Lab 5: Electrical and Neural Circuits

For more information see Section 3.5 of the textbook.

Electrical circuits are built up of individual components, each of which has two ends through which they exchange electrical signals with the rest of the system. Such a component can be characterized by the relationship between the current flowing through the device (I) and the electrical potential drop across the device (V). Current I is the amount of electrical charge \( dQ \) that flows past a given point in a small period of time \( dt \), i.e.,

\[
I = \frac{dQ}{dt}.
\]

Charge \( Q \) is measured in coulombs, current \( I \) is in amperes (1 ampere = 1 coulomb/second), and electrical potential \( V \) is measured in volts. The voltage is the potential drop across the device, e.g. the potential at the top end minus the potential at the bottom end. For practical circuits, charge cannot be created or destroyed within the component, so the current into one end is exactly equal to the current leaving the other end.

The three common circuit elements and their current-voltage models are shown on the right. For the resistor, the current and voltage are proportional with constant \( R \), called the resistance (measured in ohms):

\[
V = I \cdot R.
\]

This model is called Ohm’s Law and is also frequently written as

\[
V G = I
\]

in biology, where the conductance \( G = 1/R \) (measured in siemens).

The capacitor stores charge on two parallel plates separated by an insulator. The voltage \( V \) across a capacitor is proportional to the stored charge \( Q \), so that

\[
Q = C \cdot V,
\]

where \( C \) is the capacitance (measured in farads F). The current \( I \) through the capacitor is then

\[
I = \frac{dQ}{dt} = C \frac{dV}{dt}.
\]

The inductor, usually a coil of wire, stores energy in a magnetic field when electric current flows through it. It is characterized by its inductance \( L \), in units of henrys. From electrophysics, there is a voltage drop across the inductor whenever the current flow changes, given by

\[
V = L \frac{dI}{dt}.
\]
These circuit elements are combined to form circuits. Physical principles governing electrical circuits were formulated by G.R. Kirchhoff in 1859. They are the following:

1. **Kirchhoff’s current law**: The algebraic sum of the currents flowing into any junction point must be zero. This implies that the same current passes through all elements in each circuit of the figure above.

2. **Kirchhoff’s voltage law**: The algebraic sum of the instantaneous changes in potential (voltage drops) around any closed loop must be zero.

![Diagrams of RL and RC circuits](image)

1. If we let \( E(t) \) denote the voltage supplied to the circuit through the voltage source at time \( t \), then applying Kirchhoff’s voltage law to the RL circuit (Figure (a) above) gives

\[
E(t) = V_L + V_R,
\]

where \( V_L \) and \( V_R \) are the **voltage drops** across the inductor and resistor, respectively. **Substitute the circuit element expressions for** \( V_L \) **and** \( V_R \) **into the above to write the linear differential equation for the current** \( I \) **as a function of** \( t \), i.e., \( \frac{dI}{dt} \).
2. Find the integrating factor and set up the solution to the differential equation for \( I(t) \) in terms of an integral for an arbitrary voltage source \( E(t) \). Also include an arbitrary initial condition \( I(0) \).

3. Common voltage sources are batteries, for which \( E(t) = \) constant, or sinusoids that mimic AC power. An RL circuit with a \( R = 1 \) ohm resistor and a \( L = 0.01 \) henry inductor is driven by a voltage \( E(t) = \sin 100t \) volts. Use your expression from question 2 to solve the corresponding initial value problem with \( I(0) = 0 \) for \( I(t) \). Plot both your exact solution and a numerical solution to check your work.

4. An application of the RL circuit is the spark plug of a combustion engine. If a voltage source establishes a nonzero current in an inductor and the source is suddenly disconnected, the rapid change of current produces a high \( \frac{dI}{dt} \) and the inductor generates a voltage surge sufficient to cause a spark across the terminals—thus igniting the gasoline.

Model this surge by plotting the numerical solution to the differential equation for the RL circuit with piecewise-constant voltage source

\[
E(t) = \begin{cases} 
1 & 0 < t < 0.2 \\
0 & 0.2 < t 
\end{cases}
\]

and initial current \( I(0) = 0 \) up to time \( t = 2 \). You can implement such a function in MATLAB with:

\[
E = @(t) 1.*(0<t)\&&(t<0.2));
\]
5. Now consider the RC circuit in Figure (b) above. Assuming that there is no voltage source, i.e., $E(t) = 0$, and applying Kirchhoff’s current law gives

$$I_R + I_C = 0,$$

where $I_R$ and $I_C$ are the currents flowing into the resistor and capacitor, respectively. Substitute the circuit element expressions for $I_R$ and $I_C$ into the above to write the linear differential equation for $dV/dt$, where $V$ is the voltage drop across each circuit element (which is the same for both the capacitor and resistor).

6. An RC circuit with a $R = 1$ ohm resistor and a $C = 0.001$ farads capacitor is not driven by any voltage ($E(t) = 0$ volts). However, the capacitor is charged so that the initial voltage drop across it is $V(0) = 10$. Solve the differential equation from problem 5 with $V(0) = 10$. Plot both your exact solution and a numerical solution to check your work.
Electrical Circuit Model of a Neuron

Neurons are cells that send electrical signals to one another, and they are often modeled by RC circuits. The cell membrane separates the inside of the neuron from the outside. The membrane is an insulator that separates internal and external conducting solutions, which forms a significant electrical capacitor \( C \sim 0.1 - 1 \, \text{nF} \). The membrane also contains many conducting channels which are permeable to specific ions, which act like resistors \( R \sim 10 - 100 \, \text{MΩ} \).

The membrane potential can be measured by inserting a microelectrode into the inside of a cell’s membrane and comparing the voltage inside relative to outside the membrane. This voltage is called the membrane potential. The inside of the cell is negative with respect to the outside. The difference in voltage between the inside and the outside of the cell is due to a difference in ion concentrations. This difference exists because the membrane is not completely permeable to these ions. At equilibrium, the membrane potential of a typical neuron is about \( V_E = -70 \text{mV} \).

Electrically, the membrane is often represented as a resistor (representing the conductive pathways for ions) in series with a source (representing the reversal potential for the conductivity pathway), together in parallel with a capacitor (representing the capacity to separate and store charges). The following figure shows this model for the neuron’s membrane potential with a membrane capacitance \( C \) and a leakage ion channel with resistivity \( R_L \) and reversal potential \( E_L \).

Using Kirchhoff’s current law, the differential equation for the voltage drop across the cell membrane in the above electric circuit model is

\[
C \frac{dV}{dt} + \frac{V - E_L}{R} = 0
\]

7. Find the steady state (critical point) of the differential equation and determine its stability. Sketch solution trajectories.
8. How does this model behavior compare to the RC circuit model behavior in problem 6?

9. How does the equilibrium potential computed in problem 7 compare to the equilibrium potential $V_E = -70 \text{ mV}$ given in the text above? Explain what you think the reversal potential $E_L$ means in this context.

10. Explain each term in the differential equation above and how they relate to the figure.

11. Neuroscientists often stimulate neurons with electrical impulses to study how the neurons respond. We can inject current $I(t)$ into our model neuron by including it in series with our resistor and capacitor. The differential equation becomes:

$$C \frac{dV}{dt} + \frac{V - E_L}{R} - I(t) = 0.$$ 

Note that the current $I(t)$ is negative since it’s being forced into the cell.

**How does the voltage of our simple neuron respond to a sinusoidal input current?**

Let $R = 1\Omega$, $C = 0.001 \text{ F}$, $V_E = -70\text{ mV}$, $I(t) = \sin(100t)$. Solve the differential equation with $V(0) = -45 \text{ mV}$ by hand or numerically, and plot the solution up to time $t = 10$.

12. Often the stimulation is just a constant electrical impulse for a short amount of time. Model the voltage response of a neuron to the piecewise-constant current

$$I(t) = \begin{cases} 20.5 & 0 < t < 0.02 \\ 0 & 0.02 < t \end{cases}$$

starting at rest $V(0) = V_E$ by plotting the numerical solution up to time $t = 0.04$. 

Second-Order Linear ODE: RLC Circuit

For more information see Section 5.7 of the textbook. If an inductor and a capacitor both appear in a circuit, the governing differential equation will be second order. The schematic to the right illustrates the RLC circuit, a closed loop of a resistor, inductor, capacitor, and voltage source. RLC circuits have many applications as oscillator circuits. Radio receivers and television sets use them for tuning to select a narrow frequency range from ambient radio waves.

According to Kirchhoff’s current law, the same current $I$ passes through each circuit element, therefore the voltage drops across the resistor $V_R$, inductor $V_L$, and capacitor $V_C$ are

$$V_R = RI, \quad V_L = L \frac{dI}{dt}, \quad V_C = \frac{1}{C}q.$$  

Using Kirchoff’s voltage law, $V_R + V_L + V_C = E$, so the differential equation governing the RLC circuit is

$$L \frac{dI}{dt} + RI + \frac{1}{C}q = E(t).$$

13. Note that the current $I$ through the capacitor equals the instantaneous rate of change of its charge $q$, so that $I = dq/dt$. Differentiate both sides of the governing RLC circuit equation with respect to $t$ and use $I = dq/dt$ to obtain a second-order differential equation for the current $I(t)$.

14. The series RLC circuit above has a voltage source given by $E(t) = \sin(100t)$ volts (V), a resistor of 0.02 ohms (Ω), an inductor of 0.001 henrys (H), and a capacitor of 2 farads (F). (These values are selected for numerical convenience; typical capacitance values are much smaller.) If the initial current and the initial charge on the capacitor are both zero (i.e., $I(0) = 0$, $q(0) = 0$), find $I'(0)$ by substituting the values for $L$, $R$, and $C$ into the governing RLC circuit equation and equating the two sides at $t = 0$. 
15. Use the initial condition $I(0) = 0$ and $I'(0)$ from problem 14 to solve the second-order differential equation and determine the current $I(t)$ in the circuit for $t > 0$.

16. Note that your solution current in 15 has two components: a transient current that tends to zero as $t \to \infty$, and a sinusoidal steady-state current that remains. Plot your solution up to $t = 1$. Can you see these two solution components?

17. After finishing your write-up for this lab, answer the following questions. **What was your favorite part of this lab?**  **What was the easiest part of this lab?**  **The hardest?**  **What was confusing?**  **What was interesting?**