, so they must lie in a p-torus of H, giving us our result. □

Proof of the theorem. We must show that if E is a p-torus of rank greater than $r(H)+1$ then the centralizer of E lies entirely in H^p.

First, we will show that E lies in H^p. If not, by the first lemma E is generated by $E' < H^p$ and an element g of $H \wr \mathbb{Z}/p - H^p$. But E' must then be a p-torus of rank greater than $r(H)$ that commutes with an element of $H \wr \mathbb{Z}/p - H^p$, which contradicts the second lemma.

Now the second lemma finishes the proof. □

Lemma 4.3.4. *If $H \to G$ is i-trivial, then $H^n \to G^n$ is $(n-1)r(H)+i$ trivial.*

Recall that if H is i-trivial in G by definition i is less than or equal to the p-rank of G, so the p-rank of H is equal to the p-rank of G.

Proof. The i-triviality condition ensures that for E a p-torus of rank greater than or equal to $(n-1)r(H)+i$ in G^n, the rank of image $\pi_j E$ is greater than or equal to i for each j (π_j is the j^{th} projection map). So, if $c \in G^n$ commutes with E, then as $\pi_j(c)$ commutes with $\pi_j E$, so $\pi_j(c) \in H$, and $c \in H^n$. □

Lemma 4.3.5. *If H is normal in G, then H^p is normal in $G \wr \mathbb{Z}/p$.*

Proof. This follows because H^p is invariant under the $\mathbb{Z}/p$ action on G^p and is normal in G^p. ☐

Corollary 4.3.6. *If H is i-trivial in G, and $(p-1)r(H)+i$ is greater than $r(G)+2$, then H^p is $(p-1)r(H)+i$-trivial in $G \wr \mathbb{Z}/p$*

Lemma 4.3.7. *Let $p \geq 5$, and let $W(n)$ denote the $n-1$-fold iterated wreath product of $\mathbb{Z}/p$ with itself, with $W(1) = 1 \wr \mathbb{Z}/p = \mathbb{Z}/p$. Then for $n > 1$, $W(n)$ has a unique, normal, maximal rank p-torus $E(n)$ of rank p^{n-1}, $E(n)$ is $p^{n-1}-p+3$-trivial in $W(n)$, and $W(n)/E(n) \cong W(n-1)$.*

Proof. First, we will define $E(n)$, show that it is normal, has maximal rank, and that the quotient is $W(n-1)$. Then we will show that it is $p^{n-1}-p+3$ trivial, which implies that it is the unique maximal p-torus.

We define $E(n)$ inductively: $E(1) = W(1) = \mathbb{Z}/p$, and for $n > 1$ $E(n) = (E(n-1))^p < W(n-1)^p < W(n)$. This shows that $E(n)$ is normal by 4.3.5. To see that $E(n)$ has maximal rank, we can note that $W(n)$ is the p-Sylow of S_{p^n}, where it is clear that a maximal rank p-torus has rank p^{n-1}.

To determine that the quotient is $W(n-1)$, we again proceed inductively (noting that this is true for $n = 2$). Note that we have the map of extensions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & W(n-1)^p & \longrightarrow & W(n) & \longrightarrow & \mathbb{Z}/p & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & W(n-1)^p/E(n) & \longrightarrow & W(n)/E(n) & \longrightarrow & \mathbb{Z}/p & \longrightarrow & 1 \end{array}$$

So $W(n)/E(n)$ fits into an extension of $\mathbb{Z}/p$ by $W(n-1)^p/E(n-1)^p$, which is $W(n-2)^p$ by hypothesis. Then a section $\mathbb{Z}/p \to W(n)$ determines a section of $W(n)/E(n) \to \mathbb{Z}/p$, and the induced action of $\mathbb{Z}/p$ on $W(n-1)^p/E(n-1)^p$ cyclically permutes the factors, so $W(n)/E(n)$ is $W(n-1)$.

Now, it remains to show that $E(n)$ is $p^{n-1}-p+3$ trivial. First, this is true for $n = 2$. We have that $W(2) = \mathbb{Z}/p \wr \mathbb{Z}/p$ and $E(2) = \mathbb{Z}/p^p$, and this is 3 trivial by 4.3.1. Now, suppose it is true for $W(n-1)$, and consider $W(n)$. Since $r(E(n-1)) = r(W(n-1))$, 4.3.6 tells us that $E(n) = E(n-1)^p$ is $(p-1)p^{n-2} + p^{n-2} - p + 3$ trivial in $W(n)$, and $(p-1)p^{n-2} + p^{n-2} - p + 3 = p^{n-1} - p + 3$. □

Now, applying our result 4.2.11 about spectral sequences for i-trivial extensions, we have the following:

Proposition 4.3.8. *There is a spectral sequence*

$$H^i(W(n-1), \mathcal{H}^j(F_{p^{n-1}-p+3}H_S^*(E(n)\backslash U(V))) \Rightarrow \mathcal{H}^{i+j}(F_{p^{n-1}-p+3}H_S^*(W(n)\backslash U(V))).$$

Note that $\mathcal{H}^j(F_{p^{n-1}-p+3}H_S^*(E(n)\backslash U(V)))$ is zero except for $j = p^{n-1}$ and possibly for $j = p^{n-1} - p + 3$, that $\mathcal{H}^{i+j}(F_{p^{n-1}-p+3}H_S^*(W(n)\backslash U(V)))$ is zero for $i + j < p^{n-1} - p + 3$ and $i + j > p^{n-1}$, and that $\mathcal{H}^{i+j}(F_{p^{n-1}-p+3}H_S^*(W(n)\backslash U(V))) = \mathcal{H}^{i+j}(H_S^*(W(n)\backslash U(V)))$ for $i + j > p^{n-1} - p + 3$. The last equality follows from 2.5.1.

This tells us that the spectral sequence is concentrated in two rows, so the only differential is a $d_{p-3+1} : E_{p-3+1}^{i,p^{n-1}} \to E_{p-3+1}^{i+p-3+1,p^{n-1}-(p-3)}$. Therefore this differential must be an isomorphism for $i > 0$ and a surjection for $i = 0$, which tells us that:

$$\mathcal{H}^i H_S^*(W(n)\backslash U(V)) = H^{i-(p^{n-1}-(p-3))}(W(n-1), \mathcal{H}^{p^{n-1}-(p-3)}(F_{p^{n-1}-(p-3)}H_S^*(E(n)\backslash U(V)))) \quad (4.3.2)$$

for $p^{n-1} - (p-3) < i < p^{n-1}$, and:

$$\mathcal{H}^{p^{n-1}} H_S^*(W(n)\backslash U(V)) = H^{(p-3)}(W(n-1), \mathcal{H}^{p^{n-1}-(p-3)}(F_{p^{n-1}-(p-3)}H_S^*(E(n)\backslash U(V)))) \oplus (\ker : d_{p-2} : E_2^{0,p^{n-1}} \to E_2^{p-2,p^{n-1}-(p-3)}). \quad (4.3.3)$$

We would like to have some expression of $\mathcal{H}^i(H_S^*W(n)\backslash U(V))$ that doesn't reference the Duflot filtration, which we can achieve if we can relate $H^{i-(p^{n-1}-(p-3))}(W(n-1), \mathcal{H}^{p^{n-1}-(p-3)}(F_{p^{n-1}-(p-3)}H_S^*(E(n)\backslash U(V))))$ to the group cohomology of $W(n-1)$ with coefficients in $\mathcal{H}^{p^{n-1}}H_S^*(E(n)\backslash U(V))$. Fortunately, we are able to do this.

Recall how the spectral sequence at hand is constructed: it is one of the hypercohomology spectral sequences for $W(n-1)$ acting on the Duflot complex for one level of $H^*_S(E(n)\backslash U(V))$. We know what it is converging to since the Duflot complex for this level of $H^*_S E(n)\backslash U(V)$ is a complex of free $W(n-1)$ modules, and the fixed point set is the Duflot complex for $H^*_S W(n)\backslash U(V)$.

If instead of taking the hypercohomology spectral sequence we constructed a hyper-Tate cohomology spectral sequence, we would have:

$$E_2^{i,j} = \widehat{H}^i(W(n), \mathcal{H}^j(F_{p^{n-1}-(p-3)}(H^*_S E(n)\backslash U(V)))).$$

This agrees with the E_2 page of the previous spectral sequence for $i > 0$, and it converges to 0 because the Tate cohomology of a free module is zero. This tells us that for all i, $\widehat{H}^i(W(n-1), \mathcal{H}^{p^{n-1}}(H^*_S E(n)\backslash U(V))) \xrightarrow{\sim} \widehat{H}^{i+p-2}(W(n-1), \mathcal{H}^{p^{n-1}-(p-3)}(F_{p^{n-1}-(p-3)} H^*_S E(n)\backslash U(V)))$.

This gives us the following computation for the local cohomology of $H^*_S W(n)\backslash U(V)$.

For a G-module A, let $N : A \to A^G$ denote the map $a \mapsto \sum_{g\in G} g \cdot a$, and recall that the kernel of the map $H^0(G, A) \to \widehat{H}^0(G, A)$ is image N.

Theorem 4.3.9. *For $0 < i < p - 3$,*

$$\mathcal{H}^{p^{n-1}-(p-3)+i} H^*_S(W(n)\backslash U(V)) \cong \widehat{H}^{i-(p-2)}(W(n-1), \mathcal{H}^{p^{n-1}} H^*_S E(n)\backslash U(V)).$$

This isomorphism is as $H^*_S W(n)\backslash U(V)$-modules.

For the top local cohomology, we have that $\mathcal{H}^{p^{n-1}}(H^*_S W(n)\backslash U(V))$ is isomorphic as vector spaces to $\widehat{H}^{-1}(W(n-1), \mathcal{H}^{p^{n-1}}(H^*_S E(n)\backslash U(V))) \oplus N(\mathcal{H}^{p^{n-1}}(H^*_S E(n)\backslash U(V)))$.

Proof. Everything follows from our computation with Tate cohomology, the only part that isn't immediate is the second half of the direct sum in the top local cohomology, and this comes from our identifying the kernel of the map d_{p-2} on $E_2^{0,p^{n-1}}$ of the original spectral sequence with the kernel from H^0 to $\widehat{H}^0$. □

By [DGI06] there is a local cohomology spectral sequence $\mathcal{H}^*(H^*_S E(n)\backslash U(V)) \Rightarrow (H^*_S E(n)\backslash U(V))^*$. But $H^*_S E(n)\backslash U(V)$ is Cohen-Macaulay, so the spectral sequence collapses and we have that:

$$\mathcal{H}^{p^{n-1}} H^*_S E(n)\backslash U(V) = \Sigma^{-p^{n-1}+d}(H^*_S E(n)\backslash U(V))^*$$

Here $d = \dim U(V)$.

So, we can rewrite the computation of the top local cohomology modules of $H_S^* W(n)\backslash U(V)$ without reference to local cohomology.

Theorem 4.3.10. *Denote the dimension of* $U(V)$ *by* d.

For $0 < i < p-3$*, we have an isomorphism* $H_{W(n)}^*$*-modules:*

$$\mathcal{H}^{p^{n-1}-(p-3)+i} H_S^*(W(n)\backslash U(V)) = \widehat{H}^{i-(p-2)}(W(n-1), \Sigma^{-p^{n-1}+d}(H_S^* E(n)\backslash U(V))^*).$$

For the top local cohomology, we have that $\mathcal{H}^{p^{n-1}}(H_S^* W(n)\backslash U(V)) \cong \widehat{H}^{-1}(W(n-1), \Sigma^{-p^{n-1}+d}(H_S^* E(n)\backslash U$ $N(\Sigma^{-p^{n-1}+d}(H_S^* E(n)\backslash U(V))^*)$*. This isomorphism is only as graded* $\mathbb{F}_p$ *vector spaces, there is an extension problem to solve to compute the* $H_{W(n)}$*-module structure.*

As mentioned in the introduction, we can now show that there are groups whose cohomology has arbitrarily long sequences of nonzero local cohomology modules.

Corollary 4.3.11. *For* $0 \leq i < p-3$*,* $\mathcal{H}^{p^{n-1}-i}(H_{W(n)}^*) \neq 0$.

Proof. That the top local cohomology is nonzero is immediate from the fact that the Krull dimension is equal to the top degree in which local cohomology is nonvanishing, so we just need to show the result for $0 < i < p-3$.

By 4.1.1 is enough to show that the result is true with $H_S^*(W(n)\backslash U(V))$ in place of $H_{W(n)}^*$.

For this, we can use 4.3.10. This tells us that:

$$\mathcal{H}^{p^{n-1}-(p-3)+i}(H_S^* W(n)\backslash U(V)) = \widehat{H}^{i-(p-2)}(W(n-1); \Sigma^{-p^{n-1}+d}(H_S^* E(n)\backslash U(V))^*).$$

Recall that local cohomology is bigraded. We have that:

$$\mathcal{H}^{p^{n-1}-(p-3)+i, -p^{n-1}+d}(H_S^* W(n)\backslash U(V)) = \widehat{H}^{i-(p-2)}(W(n-1); \mathbb{F}_p).$$

But $H_{W(n-1)}^*$ is nonzero in all degrees since it is a p-group, so we are done. □

We have proved:

Proposition 4.3.12. *For each $p \geq 5$, for each n there exists a p-group G and an i so that $\mathcal{H}^{i+j}(H^*_G) \neq 0$ for all $0 < j < n$, and so that $i + j$ is not the dimension of an associated prime.*

The regularity theorem states that for each finite group G, $\mathcal{H}^{i,j} H^*_G = 0$ for $j > -i$ and for some i $\mathcal{H}^{i,-i} H^*_G \neq 0$. In fact, it is known that if r is the dimension H^*_G, then $\mathcal{H}^{r,-r} H^*_G \neq 0$, and it is conjectured that for all other i we have that $\mathcal{H}^{i,-i} H^*_G = 0$.

In fact, for the groups $W(n)$ in the range we have been studying more is true.

Corollary 4.3.13. *For $0 \leq i < p-3$, $\mathcal{H}^{p^{n-1}-i,j} H^*_{W(n)} = 0$ for $j > -p^{n-1}$.*

Proof. For $i = 0$ the result is true by the regularity theorem, so we just need to show the result for $i > 0$.

Recall that by 4.1.1, $H^*_S W(n)\backslash U(V) \cong H^*_{W(n)} \otimes H^* W(n)\backslash U(V)$ as $H^*_{W(n)}$ modules. So, $\mathcal{H}^*(H^*_S(W(n)\backslash U(V)) \cong \mathcal{H}^*(H^*_{W(n)}) \otimes H^* W(n)\backslash U(V)$. So, as $W(n)\backslash U(V)$ is an oriented N dimensional manifold (where $N = \dim U(V)$) the top nonvanishing degree of its cohomology is N. Therefore the top nonzero degree of $\mathcal{H}^i H^*_S W(n)\backslash U(V)$ is N plus the top nonzero degree of $\mathcal{H}^i H^*_{W(n)}$.

So, we need to show that for $0 < i < p-3$, $\mathcal{H}^{p^{n-1}-i,j+N} H^*_S W(n)\backslash U(V) = 0$ for $j > -p^{n-1}$. For this, by 4.3.10 we have that $\mathcal{H}^{p^{n-1}-(p-3)+i} H^*_S(W(n)\backslash U(V)) = \widehat{H}^{i-(p-2)}(W(n-1), \Sigma^{-p^{n-1}+N}(H^*_S E(n)\backslash U(V))^*)$. But since $H^*_S E(n)\backslash U(V)$ is concentrated in positive degrees, its dual is concentrated in negative degrees, so $\Sigma^{-p^{n-1}+N}(H^*_S E(n)\backslash U(V))^*)$ is zero above degree $-p^{n+1} + N$, and the result follows. □

Chapter 5

STRATIFICATION OF COCHAINS

5.1 Introduction

One of the exciting reasons to study the geometry of group cohomology rings is because of a non-obvious connection to representation theory. This story starts in stable homotopy theory, with the nilpotence theorem of Devinatz, Hopkins, and Smith [DHS88].

Using the nilpotence theorem, Hopkins and Smith [HS98] were able to classify the thick subcategories of the compact objects in the stable homotopy category. Even though the stable homotopy category is tremendously complicated, its thick subcategories can be described and have a relatively simple structure: roughly speaking they look like $\operatorname{Spec}\mathbb{Z}$, but where in place of each of the prime numbers there is an infinite tower of thick subcategories.

This idea of studying the thick subcategories of complicated triangulated categories took root, and Benson, Carlson, and Rickard [BCR97] classified the thick tensor ideals of the stable module category of a finite group G, and showed that they are in bijection with specialization closed subsets of $\operatorname{Proj} H^*_G$. Later, [FP07] showed how to put the structure of a locally ringed space on the set of prime thick tensor ideals in the stable module category, and showed that the set of prime thick tensor ideals of the stable module category is isomorphic as locally ringed spaces to $\operatorname{Proj} H^*_G$.

Balmer [Bal10] associated to every essentially small tensor triangulated category $\mathcal{T}$ a locally ringed space now called the Balmer spectrum of $\mathcal{T}$, or just $\operatorname{Spec}\mathcal{T}$. In this language, the spectrum of the stable module category of G is $\operatorname{Proj} H^*_G$. This gives a representation theoretic interpretation of any geometric fact about group cohomology.

Benson, Iyengar, and Krause [BIK11c] were also able to classifying the localizing subcategories of the big stable module category of G, $\mathbf{Stmod}\,\mathbb{F}_pG$, via the notion of an action of a ring on a triangulated category "stratifying" the triangulated category. Some of the precursors to these ideas are contained in the work of Hovey, Palmieri, and Strickland in [HPS97], which gives a framework for translating results and techniques from stable homotopy theory to the setting of tensor triangulated categories. In their work, Benson, Iyengar, and Krause study an enlargement of $\mathbf{Stmod}\,\mathbb{F}_pG$, the homotopy category of complexes of injective $\mathbb{F}_pG$-modules. When G is a p-group, $K(\mathrm{Inj}\,\mathbb{F}_pG)$ is equivalent as tensor triangulated categories to the derived category of C^*BG-modules, and in general the derived category of C^*BG-modules is equivalent to the localizing subcategory of $K(\mathrm{Inj}\,\mathbb{F}_pG)$ generated by an injective resolution of $\mathbb{F}_p$.

There is a long history dating back to Quillen and even earlier to use compact Lie groups to study the cohomology rings of finite groups, and the previous part of this thesis has been devoted to studying group cohomology rings using the equivariant cohomology of manifolds. However, currently there aren't applications of compact Lie groups or of equivariant cohomology to the study of modular representation theory directly.

Of course, C^*BG is a perfectly fine ring spectrum even if G is a compact Lie group, and it makes sense to study its derived category. Benson and Greenlees [BG14] and Barthel, Castellana, Heard, and Valenzuela [BCHV17] show that this category is stratified by the action of the coefficient ring, which is just H^*_G. We have that $\mathrm{Spec}(D(C^*BG)^c)$ is the homogeneous spectrum of H^*_G, denoted by $\mathrm{Spec}^h\, H^*_G$, so any geometric features of H^*_G have an interpretation in $D(C^*BG)^c$.

Consequently, we would like to understand how the Duflot filtration fits into this picture. The first step is to show that for M a G-manifold, $D(C^*(EG \times_G M))$ is also stratified by $H^*_G M$, which is the goal of this chapter.

The background material for this chapter is primarily in a series of papers of Benson, Iyengar, and Krause [BIK08, BIK11b, BIK11c, BIK11a], as well as a paper by Barthel, Heard, Castellana, and Valenzuela [BCHV17], which generalizes some of Benson, Iyengar, and Krause's work in a more homotopical setting. In the next two sections we will recall their definitions and theorems.

5.2 Local (co)homology and (co)support

We work in a good category of spectra such as the $\mathbb{S}$-modules of [EKMM97]. In such a category the commutative monoid objects are E_∞-ring spectra; we will refer to these commutative monoid objects as commutative ring spectra. This means that for R a commutative ring spectrum, the homotopy category of R-modules, or $D(R)$, has the structure of a tensor triangulated category, where the tensor is $\wedge_R$.

The homotopy groups of R inherit a ring structure and we say that R is Noetherian if $\pi_* R$ is Noetherian. The homotopy groups of R act on $D(R)$ in the sense of [BIK08]: we have maps $\pi_* R \to \pi_*(\operatorname{End}_R M)$ so that the two induced actions on $\pi_*(\hom_R(M, N))$ are compatible.

In [BIK08] a support theory is developed for a triangulated category with an action of a commutative ring. This support theory is related to the support theory developed in Chapter 6 of [HPS97] in the context of axiomatic stable homotopy theory. In what follows we will describe their definitions and conclusions in the special case of the derived category of a Noetherian commutative ring spectrum. In this setting the theory is slightly simplified.

Notation 5.2.1. We will denote our fixed Noetherian commutative ring spectrum by R, and denote $\pi_* R$ by A. We denote $D(R)$ by T.

The monoidal unit of T is R, so we denote R by 1. We use hom for the internal hom and Hom for the categorical hom, which is related to the internal one by taking the 0^{th} homotopy group.

In [BIK08] the authors show that if $\mathcal{V}$ is a specialization closed subset of $\operatorname{Spec}^h A$, then the subcategory $T_\mathcal{V}$ consisting of those objects $X \in T$ which have $\pi_* X$ supported in $\mathcal{V}$ (that is if $\mathfrak{p} \in \operatorname{Spec}^h A$ and $\pi_*(X)_\mathfrak{p} \neq 0$, then $\mathfrak{p} \in \mathcal{V}$) forms a localizing subcategory of T denoted by $T_\mathcal{V}$. This is Lemma 4.2 and Lemma 4.3 of [BIK08].

Benson, Iyengar, and Krause use Brown representability to construct a localization functor $L_\mathcal{V} : T \to T$ whose kernel is $T_\mathcal{V}$, and such that every $X \in T$ fits in a localization triangle: $\Gamma_\mathcal{V} X \to X \to L_\mathcal{V} X$.

For $\mathfrak{p} \in \operatorname{Spec}^h A$ define $\mathcal{V}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec}^h A : \mathfrak{p} \subset \mathfrak{q}\}$, and $\mathcal{Z}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec}^h A : \mathfrak{q} \not\subset \mathfrak{p}\}$.

Definition 5.2.2. We define functors $\Gamma_{\mathfrak{p}} : T \to T$ for $\mathfrak{p} \in \operatorname{Spec}^h A$ by $\Gamma_p = \Gamma_{\mathcal{V}(\mathfrak{p})} L_{\mathcal{Z}(\mathfrak{p})}$. We call these functors *local cohomology functors.*

This definition is in the beginning of Section 5 of [BIK08].

One justification of the terminology is that the objects in $\Gamma_{\mathfrak{p}} T$ are exactly those which are $\mathfrak{p}$-local and $\mathfrak{p}$-torsion.

Proposition 5.2.3 ([BIK08] Theorem 4.7, Lemma 2.4, Corollary 4.10)**.** *For $X \in T$:*

1. *We have $\pi_*(L_{\mathcal{Z}(\mathfrak{p})} X) = \pi_*(X)_{\mathfrak{p}}$.*

2. *We have $X \in \Gamma_{\mathfrak{p}} T$ if and only if $\pi_* X$ is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion.*

Benson, Iyengar, and Krause also develop a dual notion of local homology.

Lemma 5.2.4 ([BIK12] Section 4)**.** *Each functor $\Gamma_{\mathfrak{p}} : T \to T$ has a right adjoint, and $\Gamma_{\mathfrak{p}} T$ is a localizing subcategory of T.*

Definition 5.2.5 ([BIK12] Section 4)**.** For each $\mathfrak{p} \in \operatorname{Spec}^h A$ we define the *local homology functor* with respect to $\mathfrak{p}$, or $\Lambda^{\mathfrak{p}}$, to be the right adjoint of $\Gamma_{\mathfrak{p}}$.

Lemma 5.2.6 ([BIK12] Proposition 4.16)**.** *Each $\Lambda^{\mathfrak{p}} T$ is a colocalizing subcategory of T.*

Theorem 5.2.7. *[[BIK08], [BIK12] Corollary 8.3, Proposition 8.3]*

1. *For $M \in T$ we have $\Gamma_{\mathfrak{p}} M = \Gamma_{\mathfrak{p}} 1 \wedge_R M$.*

2. *For $M \in T$, we have $\Lambda^{\mathfrak{p}} M = \hom(\Gamma_{\mathfrak{p}} 1, M)$.*

Now that we have local (co)homology functors, we can define (co)support.

Definition 5.2.8. [[BIK08], [BIK12]]

1. For $M \in T$, we define the *support* of M, or $\operatorname{supp}_R M$ (or just $\operatorname{supp} M$ if T is obvious), by $\operatorname{supp} M = \{\mathfrak{p} \in \operatorname{Spec}^h A : \Gamma_\mathfrak{p} M \neq 0\}$.

2. For $M \in T$, we define the *cosupport* of M, or $\operatorname{cosupp}_R M$ (or just $\operatorname{cosupp} M$ if T is obvious), by $\operatorname{cosupp} M = \{p \in \operatorname{Spec}^h A : \Lambda_\mathfrak{p} M \neq 0\}$.

Here we gather some convenient properties of support and cosupport that hold unconditionally. We focus on support and give only a few properties of cosupport that we will subsequently use.

Theorem 5.2.9. *1. For $M \in T$, $M = 0$ if and only if $\operatorname{supp} M = \operatorname{cosupp} M = \varnothing$.*

2. For $A \to B \to C$ a triangle in T, we have $\operatorname{supp} B \subset \operatorname{supp} A \cup \operatorname{supp} C$, and $\operatorname{supp}(A \oplus C) = \operatorname{supp} A \cup \operatorname{supp} C$.

3. For $A, B \in T$, $\operatorname{supp} A \wedge_R B \subset \operatorname{supp} A \cap \operatorname{supp} B$.

4. For $A, B \in T$, $\operatorname{cosupp} \operatorname{hom}(A, B) \subset \operatorname{supp} A \cap \operatorname{cosupp} B$.

5. For nonzero $M \in \Gamma_\mathfrak{p} T$, $\mathfrak{p} \in \operatorname{cosupp} M$.

6. We have $\operatorname{supp} 1 = \operatorname{Spec}^h A$.

Proof. 1. This is [BIK08] Theorem 5.2 and [BIK12] Theorem 4.5.

2. This is Proposition 5.1 of [BIK08].

3. This follows from $\Gamma_\mathfrak{p}$ being smashing, which is Theorem 5.2.7.

4. This is Lemma 9.3 of [BIK12].

5. This follows from 4.13 of [BIK12].

6. This follows from spectral sequence 1 of Proposition 3.19 from [BHV18].

□

5.3 Stratification

Support and cosupport have particularly nice properties when the (co)localizing categories generated by each local (co)homology functor are minimal (co)localizing subcategories of T. In this case, we say that T is canonically (co)stratified. In this chapter we are particulary interested in support and stratificiation, but cosupport is unavoidable.

Definition 5.3.1 ([BIK11b] Section 4)**.** We say that T is canonically stratified if for each $\mathfrak{p} \in \operatorname{Spec}^h A$ the localizing subcategory $\Gamma_{\mathfrak{p}} T$ is a minimal localizing subcategory.

Remark 5.3.2. This is not the precise definition from [BIK11b]. There they also require that the local to global principal holds, that is that for $X \in T$ we have $\operatorname{loc}_T X = \operatorname{loc}_T\{\Gamma_{\mathfrak{p}} X : \mathfrak{p} \in \operatorname{Spec}^h A\}$. However, in [BIK11b] Theorem 7.2 they show that in our situation, where T is $D(R)$ with the canonical $\pi_* R$ action for a Noetherian ring spectrum R, this automatically holds.

They also allow that the categories $\Gamma_{\mathfrak{p}} T$ are zero, but this cannot happen in this situation for example by 6 of 5.2.9 above.

There is a convenient criterion for checking the minimality of a localizing subcategory.

Proposition 5.3.3. *[[BIK11b] Lemma 4.1] A localizing subcategory $\mathcal{C} \subset T$ is minimal if and only if for every nonzero $M, N \in \mathcal{C}$ we have that* $\hom(M, N) \neq 0$.

In the case when T is canonically stratified, then the support theory is particulary nice.

Theorem 5.3.4. *Suppose that T is canonically stratified. Then in addition to all the properties of 5.2.9, support also satisfies the following:*

1. *For $A, B \in T$,* $\operatorname{supp} A \wedge_R B = \operatorname{supp} A \cap \operatorname{supp} B$.

2. *For $A, B \in T$,* $\operatorname{cosupp} \hom(A, B) = \operatorname{supp} A \cap \operatorname{cosupp} B$.

Proof. One is Theorem 7.3 of [BIK11b], and two is Theorem 9.5 of [BIK12].

□

There are also many interesting structural consequences when stratification holds.

Proposition 5.3.5. *If T is canonically stratified by A, then:*

1. *There is a bijection between the localizing subcategories of T and subsets of* $\operatorname{Spec}^h A$*.*

2. *There is a bijection between thick subcategories of T^c and specialization closed subsets of* $\operatorname{Spec}^h A$*.*

3. *There is an isomorphism of topological spaces between the Balmer spectrum of T^c and* $\operatorname{Spec}^h A$*.*

Proof. Number one is Theorem 4.2 of [BIK11b] and number two is Theorem 6.1 of the same paper. Three follows from 2 essentially by the definition of the Balmer spectrum. □

This classification result is similar to Theorem 6.3.7 of [HPS97]; the local cohomology objects used here play the role of the $K(\mathfrak{p})$ from [HPS97].

For a space X, let C^*X be the ring spectrum $F(\Sigma^\infty X_+, H\mathbb{F}_p)$. Here F is the internal hom in $\mathbb{S}$-modules or any other convenient category of spectra, and $+$ denotes adjoining a disjoint basepoint. Note that the negative homotopy groups of C^*X are the cohomology groups of X. We are especially interested in the case that $X = BG$ for G a compact Lie group. The spectrum C^*BG inherits a commutative ring structure from the commutative ring structure on $H\mathbb{F}_p$ and the coalgebra structure on $\Sigma^\infty BG_+$. For R any commutative ring spectrum, $D(R)$ denotes the homotopy category of R-module spectra, i.e. the category of R-module spectra localized at weak equivalences.

When G is a finite group, there is a close connection between $D(C^*BG)$ and the representation theory of G.

Proposition 5.3.6. *For G a finite group, the localizing subcategory of $K(Inj(\mathbb{F}_pG))$ generated by an injective resolution of $\mathbb{F}_p$ is equivalent as a tensor triangulated category to $D(C^*BG)$. In particular, for G a finite p-group, $D(C^*BG)$ and $K(Inj(\mathbb{F}_pG))$ are equivalent as tensor triangulated categories.*

See [BK08] for a detailed discussion. The crux of the matter is the Rothenburg-Steenrod construction which gives an equivalence between the derived endomorphism ring of $H\mathbb{F}_p$ in the category of $G_+ \wedge H\mathbb{F}_p$-modules and C^*BG.

For G a finite group, it is shown in [BIK11a] that $D(C^*BG)$ is stratified by H^*_G. Their proof uses the techniques of [BIK11c].

Theorem 5.3.7 ([BIK11a]). *For G a finite group, $D(C^*BG)$ is canonically stratified.*

It follows from stratification that support and cosupport work nicely with respect to restricting to a subgroup. For $H < G$, there is a restriction map res : $C^*BG \to C^*BH$. This leads to the following adjoint triple of functors.

$$D(C^*BG) \underset{\text{ind}}{\overset{\text{coind}}{\underset{\xleftarrow[\text{res}]{}}{\rightrightarrows}}} D(C^*BH) \tag{5.3.1}$$

Here restriction is via the map $C^*BG \to C^*BH$, induction is the left adjoint to restriction defined by $M \mapsto M \wedge_{C^*BG} C^*BH$, and coinduction is the right adjoint to restriction defined by $M \mapsto \hom_{C^*BG}(C^*BH, M)$.

It is worth noting that unlike in the situation for finite groups and the stable module category, ind and coind do not always agree.

Corollary 5.3.8 (The Subgroup Theorem). *For G a finite group and $H < G$, we have that for $M \in D(C^*BG)$:*

1. $\operatorname{supp} \operatorname{ind} M = (\text{res}^*)^{-1}(\operatorname{supp} M)$

2. $\operatorname{cosupp} \operatorname{coind} M = (\text{res}^*)^{-1}(\operatorname{cosupp} M)$

Proof. This is Theorem 11.2 of [BIK11c] and Theorem 11.11 of [BIK12], with the language translated into cochains using [BK08]. These can also be derived from Proposition 3.13 of [BCHV17].

□

The subgroup theorem shows that when stratification holds support and cosupport are compatible with maps of ring spectra and the corresponding restriction, induction, and coinduction functors. Here we point out some ways in which support is compatible with arbitrary maps of Noetherian ring spectra.

For $f : R \to S$ a map of ring spectra we write f^* for the induced map $\operatorname{Spec}^h \pi_* S \to \operatorname{Spec}^h \pi_* R$. The map f induces a triple of adjoints as in equation 5.3.1.

Theorem 5.3.9. *Let $f : R \to S$ be a map of Noetherian ring spectra so that the map on homotopy groups is finite, and take $M \in D(R)$, $N \in D(S)$. Then:*

1. $\operatorname{supp}_R \operatorname{res} N = f^* \operatorname{supp}_S N$.

2. $\operatorname{supp}_R \operatorname{res} \operatorname{ind} M \subset \operatorname{supp}_R M$.

3. *For $M \in \Gamma_{\mathfrak{p}} D(R)$ with $\operatorname{coind} \Lambda_{\mathfrak{p}} M \neq 0$, then $\operatorname{cosupp}_S \operatorname{coind} M \cap (f^*)^{-1}\mathfrak{p} \neq \varnothing$.*

Number one is Proposition 3.11 of [BCHV17] and closely related to Propostion 4.3.ii of [BG14], and number two is closely related to Proposition 4.3.ii of [BG14]. Number three is essentially Prop 3.23 of [BCHV17]. For completeness we include their proofs.

Proof. 1. This follows from the fact that $\Gamma_{\mathfrak{p}} \operatorname{res} N = \bigoplus_{\mathfrak{q} \in (f^*)^{-1}\mathfrak{p}} \Gamma_{\mathfrak{q}} N$. This is in turn proved by applying Corollary 3.9 of [BCHV17] to get that $\operatorname{ind} \Gamma_p R = \bigoplus_{\mathfrak{q} \in (f^*)^{-1}\mathfrak{p}} \Gamma_{\mathfrak{q}} S$, and the fact that $\Gamma_p \operatorname{res} N = \operatorname{res}(N \wedge_S \operatorname{ind} \Gamma_{\mathfrak{p}} R)$, which follows from 2.2 of [BCHV17]. Corollary 3.9 of [BCHV17] is essentially Corollary 7.10 of [BIK12] adapated to the situation where induction does not have adjoints on both sides.

Once we have that $\Gamma_{\mathfrak{p}} \operatorname{res} N = \bigoplus_{\mathfrak{q} \in (f^*)^{-1}\mathfrak{p}} \Gamma_{\mathfrak{q}} N$ we see that if $\mathfrak{p} \in \operatorname{supp}_R \operatorname{res} N$ then there is some $\mathfrak{q}$ mapping to $\mathfrak{p}$ with $\mathfrak{q} \in \operatorname{supp}_S N$. This gives us that $\operatorname{supp}_R \operatorname{res} N \subset f^* \operatorname{supp}_S N$. The other direction also follows immediately from $\Gamma_{\mathfrak{p}} \operatorname{res} N = \bigoplus_{\mathfrak{q} \in (f^*)^{-1}\mathfrak{p}} \Gamma_{\mathfrak{q}} N$.

Note: This is essentially the proof given in [BG14]. The proof in [BCHV17] is different and appears to be more general, it doesn't use the fact that the map on homotopy groups is finite.

2. This follows immediately from the fact that $\Gamma_{\mathfrak{p}}$ is smashing. If $\mathfrak{p} \in \operatorname{supp}(\operatorname{res}\operatorname{ind} M)$, then $\Gamma_{\mathfrak{p}}(M \wedge_R S)$ is nonzero, so $\Gamma_{\mathfrak{p}} M$ is nonzero as well.

3. Note that $\hom_R(\Gamma_{\mathfrak{p}} \operatorname{res} S, M)$ is $\operatorname{res}\operatorname{coind} \Lambda^{\mathfrak{p}} M$, so by assumption $\hom_R(\Gamma_{\mathfrak{p}} \operatorname{res} S, M) \neq 0$.

   By Corollary 3.12 of [BCHV17], $\Gamma_{\mathfrak{p}} \operatorname{res} S \in \operatorname{loc}_{D(R)}\{\operatorname{res}\Gamma_{\mathfrak{q}} S : \mathfrak{q} \in (f^*)^{-1}\mathfrak{p}\}$. Now suppose that for all $\mathfrak{q} \in (f^*)^{-1}\mathfrak{p}$ we have $\mathfrak{q} \notin \operatorname{cosupp}\operatorname{coind} M$, so $\hom_S(\Gamma_{\mathfrak{q}} S, \operatorname{coind} M) = \hom_R(\operatorname{res}\Gamma_{\mathfrak{q}} S, M) = 0$. But then since $\Gamma_{\mathfrak{p}} \operatorname{res} S \in \operatorname{loc}_{D(R)}\{\operatorname{res}\Gamma_{\mathfrak{q}} S : \mathfrak{q} \in (f^*)^{-1}\mathfrak{p}\}$ we have $\hom_R(\Gamma_{\mathfrak{p}} \operatorname{res} S, M) = 0$, a contradiction.

$\square$

5.4 Stratification for Borel constructions

In [BG14] and [BCHV17], using homotopy theoretic arguments dating back to Quillen, it is shown that for G a compact Lie group, $D(C^*BG)$ is stratified by the canonical action of H^*_G.

However, it would be desirable to not only have a stratification result for BG, but for Borel constructions on manifolds with a G action. Fortunately, we can show that their arguments still hold in this generality. Fix a compact Lie group G and a finite G CW-complex X. In this section, we denote $C^*EG \times_G X$ by $C^*_G X$. Moreover, we use X as a shorthand for $D(C^*_G X)$, especially in our induction, coinduction, and restriction functors, and in our homs.

Theorem 5.4.1. *We have that $C^*_G X$ is stratified by the canonical action of $H^*_G X$.*

We will prove this by showing how to adapt the proofs [BG14] and [BCHV17] to hold in this context. First we recall the generalization of restriction to p-tori that is relevant to equivariant cohomology.

Definition 5.4.2. The *homotopy orbit category* $\mathcal{O}_G X$ is the category where the objects are homotopy classes of equivariant maps $G/H \to X$, where H is a closed subgroup of

G, and where the morphisms are homotopy commutative triangles where all the maps are equivariant.

Definition 5.4.3. The *Quillen category* $\mathcal{A}_G X$ is the full subcategory of $\mathcal{O}_G X$ spanned by objects of the form $G/E \to X$, where E is a p-torus.

We now recall the Quillen stratification theorem. For $G/E \to X \in \mathcal{A}_G X$, let $V^+_{G/E\to X}$ be the subspace of $\operatorname{Spec}^h H^*_G G/E$ consisting of primes that are not in the image of $\mathrm{res}^* : \operatorname{Spec}^h H^*_G G/E' \to \operatorname{Spec}^h H^*_G G/E$ for any $G/E' \to X \in \mathcal{A}_G X$ admitting a map to $G/E \to X$.

Theorem 5.4.4. *[Qui71] Let A be a set of isomorphism classes of objects of $\mathcal{A}_G X$. We have that* $\operatorname{Spec}^h H^*_G X$ *is isomorphic as sets to* $\bigsqcup_{x\in A} V^+_x$.

In particular, for each $\mathfrak{p} \in \operatorname{Spec}^h H^*_G X$*, there is a unique up to isomorphism orbit in* $\mathcal{A}_G X$ *minimal with respect to* $\mathfrak{p} \in \operatorname{image} \mathrm{res}^* : \operatorname{Spec}^h H^*_G G/E \to \operatorname{Spec}^h H^*_G X$.

Definition 5.4.5. For $\mathfrak{p} \in \operatorname{Spec}^h H^*_G X$, an *originator* is a choice of a minimal orbit $G/E \to X$ together with $\mathfrak{q} \in \operatorname{Spec}^h H^*_G G/E$ so that $\mathrm{res}(\mathfrak{q}) = \mathfrak{p}$, and we say that the pair $(G/E \to X, \mathfrak{q})$ originates $\mathfrak{p}$.

Remark 5.4.6. This category $\mathcal{A}_G X$ is not the precise category used to formulate the stratification theorem in [Qui71], but those results are valid with this category in place of the one that Quillen uses.

Remark 5.4.7. Each object of $\mathcal{A}_G X$ induces a triple of adjoints:

$$D(C^*_G X) \underset{\text{ind}}{\overset{\text{coind}}{\rightleftarrows}} D(C^*_G G/E) \quad (\text{res}: D(C^*_G G/E) \to D(C^*_G X))$$

These all come from the map of ring spectra $C^*_G X \to C^*_G G/E$ induced by the equivariant map $G/E \to X$.

The two keys facts that drive the proof of stratification for $D(C^*_G X)$ are the Quillen stratification theorem stated above, and an analogue of Chouinard's theorem, which in its classical form states that a modular representation of a finite group G is projective if and only if it is projective upon restricting to every p-torus of G.

Proposition 5.4.8. *Chouinard's theorem holds for $D(C_G^*X)$, in the sense that for $M \in D(C_G^*X)$ the following are equivalent.*

1. *M is equivalent to* 0.
2. *For every $G/E \to X \in \mathcal{A}_G X$,* $\operatorname{ind}_X^{G/E} M \sim 0$.
3. *For every $G/E \to X \in \mathcal{A}_G X$,* $\operatorname{coind}_X^{G/E} M \sim 0$.

For X a point this is Theorem 3.1 in [BG14], see also 4.17 of [BCHV17]. The proof of Benson and Greenlees applies immediately to this more general situation; for completeness we include their argument and fill in some details in a series of lemmas.

The proof uses the descent technique we have used throughout the thesis. For $G \hookrightarrow U(n)$ a representation, let F be the G-space $U(n)/S$, where S is the maximal diagonal p-torus, so G acts diagonally on $X \times F$.

Lemma 5.4.9. *The map $C_G^*X \to C_G^*X \times F$ induced by the projection $X \times F \to X$ admits a retract as C_G^*X-modules.*

Proof. As shown in [Qui71] and discussed elsewhere in this thesis, the Serre spectral sequence associated to the bundle $F \to EG \times_G X \times F \to EG \times_G X$ collapses, so $H_G^*X \times F$ is a finitely generated free H_G^*X-module. So, there are cohomology classes $x_1, \ldots, x_n$ in $H_G^*X \times F$ (where $|x_i| = j_i$) so that the map: $\bigoplus_{i=1}^n \Sigma^{j_i}(H_G^*X)_i \to H_G^*X \times F$ induced by mapping $1 \in (H_G^*X)_i$ to x_i is an isomorphism of H_G^*X-modules.

Each class x_i is induced by a map $\eta_i : \mathbb{S}^{-j_i} \to C_G^*X \times F$, so consider the map $\Sigma^{-j_i}C_G^*X \xrightarrow{\sim} C_G^*X \wedge \mathbb{S}^{-j_i} \to C_G^*X \wedge C_G^*X \times F \to C_G^*X$ where the first map is $id \wedge \eta$ and the second is the action map. Upon taking homotopy groups this is the map induced by mapping $1 \mapsto x_i$, so we see that $\bigoplus_{i=1}^n \Sigma^{-j_i}C_G^*X$ is equivalent as modules over C_G^*X to $C_G^*X \times F$, which yields the desired result. $\square$

Lemma 5.4.10. *We have that* $C_G^*X \times F \in \mathrm{Thick}_{X\times F}\{C_G^*G/E : G/E \to X \times F \in \mathcal{A}_G X \times F\}$

Proof. We will show that $\Sigma^\infty(EG\times_G(X\times F))_+$ is a finite sequence of homotopy pushouts of suspensions of the $\Sigma^\infty(EG\times_G G/E)_+$, where the maps defining the colimit are all compatible with the maps $\Sigma^\infty EG\times_G G/E_+ \to \Sigma^\infty EG\times_G X_+$. Since homotopy pushouts can be expressed with cones, the claimed result then follows from taking cochains.

Since the isotropy groups for $X\times F$ are all p-tori, $X\times F$ is a finite G-CW complex built of cells of the form $G/E\times D^n$ where E is a p-torus. In other words it is built from finitely many iterated homotopy colimits of the following form.

$$\begin{array}{ccc} \coprod_j (G/E_j\times S^n) & \longrightarrow & X\times F_n \\ \downarrow & & \downarrow \\ \coprod_j G/E_j & \longrightarrow & X\times F_{n+1} \end{array}$$

Here $X\times F_n$ is the n-skelteton of $X\times F$. All the maps in these diagrams are G-equivariant.

Applying the Borel construction preserves homotopy colimits, so we have:

$$\begin{array}{ccc} \coprod_j (EG\times_G (G/E_j\times S^n)) & \longrightarrow & EG\times_G (X\times F_n) \\ \downarrow & & \downarrow \\ \coprod_j EG\times_G G/E_j & \longrightarrow & EG\times_G (X\times F_{n+1}) \end{array}.$$

Adding a disjoint basepoint and taking suspension spectra gives:

$$\begin{array}{ccc} \bigvee_j \Sigma^\infty(EG\times_G G/E_j\times S^n)_+ & \longrightarrow & \Sigma^\infty(EG\times_G (X\times F_n))_+ \\ \downarrow & & \downarrow \\ \bigvee_j \Sigma^\infty(EG\times_G G/E_j)_+ & \longrightarrow & \Sigma^\infty(X\times F_{n+1})_+ \end{array}.$$

We also have that each $\Sigma^\infty(EG\times_G G/E_j\times S^n)_+$ is a cone of suspensions of $\Sigma^\infty(EG\times_G G/E_j)_+$, compatibly with the structure maps. We see that $\Sigma^\infty EG\times_G(X\times F)_+$ is obtained from a finite number of homotopy pushouts of the $\Sigma^\infty(G/E_j)_+$, compatibly with the maps to $\Sigma^\infty EG\times_G(X\times F)_+$, and we are done. □

Lemma 5.4.11. *We have that* $C^*_G X\in \mathrm{Thick}_X\{C^*_G G/E : G/E\to X\in\mathcal{A}_G X\}$.

Proof. Since by 5.4.9 we have that $C^*_G X \in \mathrm{Thick}_{C^*_G X}\, C^*_G X\times F$, it is enough to show that $C^*_G X\times F \in \mathrm{Thick}_X\{C^*_G G/E : G/E \to X\in\mathcal{A}_G X\}$. But all of the objects in

$\mathrm{Thick}_{X\times F}\{C^*_G G/E : G/E \to X \times F \in \mathcal{A}_G X \times F\}$ are in $\mathrm{Thick}_X\{C^*_G G/E : G/E \to X \in \mathcal{A}_G X\}$ when pulled back along the map $C^*_G X \to C^*_G X \times F$, so this follows from 5.4.10. □

Lemma 5.4.12. *Let T be a tensor triangulated category and let $\mathcal{C} = \mathrm{Thick}\{A_1, \ldots A_n\}$.*

1. *If for all i we have that $\mathrm{Hom}_T(A_i, M) = 0$, then for all $J \in \mathcal{C}$ we have $\mathrm{Hom}_T(J, M) = 0$.*

2. *If for all i we have that $A_i \otimes M = 0$, then for all $J \in \mathcal{C}$ we have $J \otimes M = 0$.*

Proof. If we have a triangle $A \to B \to C$ and $\mathrm{Hom}_T(A, M) = \mathrm{Hom}_T(B, M) = 0$ then by the long exact sequence of a triangle $\mathrm{Hom}_T(C, M) = 0$ as well. Similarly, tensoring with M gives a triangle $A \otimes M \to B \otimes M \to C \otimes M$, and if the first two are zero so is the third.

Also, if A is a retract of B and $\mathrm{Hom}_T(B, M) = 0$, then the identiy map $\mathrm{Hom}_T(A, M)$ factors through 0, so is 0. Similarly, if $B \otimes M = 0$ then $A \otimes M = 0$, so we have shown that these properties are closed under triangles and retracts, giving us the desired result. □

Proof of 5.4.8. One obviously implies two and three, so we only need to show that two and three imply one.

Suppose that for all $G/E \to X \in \mathcal{A}_G X$ we have $\mathrm{ind}_X^{G/E} M \sim 0$, or $C^*_G G/E \wedge_{C^*_G X} M \sim 0$. Then by 5.4.12 and 5.4.11 we have that $C^*_G X \wedge_{C^*_G X} M \sim M \sim 0$. This shows that two implies one.

For three implies one, suppose that for all $G/E \to X \in \mathcal{A}_G X$ we have $\mathrm{coind}_X^{G/E} M \sim 0$, or $\mathrm{Hom}_X(C^*_G G/E, M) \sim 0$. Then again by 5.4.12 and 5.4.11 we have that $M \sim \mathrm{Hom}_X(C^*_G X, M)$ is equivalent to 0, so we are done.

□

Now that we have Chouinard's theorem, we can use stratification for $D(C^*_G G/E)$ to prove stratification for $D(C^*_G X)$. At this point we could appeal to Theorem 3.24 of [BCHV17], but for completeness we include a variant of their argument.

Proposition 5.4.13. *If $M \in \Gamma_{\mathfrak{p}} D(C^*_G X)$ is nonzero and $\mathfrak{q} \in \operatorname{Spec} H^*_G G/E$ originates $\mathfrak{p}$ in the sense of 5.4.4, then $\mathfrak{q} \in \operatorname{supp}_{G/E} \operatorname{ind}_X^{G/E} M \cap \operatorname{cosupp}_{G/E} \operatorname{coind}_X^{G/E} M$.*

This property is closely related to what the authors of [BCHV17] call *Quillen lifting*, see Definition 3.15 of [BCHV17]. The part of this claim for supp is proved by arguing as in the first part of the proof of Theorem 4.5 of [BG14], and the second part uses Propostion 3.23 of [BCHV17].

For completeness, we give the arguments– the key ingredients are 5.3.8 and 5.3.9.

Proof. Suppose that M is nonzero, so there is some $G/E' \to X \in \mathcal{A}_G X$ with $\operatorname{ind}_X^{G/E'} M$ nonzero. Therefore, by 5.2.9 there is some $\mathfrak{q}' \in \operatorname{supp}_{G/E'} \operatorname{ind}_X^{G/E'} M$. But $\operatorname{res}^* \operatorname{supp}_{G/E'} \operatorname{ind}_X^{G/E'} M \subset \operatorname{supp}_X M$ by 5.3.9 so $\operatorname{res}^* \mathfrak{q}' = \mathfrak{p}$. Now, let $(G/E \to X, \mathfrak{q})$ originate $\mathfrak{p}$. Then $G/E \to X$ fits into a triangle:

$$\begin{array}{ccc} G/E & \longrightarrow & X \\ \downarrow j & \nearrow & \\ G/E' & & \end{array}.$$

Because $D(C^*_G G/E), D(C^*_G G/E')$ are stratified, by the subgroup theorem for $D(C^* BE')$ 5.3.8 we have that for all $X \in D(C^*_G G/E')$, $\operatorname{supp}_{G/E} \operatorname{ind}(j) X = (j^*)^{-1} \operatorname{supp}_{G/E'} X$. Therefore, $\mathfrak{q} \in \operatorname{supp}_{G/E} \operatorname{ind}_X^{G/E} M$.

The argument for cosupport is similar: there is some $G/E'' \to X$ with $\operatorname{coind}_X^{G/E''} \Lambda^{\mathfrak{p}} M$ nonzero. So by 5.3.9 we have that there is some $\mathfrak{q}'' \in \operatorname{cosupp}_{G/E''} \operatorname{coind}_X^{G/E''} M$ with $\operatorname{res}^* \mathfrak{q}'' = \mathfrak{p}$. Then, by the subgroup theorem for coinduction and cosupport, we have that for the pair $(G/E \to X, \mathfrak{q})$ originating $\mathfrak{p}$, $\mathfrak{q} \in \operatorname{cosupp}_{G/E} \operatorname{coind}_X^{G/E} M$. □

Proof of 5.4.1. We need to show that each $\Gamma_{\mathfrak{p}} D(C^*_G G/E)$ is a minimal localizing subcategory. So, by 5.3.3 we need to show that for nonzero M, N in $\Gamma_{\mathfrak{p}} D(C^*_G X)$ we have that $\operatorname{Hom}_X(M, N) \neq 0$.

Choose a (unique up to isomorphism) $\mathfrak{q} \in \operatorname{Spec} H^*_G G/E$ originating $\mathfrak{p}$. Then consider $\hom_X(\operatorname{res}_X^{G/E} \operatorname{ind}_X^{G/E} M, N)$. By adjointness, this is $\hom_{G/E}(\operatorname{ind}_X^{G/E} M, \operatorname{coind}_X^{G/E} N)$. Because $D(C^*_G G/E)$ is stratified, we have by 5.2.9:

$$\operatorname{cosupp} \hom_{G/E}(\operatorname{ind}_X^{G/E} M, \operatorname{coind}_X^{G/E} N) = \operatorname{supp} \operatorname{ind}_X^{G/E} M \cap \operatorname{cosupp} \operatorname{coind}_X^{G/E} N.$$

Then, by 5.4.13 $\mathfrak{q} \in \operatorname{cosupp} \hom_{G/E}(\operatorname{ind}_X^{G/E} M, \operatorname{coind}_X^{G/E} N)$ and we conclude that:

$$\hom_X(\operatorname{res}_X^{G/E} \operatorname{ind}_X^{G/E} M, N) \neq 0.$$

But $\operatorname{res}_X^{G/E} \operatorname{ind}_X^{G/E} M \in \operatorname{loc}_X M$ (all localizing subcategories are tensor ideals in our setting, since $D(R)$ is generated by R), so we conclude that $\hom_X(M, N) \neq 0$, as we desired. $\square$

Appendix A
RECOLLECTIONS ON LOCAL COHOMOLOGY

Here we recall some of the basic facts about local cohomology modules, which are used throughout the thesis. The main source for this is [BS13], which includes a discussion of the graded case. Fix a finitely generated connected graded $\mathbb{F}_p$-algebra R. Denote the unique homogeneous maximal ideal of R by $\mathfrak{m}$. All modules will be finitely generated, and everything is meant in the graded sense. Note that [BS13] deals exclusively with commutative rings, not the graded commutative rings we have been dealing with thus far, however Appendix B lets us extend results to the graded commutative setting.

Definition A.0.1. The $\mathfrak{m}$-torsion functor of an R-module M, or $\Gamma_{\mathfrak{m}}M$, is the functor that takes M to the submodule consisting of those elements which are annihilated by some power of $\mathfrak{m}$. This functor is left exact, and its derived functors are the *local cohomology* functors, or $\mathcal{H}^i$.

Remark A.0.2. The $\mathfrak{a}$-torsion functor makes sense for any ideal $\mathfrak{a}$ of R, and usually the functor $\mathcal{H}^i$ is written as $\mathcal{H}^i_{\mathfrak{a}}$, taking into account the ideal $\mathfrak{a}$. However, in the first four chapters of this thesis, we are solely concerned with the maximal ideal, and so we omit the ideal from the notation.

One convenient property of the local cohomology functors is that they contain much of the information about depth, dimension, and associated primes.

Theorem A.0.3. *Let M be an R-module.*

1. *The depth of M is the smallest i with $\mathcal{H}^i M \neq 0$.*

2. *The dimension of M is the largest i with $\mathcal{H}^i M \neq 0$.*

3. If $\mathfrak{p} \in \operatorname{Ass}_R M$ *of dimension* d*, then* $\mathcal{H}^d M \neq 0$*.*

Proof. One and two are standard and proofs are contained in [BS13]. Three is less well known, but is in [GL00]. □

Local cohomology is very flexible in terms of which ring it is computed over.

Theorem A.0.4 (The independence theorem)**.** *If* $R \to S$ *is a finite map of connected graded* $\mathbb{F}_p$*-algebras, and* M *is an* S*-module, then* $\operatorname{res}(\mathcal{H}^i M)$ *is naturally isomorphic to* $\mathcal{H}^i \operatorname{res} M$*.*

Proof. This is also in [BS13]. □

Local cohomology can be computed via Čech complexes, which shows that it satisfies a Künneth theorem.

Theorem A.0.5. *Let* R *and* R' *be connected, graded* $\mathbb{F}_p$*-algebras and let* M *be an* R*-module and* N *and* R'*-module. Then we have that* $\mathcal{H}^*(M \otimes_{\mathbb{F}_p} N) = \mathcal{H}^* N \otimes_{\mathbb{F}_p} \mathcal{H}^* N$*.*

This implies that if $R \cong R' \otimes_{\mathbb{F}_p} N$ where N is a bounded $\mathbb{F}_p$-algebra, then $\mathcal{H}^* R = \mathcal{H}^* R' \otimes_{\mathbb{F}_p} N$.

As we have seen throughout this thesis, because of the Duflot filtration the local cohomology of a polynomial ring determines a lot about the local cohomology of group cohomology rings. Consequently we close this appendix with a computation of the local cohomology of a polynomial ring as a module over itself.

Theorem A.0.6. *Let* $R = \mathbb{F}_p[x_1, \ldots, x_n]$*, where* $|x_i| = d_i$ *(and the* d_i *are all even when the prime is odd). Then* $\mathcal{H}^n R = \Sigma^{-(d_1 + \cdots + d_n)}(R^*)$*, where here* $*$ *denotes the* $\mathbb{F}_p$*-linear dual.*

Proof. By A.0.5, it is enough to prove this when $n = 1$. For this, we can proceed directly from the definitions. The exact sequence $0 \to \mathbb{F}_p[x] \to \mathbb{F}_p[x, x^{-1}] \to \mathbb{F}_p[x, x^{-1}]/\mathbb{F}_p[x] \to 0$ is an injective resolution of $\mathbb{F}_p[x]$. This is so because we are working in the graded category, we can use the graded version of Baer's criterion to check injectivity. Note that $\mathbb{F}_p[x, x^{-1}]/\mathbb{F}_p[x] \cong \Sigma^{-d}(\mathbb{F}_p[x]^*)$. Applying $\Gamma_{\mathfrak{m}}$ to this resolution then gives the desired result. □

Appendix B

GRADED COMMUTATIVE VERSUS COMMUTATIVE GRADED RINGS

In this thesis, we have been concerned with graded commutative rings, that is graded rings R so that for for homogeneous $a, b \in R$, $ab = (-1)^{|a||b|}ba$. Unfortunately most references in commutative algebra that deal with graded rings and modules typically assume that graded rings are strictly commutative.

It is customary to either ignore the distinction between graded commutative rings and commutative graded rings, or to insist that all theorems from commutative algebra hold for graded commutative rings. It seems worthwhile to explain how to reduce concepts for graded commutative rings to commutative graded counterparts.

First of all, for graded commutative rings the notion of a left and right ideal coincide, and if M is a left $R-$module it can be converted into a right $R-$module via $m \cdot x = (-1)^{|x||m|}x \cdot m$.

A graded commutative ring R has a subring R^{ev} which is generated by the even degree elements, and R^{ev} is both graded commutative and commutative graded. All the odd degree elements of R are nilpotent, and the inclusion $R^{ev} \to R$ induces a bijection of prime ideals. The whole ring R is an R^{ev} module, and commutative algebra notions about R are the same as notions about R as an R^{ev}-module, as indicated in the following proposition. If R is Noetherian, then R is a finitely generated R^{ev}-module.

Proposition B.0.1. *Let R be a finitely generated graded commutative ring.*

1. *The depth of R is equal to the depth of R as an R^{ev}-module.*

2. *The dimension of R is equal to the dimension of R as an R^{ev}-module.*

3. *The map $R^{ev} \to R$ induces a bijection $\operatorname{Ass}_R R \to \operatorname{Ass}_{R^{ev}} R$.*

These are all consequences of the facts that the inclusion of R^{ev} denotes a bijection of primes, and the fact that all odd degree elements of R are nilpotent.

Local cohomology modules can be defined for graded commutative rings exactly as for commutative graded rings, and the local cohomology of R as a module over itself is the same as the local cohomology of R as an R^{ev} module.

Theorem B.0.2. *For all R-modules M, the inclusion R^{ev} into R induces an isomorphism of R^{ev}-modules $\operatorname{res} \mathcal{H}^* M \to \mathcal{H}^* \operatorname{res} M$. Here the left hand side denotes local cohomology as an R-module restricted to R^{ev}, and the right hand side denotes local cohomology after restricting to R^{ev}.*

Proof. Denote the maximal ideal of R by $\mathfrak{m}$ and the maximal ideal of R^{ev} by $\mathfrak{m}^{ev}$. Then $\Gamma_{\mathfrak{m}^{ev}} \operatorname{res} M$ is equal to $\operatorname{res} \Gamma_{\mathfrak{m}} M$. This is because it is clear that if $m \in \operatorname{res} M$ is annihilated by $\Gamma_{\mathfrak{m}^{ev}}$, then x is annihilated by $\Gamma_{\langle \mathfrak{m}^{ev} \rangle}$, where $\langle \mathfrak{m}^{ev} \rangle$ denotes the ideal in R generated by $\mathfrak{m}^{ev}$. However, the torsion functor is radical invariant, and the radical of $\langle \mathfrak{m}^{ev} \rangle$ is $\mathfrak{m}$.

Now, an injective R-module is $\Gamma_{\mathfrak{m}^{ev}}$-acyclic as an R^{ev}-module. This is Theorem 4.1.6 of [BS13]. The proof there is given for commutative rings, but the same proof works in this context. □

Remark B.0.3. This proof is essentially the proof of the independence theorem A.0.4. An alternative proof proceeds using the Čech complex: we can choose generators for $\mathfrak{m}^{ev}$ so that the Čech complex for $\operatorname{res} M$ with respect to the ideal generated by $\mathfrak{m}^{ev}$ as an R^{ev}-module is equal to the Čech complex of M with respect to the ideal generated by $\mathfrak{m}^{ev}$ as an R-module.

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