Lab 3: Heating and Cooling

Our goal in this lab is to formulate a mathematical model that describes the temperature profile of the human body as a function of the outside temperature and the heat regulated inside the body via homeostatic processes.

A natural approach to modeling the temperature of the body is to use compartmental analysis with a single compartment. Let $T(t)$ represent the temperature of the body at time $t$. Then the rate of change in the temperature is determined by all the factors that generate or dissipate heat.

We will consider the two main factors that affect the temperature in the body. First is the heat produced by the hydrolysis of ATP, an exothermic reaction. The thyroid gland is responsible for controlling the \textit{basal metabolic rate} (BMR), the rate of $O_2$ consumption and subsequent ATP hydrolysis. The BMR generates heat $h(t)$ kcal/time at time $t$. We can convert this to an increase in temperature $H(t)$ by division by the \textit{heat capacity} $C$ (kcal/$^\circ$C) of the body

$$H(t) = \frac{h(t)}{C}.$$  

The heat capacity of the body $C$ derives from the specific heat $c = 0.83$ kcal/($^\circ$C kg). For a typical adult body mass of $m = 70$ kg,

$$C = c \cdot m = 58 \text{ kcal/}^\circ\text{C}.$$  

1. We can average the heat generated by the BMR over a full day by measuring $t$ in days and setting $h(t) = 1800$ kcal/day. \textbf{Write the differential equation for the rate of change of body temperature per day.}

2. Solve this differential equation for $T(0) = 37^\circ$C. Plot the solution up to $t = 5$ days and decide whether this is a reasonable model.

3. What could we add to this model to make it more reasonable? Are there factors outside the body to consider?
Outside temperature $M(t)$ has an effect on the temperature in the body. Experimental evidence has shown that this fact can be modeled using **Newton's law of cooling**. This law states that the rate of change in the temperature $T(t)$ is proportional to the difference between the outside temperature $M(t)$ and the body temperature $T(t)$. Thus the rate of change in the body temperature due to outside temperature is

$$K[M(t) - T(t)].$$

The positive constant $K$ depends on the insulative properties of the skin and clothes on the body, as well as the ratio of surface area to volume of the body, but $K$ does not depend on $M$, $T$, or $t$. Hence, when the outside temperature is greater than the inside temperature, there is an increase in the body temperature (and vice-versa).

4. Write out the differential equation for $\frac{dT}{dt}$.

5. Solve this linear first-order differential equation using the integrating-factor method.

6. For the safety of the astronauts, it’s important to keep the temperature in the International Space Station constant. Given the constant BMR above of $H(t) = 1800/58 °C/day$, **how would the body temperature of an astronaut on the ISS (kept at 22°C) change over time?** Assume that $K = 3$, and plot the solution to the body temperature differential equation with initial condition $T(0) = 37°C$ up to $t = 5$ days.
7. The thyroid gland tries to keep the core body temperature at a constant 37°C by modulating the BMR. For the astronaut in the temperature-controlled (22°C) environment, how much heat must the BMR produce to maintain homeostasis?

**HINT:** In this problem, the outside temperature is constant, $M(t) = 22$. We want the body temperature to be constant as well, so that $T(t) = 37$, and at steady-state, i.e., $dT/dt = 0$. Assume that $K = 3$. What is $H(t)$?

8. With simulations, show that this level of heat production will keep the body in homeostasis (core temperature of 37°C). Plot the direction field for $dT/dt$ for $t$ between 0 and 5 days, $T$ between 30 and 40 °C. On top of the direction fields, plot numerical solutions to the differential equation with initial conditions $T(0) = 30, 32, 34, 36, 38, 40$ °C using ode45.

9. Use your differential equation solution with these values for $H(t), M(t), K$ and take the limit of $T(t)$ as $t \to \infty$. Does this level of heat production keep the body in homeostasis?
10. A baby has a much higher surface area to volume ratio than adults, so \( K \) is much bigger, say \( K = 5 \). If the baby is in a room-temperature room \( (M(t) = 22^\circ C) \), how does its body temperature change over time? Assume that the baby has the same constant BMR as the adult, \( H(t) = 1800/58 \circ/\text{day} \) (is this assumption reasonable?), and plot the solution to the body temperature differential equation with these parameters and \( T(0) = 37 \). Compare this to the first plot of adult body temperature. What differences do you see?

11. In this same environment, how much heat \( H(t) \) must the baby produce to maintain a core temperature of \( T(t) = 37^\circ C \)? Is this more or less heat per time than an adult?

12. With simulations, show that this level of heat production will keep the body in homeostasis (core temperature of \( 37^\circ C \)). Plot the direction field for \( dT/dt \) for \( t \) between 0 and 5 days, \( T \) between \( 30^\circ \) and \( 40^\circ \) C. On top of this, plot solutions to the differential equation with initial conditions \( T(0) = 30, 32, 34, 36, 38, 40^\circ \) C using ode45. How does this differ from the adult body temperature function?
13. Detective Amy Santiago is called to the scene of a crime where a dead body in a fluffy jacket has just been found. She arrives on the scene at 10:23 pm and begins her investigation. Immediately, the temperature of the body is taken and is found to be 80°F. She checks the smart thermostat and finds that the room has been kept at a constant 68°F for the past 3 days. Use Newton’s law of cooling to derive a model for the temperature $T(t)$ of the corpse. We will determine the cooling constant $K$ later.

14. After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 78.5°F. This last temperature reading was taken exactly one hour after the first one. Assuming that the victim’s body temperature was normal (98.6°F) prior to death, find the cooling constant $K$.

15. Detective Santiago checks the building’s security cameras and sees that a take-out delivery driver delivered lunch to the victim that day at 2:30 pm, a door-to-door phonebook salesperson knocked on their door at 3:20 pm, and a package was delivered at their doorstep at 4:15 pm. The victim never left their apartment that day and no other people came to their door. Who is the murder suspect? Explain your answer.

16. After finishing your write-up for this lab, answer the following questions. What was your favorite part of this lab? What was the easiest part of this lab? The hardest? What was confusing? What was interesting?