Suppose three lakes of equal volume are interconnected, as in the following figure, with a net flow of water in the directions shown. A chemical spill releases 300 kg of a chemical pollutant into Lake 1. We would like to predict the amount of pollutant in each lake as a function of time.

Let’s construct a simplified model in which each lake is assumed to be well-mixed and none of the pollutant precipitates or evaporates out. Let \( l_1(t), l_2(t), \) and \( l_3(t) \) denote the mass of pollutant in each lake at time \( t \). Each lake can be viewed as a container with identical constant volume \( V \), and we assume that the rate of flow of water between any pair of lakes is the identical constant \( r \) liters per hour, as shown in the following figure. Then the flow of kg pollutant per hour between any pair of lakes is \( r \frac{l_i}{V} \).

1. Use the figure above to write down the system of differential equations for \( l_1, l_2, \) and \( l_3 \). 
HINT: Think of the inflow and outflow terms for each lake.
2. Based on the description above, what would the initial condition \( l_1(0), l_2(0), l_3(0) \) be?

This model is a homogeneous system of linear differential equations, but it is three-dimensional. A bit of insight can allow us to reduce this system to two equations so that we can plot the phase plane. Since none of the pollutant ever escapes the system of lakes, we would expect the total amount of pollutant in all three lakes to remain constant (at 300 kg). That is, we expect the sum

\[ Q = l_1(t) + l_2(t) + l_3(t) \]

to remain constant through time. This is what is called a *conserved quantity*.

3. Show that \( Q \) is a conserved quantity by differentiating it, i.e., show that \( dQ/dt = 0 \).

4. Show that the model reduces to a two-dimensional system by making the change of variables

\[ l_3(t) = Q - l_1(t) - l_2(t). \]

5. Rewrite the two-dimensional ODE system from question 4 in matrix notation. Notice that this is no longer a homogeneous system.

6. Plot the phase plane of the system from question 5 and find the equilibrium \( l_1^*, l_2^* \).

7. Notice that the equilibrium in question 6 is not at the origin. Define the new variables

\[ x_1(t) = l_1(t) - l_1^*, \text{ and } x_2(t) = l_2(t) - l_2^*. \]

Rewrite the initial value problem in terms of \( x_1 \) and \( x_2 \). Make sure to rewrite the initial condition in these new variables. Note that your new system should now be homogeneous.

8. Solve (by hand) the initial value problem from question 7.

9. Rewrite your solution from question 8 in terms of \( l_1, l_2, \) and \( l_3 \). For \( r/V = 0.2 \), plot each \( l_i(t) \) vs. time (all on the same plot).

10. Explain, in terms of the lakes and pollutants, what you see in your plots in question 9. Does this make sense based on the compartment model in the figures above?
Solving Systems of ODEs Numerically

Recall that the solution to the first-order ODE
\[
\frac{dx}{dt} = f(t, x)
\]
with initial condition \(x(0) = a\) could be computed numerically in MATLAB using a built-in ODE solver such as ode45.

\[
f = @(t,x) f(t,x);
\]
\[
x0 = a;
\]
\[
[t,x] = ode45(f, [t0,tf], x0);
\]

A system of first-order ODEs can be computed similarly. Consider
\[
\frac{dx_1}{dt} = f_1(t, x_1, x_2, \ldots, x_n),
\]
\[
\frac{dx_2}{dt} = f_2(t, x_1, x_2, \ldots, x_n),
\]
\[
\vdots
\]
\[
\frac{dx_n}{dt} = f_n(t, x_1, x_2, \ldots, x_n).
\]

This system can be expressed as a vectorized first-order ODE
\[
\frac{d\vec{x}}{dt} = F(t, \vec{x})
\]
where \(\vec{x} = (x_1, x_2, \ldots, x_n)\) and \(F = (f_1, f_2, \ldots, f_n)\). Solutions to this problem for initial conditions \(x_1(0) = a_1, x_2(0) = a_2, \ldots x_n(0) = a_n\) can be found using ode45 with a vectorized function \(F\) and initial condition \(\vec{x}_0 = (a_1, a_2, \ldots, a_n)\).

\[
F = @(t,x) [ f1(t,x(1), x(2), \ldots, x(n)); 
\]
\[
f2(t,x(1), x(2), \ldots, x(n)); 
\]
\[
\vdots 
\]
\[
f_n(t,x(1), x(2), \ldots, x(n)); ] ;
\]
\[
x0 = [a1; a2; \ldots; an;];
\]
\[
[t,x] = ode45(F, [t0,tf], x0);
\]

11. Use your original set of 3 differential equations from problem 1 and initial condition from problem 2 to solve the system of differential equations numerically, with \(r/V = 0.2\).

12. Compare this numerical solution to your exact solution from problem 9. Use this as an opportunity to check both your code in this part and your by-hand work in the previous parts.
13. What happens if the volumes of the lakes or flow rates are not identical? Say $r/V$ is different for each lake, e.g. $r_1/V_1 = 0.1$, $r_2/V_2 = 0.2$, and $r_2/V_2 = 0.3$. Use your code from 11 to plot the numerical solution with these new parameters and determine the long-term behavior of the system. Is there a new stable steady-state $(l_1^*, l_2^*, l_3^*)$?

14. Check this new steady state by hand, i.e., check whether $dl_1/dt = 0$, $dl_2/dt = 0$, $dl_3/dt = 0$ (or close to zero, given numerical errors) at this steady state.

15. What happens if we stop the outflow of lake 3, i.e. if $r_3 = 0$? Without doing any math or using MATLAB, what should happen to the amount of pollutant in each lake?

16. Use whatever technique you like best to determine the long-term behavior of the system with $r_1/V_1 = 0.2$, $r_2/V_2 = 0.2$, and $r_3/V_3 = 0$. Is there a new stable steady-state $(l_1^*, l_2^*, l_3^*)$? Plot either the numerical or exact solutions.

17. After finishing your write-up for this lab, answer the following questions. **What was your favorite part of this lab?** **What was the easiest part of this lab?** **The hardest?** **What was confusing?** **What was interesting?**