Lab 8: Nonlinear Systems: SIR Models

In epidemiology, SIR models are compartmental models that are used to predict how infectious diseases spread through populations. They are also used to determine policies like social distancing that can minimize the spread of the disease. An SIR model divides the population into three subpopulations: Susceptible $S(t)$, Infectious $I(t)$, and Recovered/Removed $R(t)$.

Susceptible individuals can catch the disease from an infectious individual, while recovered/immune have already had the disease and are no longer infectious or susceptible. $\beta > 0$ is the infection rate, or the average fraction of the population infected per day by each infectious person. $\gamma > 0$ is the average recovery rate, or $\gamma = 1/D$, where $D$ is the average period of time (in days) an individual is infectious. This can be modeled by the following system of three nonlinear differential equations:

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dI}{dt} = \beta SI - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

1. In the SIR model, the total population remains constant: $N = S + I + R$. Show this is the case by showing $\frac{dN}{dt} = 0$.

Because neither $S$ or $I$ depend on $R$, we don’t need to look at the third equation to understand the disease dynamics. Unless otherwise asked, consider only the $S, I$ equations.

2. The critical point structure of this model is a bit different than what you may have seen before. Show that if $I = 0$, then $S$ can be any number and the system will be at equilibrium. Physically, this means that the disease dynamics will only be stable if there are no carriers of the infection.
3. The key prediction of this model is whether an epidemic will occur. This happens if the
disease spreads and the number of infectious people $I$ increases, i.e., if $dI/dt > 0$ for some time.
Show that when $\beta S/\gamma > 1$, epidemics occur.

4. Draw the direction field diagram for the phase plane equation $dI/dS$. Use the information
from problems 2 and 3 to help.

5. The quantity $\beta S/\gamma = R_{eff}$ is the effective reproduction number for the disease, the average
number of people that each infected person infects (over the course of their infectious period).
Using this definition, explain in words why $R_{eff} > 1$ leads to epidemics and $R_{eff} < 1$ does not.

6. Explain what would happen if $R_{eff} = 1$.

7. Suppose that the total population is $N = 1000$ persons, the length of the infectious period
is 5 days (so $\gamma = 1/5 = 0.2$ (days)$^{-1}$), and that the infection rate $\beta = 0.01$ (days-patients)$^{-1}$.
If the initial population has 10 infectious people and everyone else susceptible, compute the
effective reproductive number $R_{eff}$ initially (with $S(0)$). Will an epidemic occur?

8. Compute the solutions to the SIR equations numerically to verify your answer to problem
7. Plot $S$, $I$, and $R$ vs. $t$ on one graph.

9. Suppose that the total population is $N = 1000$ persons, the length of the infectious period
is 2 days (so $\gamma = 1/2 = 0.5$ (days)$^{-1}$), and that infection rate $\beta = 0.00005$ (days-patients)$^{-1}$.
If the initial population has 10 infectious people and everyone else susceptible, compute the
effective reproductive number $R_{eff}$ initially (with $S(0)$). Will an epidemic occur?

10. Compute the solutions to the SIR equations numerically to verify your answer to problem
9. Plot $S$, $I$, and $R$ vs. $t$ on one graph.
\( R_{\text{eff}} \) and the related parameter \( R_0 \) are estimated by researchers and public health officials to assess the susceptibility of local populations to epidemics. When the whole population is susceptible, \( S = N \) and \( \beta N / \gamma = R_0 \) is the basic reproduction number for the disease, the average number of people that each infected person infects in an entirely susceptible population. It is difficult to estimate “the” reproductive number of a disease, because it varies across regions, age groups, vocations, social structures, etc. The strength of mathematical models like the SIR model are in helping guide policy through logic and reasoning, not necessarily in writing down an exact model that can predict the future.

#FlattenTheCurve  
A goal of social distancing and other public health measures is to “flatten the curve”, i.e., to lower the peak number of infectious people \( I \) so as to not overwhelm the healthcare system. In questions 11-15, we’ll show how the SIR model demonstrates the outcomes of these public health measures.

11. The infection rate \( \beta \) can be modeled as

\[
\beta = \frac{pC}{N},
\]

where \( p \) is the probability of transmission of the disease, \( C \) is the average number of person-to-person contacts (for an individual), and \( N \) is the total number of people in the population. Which variable (\( p, C, \) or \( N \)), does social distancing affect and how?

12. Which variable (\( p, C, \) or \( N \)), does wearing masks affect and how?

13. Suppose that the total population is \( N = 1000 \) persons, the length of the infectious period is 5 days (so \( \gamma = 1/5 = 0.2 \) (days)\(^{-1} \)), and that infection rate \( \beta = 0.01 \) (days-patients)\(^{-1} \) (these are the same values as in problems 7-8). If the initial population has 10 infectious people and everyone else susceptible, what is the peak number of infectious people \( I \)?

14. Suppose that the government implemented strict social distancing and mask-wearing policies immediately after the initial 10 infectious cases were identified, bringing the infection rate \( \beta \) down to 0.001. Compute the new effective reproductive number \( R_{\text{eff}} \) initially (with \( S(0) \)) and the new peak number of infectious people \( I \).

15. On the same graph, plot \( I \) vs. \( t \) for: 1) no safety measures (as in problem 13), and 2) with safety measures, i.e., #Flattened (as in problem 14).
The SIR model contains the essence of nearly all disease dynamics models. There are many possible extensions, including:

More states and connections between states:

- A latent period after Exposure but before becoming infectious (‘SEIR model’)
- An asymptomatic period after becoming infectious, but before showing symptoms.
- Recovered people losing their immunity and returning to susceptible (e.g. flu)
- Different susceptible or infected populations (e.g. young vs. old)

Spatial dynamics:

- Travel, spatial spread between different regions
- Different people contact more or fewer other people

Randomness + time-varying parameters (e.g. seasonal changes).

16. Look up one of these extensions, write down their differential equations, and explain the new terms in words.

17. After finishing your write-up for this lab, answer the following questions. What was your favorite part of this lab? What was the easiest part of this lab? The hardest? What was confusing? What was interesting?