Lab 1 Worksheet

Welcome to 27B! The first half of this lab is a tutorial on the commands we’ll be using in MATLAB to model differential equations this quarter. The second half of this lab is an extended modeling problem. There is no need to write up or turn in anything related to the first half of this lab, but please work through it as it will help you with the coding for the second half. The second half (starting page 5) has 20 questions to answer and turn in as a lab write-up due the following week. You will be graded on not just whether your calculations are correct, but also on the quality of your writing and presentation. Unless otherwise indicated, answer questions in complete sentences or paragraphs, and display all plots in-line with proper axes labels and legends.

Using Mathematical Functions in MATLAB

You can define a mathematical function in Matlab using the @-syntax. The command
\[ g = @(x) \sin(x) \cdot x \]
defines the function \( g(x) = \sin(x) \cdot x \). You can then

- **evaluate the function** for a given \( x \)-value or range of \( x \) values (0,1,...,20):
  
  \[
g(0.3) \\
g(0:20)
\]

- **plot the graph of the function** over an interval:
  
  \[
  \text{plot}(0:20, g(0:20)); \\
  \text{xlabel}('x'); \text{ylabel}('g');
  \]

You can also define @-functions of several variables. The command
\[ G = @(x,y) x^4 + y^4 - 4*(x^2+y^2) + 4 \]
defines the function \( G(x,y) = x^4 + y^4 - 4(x^2 + y^2) + 4 \) of two variables. You can then

- **evaluate the function** for a given values of \( x, y \) or a rectangle of \( x, y \) values:
  
  \[
  G(1,2) \\
  [X,Y] = \text{meshgrid}(-3:.1:3,-2:.1:2) \\
  G(X,Y)
  \]

- **plot the graph of the function** as a surface over a rectangle in the \( x, y \) plane:
  
  \[
  \text{surf}(X,Y, G(X,Y)) \\
  \text{xlabel}('x'); \text{ylabel}('y'); \text{zlabel}('G')
  \]
Direction Fields

To visualize the behavior of a differential equation, we can plot its direction field. First download the file dirfield.m and put it in the same directory as your other m-files for the lab. Define an \( @\)-function \( f \) of two variables \( x, y \) corresponding to the right hand side of the differential equation \( y'(x) = f(x, y(x)) \). E.g., for the differential equation \( y'(x) = x^2 - y \) define

\[
f = @(x,y) x^2 - y
\]

You must use \( @(x,y) \) even if \( x \) or \( y \) does not occur in your formula. E.g., for the ODE \( y'(x) = y^2 \) you would use

\[
f = @(x,y) y^2
\]

To plot the direction field for \( x \) between \( x_0 \) and \( x_1 \) with a spacing of \( dx \) and \( y \) between \( y_0 \) and \( y_1 \) with a spacing of \( dy \) use

\[
\text{dirfield}(f, x_0:dx:x_1, y_0:dy:y_1)
\]

E.g., to plot the direction field for the differential equation \( y'(x) = x^2 - y \) for \( x \) and \( y \) between \( -4 \) and \( 4 \) (\( x_0 = y_0 = -4, x_1 = y_1 = 4 \)) with a spacing of \( dx = dy = 0.4 \), type

\[
f = @(x,y) x^2 - y;
dirfield(f, -4:0.5:4, -4:0.5:4)
\]

Pick a few points \( (x,y) \), compute the slope \( \frac{dy}{dx} \) by hand, and compare with the plot. The above command generates the same direction field as in Figure 1.6(a) in Section 1.3 of the textbook.
Solving an initial value problem numerically

We will be solving numerically initial value problems for many ordinary differential equations (ODEs) in this class. Let’s walk through how to do this with one of MATLAB’s built in initial value problem solvers, ‘ode45’.  

First define the @-function \( f \) corresponding to the right hand side of the differential equation \( y’(x) = f(x, y(x)) \). E.g., for the differential equation \( y’(x) = x^2 - y \) define

\[
 f=@(x,y) x^2-y
\]

To plot the numerical solution of an initial value problem: For the initial condition \( y(x_0) = y_0 \), you can compute the solution for \( x \) between \( x_0 \) and \( x_1 \) by using

\[
 [xs, ys] = ode45(f, [x0,x1], y0)
\]

For example, to plot the solution of the initial value problem \( y’(x) = x^2 - y, y(-4) = 1 \) in the interval \([-4,4]\) use

\[
 [xs,ys]=ode45(f,[-4,4],1) \\
 plot(xs,ys,’o-’)
\]

This produces the following figure.

![Plot of solution](image)

The circles mark the values which were actually computed (the points are chosen by Matlab to optimize accuracy and efficiency). The vectors \( xs \) and \( ys \) contain the coordinates of these points, to see them as a table type

\[
 [xs,ys]
\]
You can plot the solution without the circles using

\[ \text{plot(xs,ys)} \]

To combine plots of the direction field and several solution curves use the commands **hold on** and **hold off**. After obtaining the first plot, type **hold on**. Then all subsequent commands plot in the same window until the command **hold off**.

E.g., plot the direction field for \( y'(x) = x^2 - y \) and the 6 solution curves with the initial conditions \( y(-4) = 10, -20, -50, -80, -110, -140 \):

\[
f = @(x,y) x^2-y;
\]
\[
\text{figure(7); hold on;}
\]
\[
\text{for y0=10:-30:-140}
\]
\[
[ts,ys]=\text{ode45}(f,[-4,4],y0);
\]
\[
\text{plot(ts,ys, 'LineWidth', 4)};
\]
\[
\text{end}
\]
\[
\text{dirfield(f,-4:0.5:4, -4:0.5:4)};
\]
\[
\text{legend('y(-4) = 10', 'y(-4) = -20', 'y(-4) = -50',...}
\]
\[
' y(-4) = -80', 'y(-4) = -110', 'y(-4) = -140');
\]
\[
\text{xlabel('x'); ylabel('y'); hold off;}
\]

This generates the same direction field and solution curves as in Figure 1.6(b) in Section 1.3 of the textbook.

Adapted from http://terpconnect.umd.edu/~petersd/246/matlabode.html
Exponential and Logistic Growth in a Fishery

Here, we will walk through how to build a mathematical model from pure biological intuition. We've been hired by a rancher to model the population of fish in her seasonal pond, and she needs our help to determine how many fish she can sustainably fish out per day.

At the beginning of the summer, the rancher puts a few fish (equally male and female) in the pond, which has nothing else in it but plenty of fish food. Let $N$ be the number of fish in the pond. There are no significant predators, so we will assume that the only things affecting the fish numbers are natural fish fecundity and mortality.

Fecundity $f(N)$ measures how prolific the fish are at breeding. Let $r_f$ be the average chance per day that an individual fish produces an offspring, so that the number of fish born per day is the total number of fish $N$ multiplied by the individual chance of reproduction $r_f$:

$$f(N) = r_f N.$$  

1. Mortality is the rate at which fish die per day. Let $1/r_m$ be the average lifespan (in days) of a fish, so that the chance that an individual fish dies in a given day is $r_m$. **Write a function $m(N)$ for the number of fish that die per day.**

2. Let $r = r_f - r_m$ be the relative growth rate, and **write the differential equation describing the change in the population of fish per day**, i.e., $dN/dt$ where $t$ is measured in days.
3. For \( r_f = 3 \), \( r_m = 1 \), plot (all on the same graph) the numerical solution (the population as a function of time) using ode45 in MATLAB for different initial populations \( N(0) = 0, 1, 2, 3, 4, 5 \). On the same graph, plot the direction field for \( t \) between 0 and 5 and \( N \) between 0 and 100. Label the axes and denote which initial condition yields each solution curve.

4. For the following pairs of \((r_f, r_m)\) values—(3,1), (2,2), (1,2)—which do you expect to grow and which do you expect to die out? (Hint: What is the sign of \( dN/dt \) for each?) Check your answers by plotting the population as a function of time in each case, using initial population \( N(0) = 2 \).
5. (We will cover how to do this in video lecture 3. You can skip this question today and answer it after we cover this material later. If you have learned how to do this in one of your previous math classes, you can return to this question at the end of the lab today if you have time.) Solve the ODE by hand with arbitrary initial fish population $N(0) = N_0$. What happens to the fish population as $t \to \infty$? How does the limit depend on $r_f$ and $r_m$?

6. Do you think this is a reasonable model for the population growth? List a few biological constraints that would change the growth rate.
Ecologists often incorporate *intraspecific competition* (competition between members of the same species) into their population models. Our model assumes that fecundity and mortality are proportional to the number of fish in the pond. If space is limited and food is scarce, however, it would make sense for fecundity to decrease and mortality to increase as the population grows large.

7. Note that $N^2$ is the number of possible competitive interactions between fish, and let $c$ be the average reduction in the growth rate due to a competitive interaction. **Add a term (including $c$ and $N^2$) to the model to account for intraspecific competition.**

8. Show that the change of parameters $K = \frac{r}{c}$ yields the *Logistic Growth Model*:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$
9. For \( r = 2, K = 50 \), plot the population as a function of time with initial populations \( N(0) = 0, 1, 2, 3, 4, \) and 5. Also plot the direction field for \( t \) between 0 and 5, and for \( N \) between 0 and 100. **How does this compare to the plots from question 3?**

10. \( K \) is called the *carrying capacity* of the population. **Based on your plots from question 9, what do you think this means?**

11. Let’s take a step back. What do all of the variables and parameters in our model mean biologically? What units are they measured in? Write a paragraph, draw a diagram, or make a table to describe \( N, t, r, \) and \( K \) in a way that makes sense to you.
12. (We will cover how to do this in video lecture 3. You can skip this question today and answer it after we cover this material later. If you have learned how to do this in one of your previous math classes, you can return to this question at the end of the lab today if you have time.) Solve this differential equation for arbitrary initial fish population $N(0) = N_0$. What happens to the fish population as time goes on, i.e., as $t \to \infty$?

13. (We will cover how to do this in Monday lecture. You can skip this question today and answer it after we cover this material on Monday. If you have learned how to do this in one of your previous math classes, you can return to this question at the end of the lab today if you have time.) If the rancher initially places 2 fish in the pond, about how many days will it take for the population to reach 50 fish with growth rate $r = 1$?
14. Finally, we get to the question the rancher hired us to answer: How many fish can she sustainably fish out per day? To figure this out, let’s suppose the rancher fishes out $F$ fish per day, and add a term to the model to account for this constant fishing rate.

15. Plot the numerical solution to this new ODE with $r = 1$, $K = 100$, $N(0) = 50$, and $F = 20$. What do you observe about the solution?

16. Do you expect the solution to change if you change $F$? Can you think of any fishing rates $F$ that are definitely sustainable or definitely not sustainable?
17. Plot the numerical solution to the ODE with $r = 1$, $K = 100$, $N(0) = 50$, and a range of $F$ between 0 and 50 (pick as many as you think best illustrates the possibilities). Plot all your solution curves together vs. time, including a legend with $F$ labels.

18. What do you observe in the behavior of these solutions as you change the parameter $F$?

19. Estimate the maximum sustainable fishing rate $F$ for this initial population level, i.e., what is the largest number of fish $F$ the rancher can fish out each day?

20. After finishing your write-up for this lab, answer the following questions: What was your favorite part of this lab? What was the easiest part of this lab? The hardest? What was confusing? What was interesting? Any other comments about the lab are welcome!