Research Statement

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I am interested in tensor triangulated geometry, homotopy theory, and modular representation theory, and interactions between these subjects. During my PhD I focused on geometric properties of equivariant cohomology rings and using the Borel equivariant cohomology of smooth manifolds to study group cohomology rings. This led to generalizing theorems on depth and associated primes from group cohomology rings to the equivariant cohomology of smooth manifolds, as well as a method to study the local cohomology modules of group cohomology rings using equivariant cohomology. During my poctdoc at UCLA I have focused on tensor triangulated geometry, where homotopy theory and modular representation theory are two of my favoritve examples in the more general theory. In particular I have been working on understanding fields in tensor triangulated geometry, and in joint work with Paul Balmer we have classified the homological residue fields that occur in tensor triangulated categories arising in algebra, homotopy theoory, and representation theory.

1. Tensor triangulated geometry

Tensor triangulated geometry (tt geometry) is the study of of tensor triangulated categories (tt categories) using geometric methods. One of the organizing principles is the *Balmer Spectrum* [Bal10], this is a space asscociated to a tensor triangulated category that is the home of the universal support theory for the category, and understanding this space gives information about the tensor triangulated category, such as a classification of thick tensor ideals. Theorems that classify the thick tensor ideals of a tensor triangulated category, such as in the case of the stable homotopy category [HS98] or the stable module category of a group [BCR96], can be interpreted as theorems about the Balmer spectrum of these categories. The thrust of tt geometry can be encapulated in two open ended questions:

(1) Given a tt category $\mathcal{T}$, what is the Balmer spectrum of $\mathcal{T}$?
(2) What can the Balmer spectrum of $\mathcal{T}$ (and associated constructions) tell you about $\mathcal{T}$?

My research involves both of these questions.

1.1. **Homological Residue Fields.**

A good theory of residue fields for tt categories does not yet exist, but key examples such as the derived category of a ring, the stable homotopy category, and the stable module category of a finite group all have functors that act like residue field functors. Homological residue fields were introduced by Balmer, Krause, and Stevenson [BKS19] as the abelian avatar of the hypothetical tensor triangulated residue field functor, they are symmetric monoidal functors from a tt category $\mathcal{T}$ to an abelian category with no nontrivial localizing tensor ideals, and they have the advantage that there is always at least one such homological residue field for each tensor triangulated prime.

To construct a homological residue for a compactly-rigidly category $\mathcal{T}$, one takes the category of $\mathcal{T}^c$-modules, that is the category of additive functors from $\mathcal{T}^c$ to abelian groups. A homological residue field is a quotient of $\mathcal{T}^c$-modules by a homological prime $\mathcal{B}$, and if we have a tensor triangulated residue field functor $\mathcal{T} \to \mathcal{F}$ we have a diagram
$$\begin{tikzcd} \mathcal{T} \arrow[r] \arrow[d, shift right, "F"'] & \text{Mod-}\mathcal{T}^c \arrow[d, shift right, "\hat{F}"'] \\ \mathcal{F} \arrow[r] \arrow[u, shift right, "U"'] & \text{Mod-}\mathcal{F}^c \arrow[u, shift right, "\hat{U}"'] \end{tikzcd}$$
from which we can contruct a homological prime $\mathcal{B}$ and an homological residue field $\bar{\mathcal{A}}_{\mathcal{B}}$.

Homological residue fields are useful, for example Balmer in [Bal20b] [Bal20a] has used them to prove a theorem characterizing maps that are tensor nilpotent in a general tt category and also to develop a theory of supports for not necessarily compact objects.

Homological residue fields are useful but their definition is rather abstract and a priori it is difficult to do calculations with them, and some of my joint work with Paul Balmer has been in connecting homological residue fields with the tt residue fields that exist in some examples.

Each homological residue field $\bar{\mathcal{A}}_{\mathcal{B}}$ for a tt category $\mathcal{T}$ is governed by a pure-injective object $E_{\mathcal{B}}$ of $\mathcal{T}$, and one way of understanding $\bar{\mathcal{A}}_{\mathcal{B}}$ is to understand this object $E_{\mathcal{B}}$. We show that in the standard examples the object $E_{\mathcal{B}}$ is a familiar object related to the tt residue field.

Theorem 1 ([BC20])**.** (1) *For $\mathcal{B}$ the homological prime of $\mathrm{Spc}^{\mathrm{h}}(\mathrm{SH}^c)$ corresponding to $K(p,n)$, we have an isomorphism $E_{\mathcal{B}} \simeq K(p,n)$ in* SH*.*

(2) *Let R be a ring with derived category $\mathcal{T} = \mathrm{D}(R)$ and let $\mathcal{B}$ be a homological prime corresponding to the prime $\mathfrak{p}$ of $\mathrm{Spec}(R) \cong \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(R))$. Then the pure-injective object $E_{\mathcal{B}}$ is isomorphic to the residue field $\kappa(\mathfrak{p})[0]$.*

One deficiency of this theorem is that it does not cover the case of the stable module category of a group. We have partial results in this direction with restrictive hypotheses. The reason is that the tt residue field functor in question, the π-points of [FP07], give functors on stable module categories that aren't monoidal, so the theory that connects the tt residue field with a homological residue field does not work, and resolving this difficulty is a high priority.

Project 1. For $\bar{\mathcal{A}}_{\mathcal{B}}$ a homological prime of $\mathrm{Stab}\, kG$, for G an arbitrary finite group, determine the associated pure injective object $E_{\mathcal{B}}$.

Another approach to understanding the homological residue field $\bar{\mathcal{A}}_{\mathcal{B}}$ arrising from tensor triangulated residue field $\mathcal{F}$ is to directly relate the category $\mathcal{F}$ and $\bar{\mathcal{A}}_{\mathcal{B}}$, and Balmer and I do this as well.

Theorem 2 ([BC20])**.** *Let $F\colon \mathcal{T} \to \mathcal{F}$ be a monoidal functor with right adjoint U, where $\mathcal{F}$ is a tensor-triangulated field. The category of comodules for the comonad $\widehat{FU}$ on the functor category of $\mathcal{F}^c$-modules is equivalent to the homological residue field $\bar{\mathcal{A}}_{\mathcal{B}}$ corresponding to F.*

Project 2. Understand the comonad $\widehat{FU}$ and its category of comodules in the standard examples, and determine in what generality $\widehat{FU}$ comodules are comodules for some coalgebra. This is the case for the stable homotopy category and for the derived category of a ring, and I am currently pursuing this further in joint work with Greg Stevenson.

Project 3. Determine the structure of the pure injective object $E_{\mathcal{B}}$ associated to a homological prime $\mathcal{B}$. It is always a "weak ring" object by [BKS19] and is a ring object in all of the examples that have been computed thus far– it is perhaps always a ring object? If not, are there conditions on $\mathcal{T}$ that guarantee every $E_{\mathcal{B}}$ is a ring object?

1.2. **Stratification for cochain algebras.** Another approach to understanding the large scale structure of tt categories, especially those of an "algebraic nature" is the support theory and associated stratification theory of Benson, Iyengar, and Krause [BIK11b]. This is a support theory for a tt category $\mathcal{T}$ with values in the spectrum of a Noetherian ring equipped with a map to the homotopy groups of the unit of $\mathcal{T}$, and there are conditions for when this support theory classifies the localizing subcategories of $\mathcal{T}$, and also the thick subcategories of $\mathcal{T}^c$, and hence the Balmer spectrum. In this case we say that the category $\mathcal{T}$ is stratified by the ring R.

Benson, Iyengar, and Krause [BIK11a] show that the derived category of modules for $\mathbb{F}_p$-cochains on the classifying space of a finite group is stratified by $H^*(BG, \mathbb{F}_p)$, initiating a study of when the derived category of $\mathbb{F}_p$-cochains on a space is stratifed by the $\mathbb{F}_p$ cohomology ring.

Benson and Greenlees [BG14] showed that stratification holds for connected compact Lie groups, and Barthel, Castellana, Heard, and Valenzuela [BCHV19] showed that stratification holds for all compact Lie groups, as well as for homotopical generalizations of compact Lie groups.

Building on the work of Benson and Greenlees and of Barthel, Castellana, Heard, and Valenzuela, I show that stratification holds for the Borel construction of a compact Lie group G acting on a finite G-CW complex.

Theorem 3 ([Cam18]). *Let G be a compact Lie group and and X a finite G-CW complex. Then the category $D(\mathrm{Mod} - C^*(EG \times_G X, \mathbb{F}_p))$ is stratified by the Borel equivariant cohomology $H^*_G(X, \mathbb{F}_p)$.*

In particular, the Balmer spectrum $\mathrm{Spc}\, D(\mathrm{Mod} - C^*(EG \times_G X, \mathbb{F}_p)^c$ is isomorphic to the homogeneous spectrum $\mathrm{Spec}^h\, H^*_G(X, \mathbb{F}_p)$.

The most interesting case for me is the case where $G \to U(n)$ is a faithful representation, S is a maximal elementary abelian p-subgroup of $U(n)$, and G acts on $U(n)/S$. In this case, the equivariant $U(n)/S$ is intimately related to the cohomology ring $H^*(BG, \mathbb{F}_p)$ and cochains on $EG \times_G U(n)/S$ are closely related to chains on BG.

In my work in group cohomology I use a geometrically defined filtration of $H^*_G(U(n)/S, \mathbb{F}_p)$ due to Duflot [Duf83b] to study the local cohomology of $H^*(BG, \mathbb{F}_p)$. This filtration of $H^*_G(U(n)/S, \mathbb{F}_p)$ lifts to a filtration of $C^*(EG \times_G U(n)/S, \mathbb{F}_p)$, and it is natural to ask what this filtration tells you about $\mathrm{Mod} - C^*(EG \times_G U(n)/S, \mathbb{F}_p)$ and about $\mathrm{Mod} - C^*(BG, \mathbb{F}_p)$. Because $\mathrm{Mod} - C^*(BG, \mathbb{F}_p)$ is closely related to the stable module category of $\mathbb{F}_pG$, this would give an approach to study modular representation theory using equivariant topology, mirroring the great successes of using equivariant cohomology to study group cohomology.

Project 4. Describe the structure the Duflot filtration puts on $D(C^*(EG \times_G U(n)/S, \mathbb{F}_p))$, and use this to study $D(C^*(BG, \mathbb{F}_p))$ and $\mathrm{StMod}\, kG$.

2. Cohomology of classifying spaces and equivariant cohomology

I am interested in the the modular cohomology rings of classifying spaces of compact Lie groups. For finite groups these rings can be described purely algebraically as $\mathrm{Ext}^*_{\mathbb{F}_pG}(\mathbb{F}_p, \mathbb{F}_p)$, but there is a long and fruitful history of studying these rings using techniques from algebraic topology. For G a finite group the Balmer spectrum of the stable module category of $\mathbb{F}_pG$ is $\mathrm{Proj}\, H^*(BG, \mathbb{F}_p)$, so understanding $H^*(BG, \mathbb{F}_p)$ from a geometric point of view has direct consequences in representation theory. In my research, I've used a filtration due to Duflot [Duf83b] on the Borel equivariant cohomology ring of a smooth manifold with an elementary abelian p-group action to study group cohomology rings. I've developed a framework that gives unified proofs of Quillen's theorem on dimension, Duflot's theorem on depth, and Symonds' regularity theorem, as well as new restrictions on depth and associated primes for H^*BG, and new local cohomology computations.

2.1. Detection and Associated Primes.

The Duflot filtration is a filtration by the natural numbers, but I show how it can be refined to be a filtration by a certain poset relating to fixed point data. This leads to an alternative description of the Duflot filtration, and also yields some restrictions on associated primes. This extends to compact Lie groups several results that were only known for finite groups, and provides new proofs for the finite group case.

Recall that Duflot proved [Duf83a] that all associated primes of H^*BG come from restricting to elementary abelian p-subgroups.

Theorem 4 (Cameron, [Cam17]). *Let G be a compact Lie group and $E < G$ an elementary abelian p-subgroup. The following are equivalent:*

1. *E represents an associated prime in $H^*(BG, \mathbb{F}_p)$*
2. *The Duflot bound for depth is sharp for $H^*B(C_GE, \mathbb{F}_p)$*

This follows from the work of Okuyama [Oku10] of Kuhn [Kuh13] for finite groups, so this theorem is new in the compact Lie group case. The recent work of Heard [Hea20] gives another approach to these ideas using the technology of unstable modules over the Steenrod algebra, but the approach using the Duflot filtration, or perhaps a combination of the Duflot filtration and unstable algebra technology is very promising.

This theorem shows that understanding when the Duflot bound for depth is sharp is equivalent to understanding what elementary abelian p-subgroups give associated primes, which raises the following questions.

Project 5. Find a group theoretic criterion for determining when the Duflot bound for depth is sharp.

Project 6. Classify the elementary abelian p-groups giving associated primes of $H^*(BG, \mathbb{F}_p)$.

Because $\operatorname{Spc} \operatorname{Stab} \mathbb{F}_p G^c$ is isomorphic to $\operatorname{Spec}^h H^*(BG, \mathbb{F}_p)$ as locally ringed spaces, you could instead try to understand the associated primes of $\operatorname{Spc} \operatorname{Stab} \mathbb{F}_p G^c$, or indeed the associated primes of the spectrum of any tt category.

Project 7. Develop a theory of associated primes in tensor triangulated geometry.

2.2. **Vanishing and Nonvanishing of Local Cohomology.** Local cohomology is known to vanish in degrees below the depth and above the dimension, and it is nonvanishing at degrees equal to the depth, dimension, and at any degree equal to the dimension of an associated prime. Denote local cohomology with respect to the ideal of positive degree elements by $\mathcal{H}$.

The seminal regularity theorem of Symonds [Sym10] concerns the vanishing of $\mathcal{H}^{i,j} H^*BG$ for $j > -i$, but little is known about the vanishing and nonvanishing of the entire $\mathcal{H}^i(H^*BG)$ beyond what is known for general graded rings. This motivates the following question, if the answer to this question is no (as it currently appears based on computational evidence) this place great restrictions on group cohomology rings among all finite $\mathbb{F}_p$-algebras:

Project 8. Determine whether or not there is any compact Lie group G with $\operatorname{depth} H^*BG = d$ and $\dim H^*BG = r$ and some $d < i < r$ with $\mathcal{H}^i H^*BG = 0$.

In other words, in the range where local cohomology can be either zero or nonzero, it is not currently known if there is a group where local cohomology is ever nonzero.

I have made some partial progress on this question: in [Cam17] I show that for each n there exists a G and an i so that $\mathcal{H}^{i+j}(H^*BG) \neq 0$ for all $0 < j < n$, and so that $i + j$ is not the dimension of an associated prime.

The groups that give examples of this phenomenon are the Sylow p-subgroup of S_{p^n}, denoted by $W(n)$. By taking advantage of the geometry of bundles associated to certain extensions involving $W(n)$, I use the machinery I develop in [Cam17] to compute the local cohomology of $W(n)$ in terms of the group cohomology of $W(n-1)$, and from this learn the following:

Theorem 5 ([Cam17])**.** *For $0 \leq i < p-3$, $\mathcal{H}^{p^{n-1}-i}(H^*B(W(n)) \neq 0$.*

In other words:

Proposition 6 ([Cam17])**.** *For each n there exists a G and an i so that $\mathcal{H}^{i+j}(H^*_G) \neq 0$ for all $0 < j < n$, and so that $i + j$ is not the dimension of an associated prime.*

2.3. **More exotic groups.** My theorems on the cohomology rings of classifying spaces focus on the compact Lie group case, but there are interesting cases beyond compact Lie group such as p-compact groups where much of the theory works the same, but essential theorems involving local cohomology such as Symonds' regularity theorem are unknown.

Project 9. Determine whether or not the regularity theorem holds for p-compact groups: that is, if $\mathcal{G}$ is a p-compact groups of dimension d, is $\operatorname{reg} H^*(B\mathcal{G} = -d$? Answering this question would likely involve a more homotopical formulation of the Duflot filtration, and might also involve unstable algebra technology.

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