In epidemiology, SIR models are compartmental models that are used to predict how infectious diseases spread through populations. They are also used to determine policies like social distancing that can minimize the spread of the disease. An SIR model divides the population into three subpopulations: Susceptible $S(t)$, Infectious $I(t)$, and Recovered/Removed $R(t)$. 

$\beta > 0$ is the infection rate, or the average fraction of the population infected per day by each infectious person. $\gamma > 0$ is the average recovery rate, or $\gamma = 1/D$, where $D$ is the average period of time (in days) an individual is infectious. This can be modeled by the following system of three nonlinear differential equations:

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dI}{dt} = \beta SI - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

Because neither $S$ or $I$ depend on $R$, we don’t need to look at the third equation to understand the disease dynamics. Unless otherwise asked, consider only the $S, I$ equations.

**Question 1** Find the $S$ and $I$ nullclines for the above system and plot them in the $S - I$ phase plane. (If you use pplane to plot, still find the nullclines by hand)
The key prediction of this model is whether an epidemic will occur. This happens if the disease spreads and the number of infectious people \( I \) increases, i.e., if \( dI/dt > 0 \) for some time. This means that when \( \beta S/\gamma > 1 \), epidemics occur.

**Question 2** The quantity \( \beta S/\gamma = R_{\text{eff}} \) is the effective reproduction number for the disease, the average number of people that each infected person infects (over the course of their infectious period).\(^2\) Using this definition, explain in words why \( R_{\text{eff}} > 1 \) leads to epidemics and \( R_{\text{eff}} < 1 \) does not. What would happen if \( R_{\text{eff}} = 1 \)?

**Question 3** Suppose that the total population is \( N = 1000 \) persons, the length of the infectious period is 5 days (so \( \gamma = 1/5 = 0.2 \) (days\(^{-1}\)), and that the infection rate is \( \beta = 0.01 \) (days-persons\(^{-1}\)). If the initial population has 10 infectious people and everyone else susceptible, compute the effective reproductive number \( R_{\text{eff}} \) initially (with \( S(0) \)). Will an epidemic occur?

**Question 4** Use pplane or Euler’s method to numerically compute the solutions to the SI equations to verify your answer to problem 3. Give the plot of \( S(t) \) and \( I(t) \).

**Question 5** Suppose that the total population is \( N = 1000 \) persons, the length of the infectious period is 2 days (so \( \gamma = 1/2 = 0.5 \) (days\(^{-1}\)), and that infection rate \( \beta = 0.00005 \) (days-persons\(^{-1}\)). If the initial population has 10 infectious people and everyone else susceptible, compute the effective reproductive number \( R_{\text{eff}} \) initially (with \( S(0) \)). Will an epidemic occur?

**Question 6** Use pplane or Euler’s method to numerically compute the solutions to the SI equations to verify your answer to problem 5. Give the plot of \( S(t) \) and \( I(t) \).
A goal of social distancing and other public health measures is to “flatten the curve”, i.e., to lower the peak number of infectious people $I$ so as to not overwhelm the healthcare system. In questions 7-11, we’ll show how the SIR model demonstrates the outcomes of these public health measures.

**Question 7** The infection rate $\beta$ can be modeled as

$$\beta = \frac{pC}{N},$$

where $p$ is the probability of transmission of the disease, $C$ is the average number of person-to-person contacts (for an individual), and $N$ is the total number of people in the population. Which variable ($p$, $C$, or $N$), does social distancing affect and how?

**Question 8** Which variable ($p$, $C$, or $N$), does wearing masks affect and how?

**Question 9** Suppose that the total population is $N = 1000$ persons, the length of the infectious period is 5 days (so $\gamma = 1/5 = 0.2$ (days)$^{-1}$), and that infection rate $\beta = 0.01$ (days$\cdot$persons)$^{-1}$ (these are the same values as in problems 3-4). If the initial population has 10 infectious people and everyone else susceptible, what is the peak number of infectious people $I$?

**Question 10** Suppose that the government implemented strict social distancing and mask-wearing policies immediately after the initial 10 infectious cases were identified, bringing the infection rate $\beta$ down to 0.001. Compute the new effective reproductive number $R_{eff}$ initially (with $S(0)$) and the new peak number of infectious people $I$.

**Question 11** To compare these epidemics, plot $I$ vs. $t$ for: 1) no safety measures (as in problem 9), and 2) with safety measures, i.e., #Flattened (as in problem 10).