In this lab, we’ll explore autonomous differential equations a bit more in the context of Newton’s Law of cooling problems. Specifically, we’ll

1. find solutions to an autonomous differential equation
2. understand long term behavior of the solution
3. use Euler’s method to approximate a solution

Setup

In preparation for Thanksgiving, Carter places his turkey roast in the oven and lets it bake until it reaches a temperature of 400 degrees Fahrenheit and then takes it out to cool. Unfortunately, he immediately forgets about it and lets it cool for an inappropriate amount of time. In this problem, we’ll model the temperature of the temperature of the turkey and try to figure out if it will be too cold for Carter to eat during his Thanksgiving Zoom meeting.

We’ll assume the temperature of the turkey (which we’ll call $P$) at $t$ minutes after taking it out of the oven can be modeled by Newton’s Law of Cooling, which says, the temperature $P$ satisfies

$$\frac{dP}{dt} = \alpha (A - P),$$

where $A$ is the ambient temperature, which we’ll assume is a brisk 50 degrees Fahrenheit and $\alpha$ is a parameter relating to how quickly the turkey cools\(^1\), which we’ll say is $\alpha = 0.1$.

**Question 1** As with most differential equations in the class, an initial condition is necessary. Somewhere in the previous information I’ve hidden the appropriate initial condition, so what is $P(0)$?

**Question 2** Although we won’t solve the differential equation yet, is $dP/dt$ positive or negative? That is, is $P > A$ or $P < A$? Why does this make sense? \(^2\)

**Question 3**

Find the solution to the differential equation with this initial condition using integration via separation of variables.

\(^1\) often called a cooling constant. note it has units 1/time, also known as a rate

\(^2\) it’s also worth thinking about if the temperature of the turkey was below that of the ambient temperature, as if Carter put it in the freezer during a bout of sleepwalking. What happens then?
**Question 4**

Say Carter goes to take a quick nap but ends up falling asleep for a very long time. What happens to the temperature of the turkey as \( t \to \infty \)? Why does this make sense?

Make a plot of the function that demonstrates this behavior.

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**Euler’s Method**

Although a previous part gave the solution to this differential equation, it’s not always the case that we can write something like this down and therefore want a way of approximating this. One way of doing so is Euler’s method. Here I’ll give a brief reminder of what Euler’s method says.

Suppose we have some differential equation that looks like

\[
\frac{dP}{dt} = f(P), \quad P(0) = P_0.
\]

We can then use a tangent line approximation to the true solution \( P(t) \) to approximate the value at some time close to our initial point. Say, our timesteps are of size \( \Delta t \), then the first point we approximate would then be \(^3\)

\[
P(\Delta t) \approx \hat{P}(\Delta t) = P(0) + P'(0)\Delta t,
\]

but note that we know \( P' = f \), so we know everything in this equation! How exciting. Repeating this procedure, calling \( t_{\text{new}} = t_{\text{old}} + \Delta t \), we can then update using the same tangent line idea

\[
\hat{P}_{\text{new}} = \hat{P}_{\text{old}} + P'(\hat{P}_{\text{old}})\Delta t.
\]

**Question 5**

Complete the code `eulers_method.r` and show the resulting produced by utilizing Euler’s method for this problem. Take timesteps of \( dt = 5 \) minutes and go until \( t = 30 \) minutes. How good of an approximation is this?

**Question 6** Try a smaller value of \( dt \) of your choice. Does the approximation get better or worse? Why?
Murder Mystery

Question 7 Detective Amy Santiago is called to the scene of a crime where a body has just been found. She arrives on the scene at 10:23 pm and begins her investigation. Immediately, the temperature of the body is taken and found to be 80°F. She checks the smart thermostat and finds that the room has been kept at a constant 68°F for the past 3 days. Use Newton’s law of cooling to derive a model for the temperature $H(t)$ of the corpse. We will determine the cooling constant $\alpha$ later.

Question 8 After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 78.5°F. This last temperature reading was taken exactly one hour after the first one. Assuming that the victims body temperature was normal (98.6°F) prior to death, find the cooling constant $\alpha$.

Question 9 Detective Santiago checks the building’s security cameras and sees that a take-out delivery driver delivered lunch to the victim that day at 2:30 pm, a door-to-door phonebook salesperson knocked on their door at 3:20 pm, and a package was delivered at their doorstep at 4:15 pm. The victim never left their apartment that day and no other people came to their door. Who is the murder suspect? Explain your answer.\footnote{Figure out how long it would take for the body to cool from 98.6°F (at time of death) to 80°F (at 10:23 pm).}