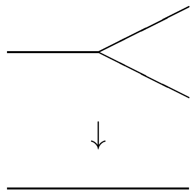
Connectivity of Fibres

Kai-Uwe Bux

preliminary version, 12th September 1999
subjected to change

In [2], Quillen pointed out how to relate homotopical and homological properties of two spaces E and X when a map $\pi : E \to X$ is given whose fibres are highly connected. In this paper we will address the complementary problem: Given a map between highly connected spaces, can we deduce vanishing of homotopy or homology groups of the fibre up to a certain dimension?

Even the trivial example of a 3-valent tree mapping to a line shows that in general fibres need not even be connected even if E and X are contractible. Hence we have to impose some assumptions on the map.



Let $\pi : E \to X$ be a simplicial map between simplicial complexes. For each closed simplex $e \in E$ and each coface $x \geq \pi(e)$, we define the *relative link* to be the poset

$$\mathrm{Lk}_x^\pi(e) := \{\bar{e} \geq e \mid \pi(\bar{e}) = x\}$$

of all cofaces of e that are mapped *onto* x with the order relation induced by the face relation in E. Recall that the geometric realization of a poset is defined to be the geometric realization of its associated flag complex which is a simplicial complex. Points in the geometric realization of $\mathrm{Lk}_x^\pi(e)$ represent directions starting at the barycentre of e whose π-projection in X are pointing towards the barycentre of x. Note that $\mathrm{Lk}_{\pi(e)}^\pi(e)$ is a contractible poset since it contains e as a unique minimal element.

Recall that a space is called m-connected if, for every $-1 \leq l \leq m$, every continous map defined on the l-sphere extends to a map defined on the $(l+1)$-ball. Here the (-1)-sphere is empty and bounds a point. So being (-1)-connected is being non-empty. In Section 6 we will prove Theorem 6.1 which implies out main result

Theorem A. *Let $\pi : E \to X$ be a simplicial map. Suppose that for every simplex $e \in E$ and for every coface $x \geq \pi(e)$ the relative link $\mathrm{Lk}_x^\pi(e)$ is $(m-1-\dim e)$-connected. Let $\bullet_X$ be a vertex in X, and let F be the fibre $\pi^{-1}(\bullet_X)$ over $\bullet_X$. Then there is an exact sequence*

$$\pi_m F \to \pi_m E \to \pi_m X \to \pi_{m-1} F \to \cdots \to \pi_l F \to \pi_l E \to \pi_l X \to \cdots \to \pi_0 X$$

of homotopy groups.

Hence, if E and X are themselves highly connected it follows that the Fibre F is highly connected as well.

In Section 1 we will prove a special case of the main theorem exhibiting the connection of the result to Morse theory in the sense of M. Bestvina and N. Brady as laid down in [1]. Then we outline the idea of proof for the theorem above in Section 2. This programme is carried out in the following sections.

I want to thank Prof. Marc Burger and the Forschungsinstitut für Mathematik at the ETH-Zürich for their kind hospitality. Most of this research was brought about while the author was guest of the Forschungsinstitut that provided a very warm and fruitfull atmosphere.

1 Connections to Bestvina-Brady-Morse theory

Our approach is inspired by Morse theory on cell complexes as developed by M. Bestvina and N. Brady in [1]. We will see, how their results fit into this framework. They used ascending and descending links of a real-valued Morse function to deduce connectivity of its fibres in order to establish finiteness properties of groups acting fibre-preserving on a space. The starting point of this paper was the attempt to understand what they have been doing in order to generalize their arguments to non real-valued Morse functions.

Before discussing how to prove Theorem A in full generality, we shall confine ourselves to a special case that will exhibit the connection to Bestvina-Brady-Morse theory.

Theorem 1.1. *Let $\pi : E \rightarrow X$ and F as in Theorem A and suppose that $X = \bigcup_{i\geq 1} X_i$ is the ascending union of a sequence $\bullet_X = X_1 \subset X_2 \cdots$ of finite subcomplexes each of which can be collapsed to its predecessor. Then X is contractible and the inclusion $F \hookrightarrow E$ induces isomorphisms $\pi_l F \hookrightarrow\!\!\!\!\rightarrow \pi_l E$ for $l < m$ and an epimorphism $\pi_m F \twoheadrightarrow \pi_m E$.*

When X is just a simplicially decomposed real line, this reduces to Bestvina-Brady-Morse theory for simplicial complexes. The notion of ascending and descending links are easily seen to coincide with our notion of relative links up to barycentric subdivision.

Proof. Without loss of generality, we assume that X_i is obtained from X_{i+1} by collapsing the free face x_{i+1} into its coface $\tilde{x}_{i+1}$.

Put $E_i := \pi^{-1}(X_i)$. Consider a simplex $e \in E_{i+1}$ with $x_{i+1} = \pi(e)$ of maximum dimension. Of course, this simplex need not be free. However, since it is of maximal dimension among all simplices above x_{i+1}, all its cofaces are mapped onto $\tilde{x}_{i+1}$ and contribute therefore to the relative link $\mathrm{Lk}^{\pi}_{\tilde{x}_{i+1}}(e)$.

Imagine e and all its cofaces were removed from E_{i+1} and call the resulting complex E'_{i+1}. The inverse process of reinserting the simplices amounts, up to a deformation, to cone off a subspace of E'_{i+1} of the form $\mathrm{Lk}^{\pi}_{\tilde{x}_{i+1}}(e) * \partial e$ which is $(m-1)$-connected by assumption. Hence coning off this subspace does not alter homotopy groups in dimension $< m$ and does not introduce non-trivial homotopy elements in dimension m.

Applying this argument to all simplices above x_{i+1} from high dimensional ones to low dimensions, we can get rid of all of them and see that the inclusion $E_i \hookrightarrow E_{i+1}$ induces isomorphisms $\pi_l E_i \rightarrow \pi_l E_{i+1}$ for $l < m$ and a surjection for $l = m$.

The argument of the preceding paragraph shows that at each step the homotopy groups exhibit the required behaviour. So summing up we conclude that the inclusion $F = E_1 \hookrightarrow E$ induces isomorphisms $\pi_l F \to \pi_l E$ for $l < m$ and a surjection for $l = m$. **q.e.d.**

2 Idea of proof

Denote by $\mathbb{B}^k$ the k-dimensional ball identified with the northern hemisphere. So we have

$$\mathbb{B}^k \subset \mathbb{S}^k = \mathbb{B}^k \cup -\mathbb{B}^k = \partial \mathbb{B}^{k+1} \subset \mathbb{B}^{k+1}$$

Let us call a map $\pi : E \to X$ of topological spaces <u>m-lifting</u>, if for every $l \leq m$ and for every pair $(\psi, \tilde{\psi})$ making the outer square in

$$\begin{tikzcd} \mathbb{B}^l \arrow[r, "\tilde{\psi}"] \arrow[d, hook] & E \arrow[d, "\pi"] \\ \mathbb{B}^{l+1} \arrow[r, "\psi"] \arrow[ur, dotted, "\tilde{\Psi}"'] & X \end{tikzcd} \tag{1}$$

commute, the lifting problem has a solution $\tilde{\Psi}$ producing two commutative triangles. This is, π has a solvable homotopy lifting problem for the balls up to dimension m. Hence, by definition, π is a Serre-fibration if and only if it is ∞-lifting.

It is well known that a Serre fibration gives rise to a long exact sequence of homotopy groups. Inspecting the proof it is clear, that an m-lifting map gives rise to a finite exact sequence of homotopy groups. So the first idea to prove Theorem A is to establish that π is m-lifting.

We can use the topological setting, in order to outline the main steps of this procedure. The following two propositions reduce the task of proving a map π to be m-lifting to local considerations. We will merely outline the proofs since these statements only serve as signposts.

Proposition 2.1. *Let $\pi : E \to X$ be a map of topological spaces and suppose that $X = \bigcup_{j\in I} X_j$ is an open cover such that, for each $j \in I$, the restriction of π to $E_j := \pi^{-1}(X_j)$ is m-lifting. Then π is m-lifting.*

Proof. Suppose we are given a pair $(\psi, \tilde{\psi})$ making the outer square of (1) commute. Identify $\mathbb{B}^{l+1}$ with the $(l+1)$-simplex and think of $\mathbb{B}^{l} \subset \mathbb{B}^{m+1}$ as consisting of all but one face, the free face. You should think of the process of extending $\tilde{\psi}$ to all of $\mathbb{B}^{l+1}$ as a process that is inverse to collapsing the free face of the simplex. By iterated barycentric subdivision, eventually the simplices in the subdivision become small, that is, each of them has an image that is entirely contained in one of the open sets X_j. On the other hand, there is still a strategy for collapsing the subdivided simplex $\mathbb{B}^{l+1}$ to $\mathbb{B}^{l}$. The inverse strategy can be used to construct a lifting, because now, all the lifting problems are small whence they have solutions by hypothesis. **q.e.d.**

Proposition 2.2. *Let $\pi : E \to X$ be a map of topological spaces and suppose that $E = \bigcup_{j\in I} E_j$ is an open cover such that, for each non-empty finite subset $J \subseteq I$ the restriction of π to the intersection $E_J := \bigcap_{j\in J} E_j$ is $(m+1-|J|)$-lifting. Then π is m-lifting.*

Proof. For $l \leq m$, we have to solve the lifting problem (1) posed by a pair of functions $(\psi : \mathbb{B}^{l+1} \to X, \tilde{\psi} : \mathbb{B}^{l} \to E)$. We identify $\mathbb{B}^{l+1}$ with the product $\Delta^{l} \times [0,1]$ and $\mathbb{B}^{l} \subset \mathbb{B}^{l+1}$ with $\Delta^{l} \times \{0\}$.

Fix a subdivision of Δ^{l} into convex polyhedral cells satisfying the following two conditions: First, for each l-cell c_l there is an index $j_{c_l} \in I$ such that $\tilde{\psi}(c_l \times \{0\}) \subseteq E_{j_{c_l}}$–this can be achieved by passing to a subdivision. Second, every k-cell is a face of at most $1+l-k$ l-cells. Such a subdivision can be obtained, for example, as the dual of a sufficiently iterated barycentric subdivision. Let $\Delta^{(k)}$ denote the k-skeleton of the subdivision.

Suppose, $\tilde{\Psi}$ is already defined on $\Delta^{l} \times \{0\} \cup \Delta^{(k-1)} \times [0,1]$ such that for each k-cell c_k and each maximum coface c_l we have

$\tilde{\Psi}(\partial c_k \times [0,1]) \subseteq E_{j_{c_l}}$. In this case, the hypotheses ensure that we can solve the lifting problem for $c_k \times \{0\} \cup \partial c_k \times [0,1] = \mathbb{B}^k \subset \mathbb{B}^{k+1} = c_k \times [0,1]$ extending $\tilde{\Psi}$ to $\Delta^l \times \{0\} \cup \Delta^{(k)} \times [0,1]$ preserving the assumption.

Starting with $k = 0$, we can hence find inductively a solution to the lifting problem. **q.e.d.**

So a first idea of proving Theorem A is to consider the cover of X by open stars of vertices. So fix a vertex $x_0 \in X$. The preimage of its open star is itself covered by the open stars of those vertices $e_0 \in E$ being mapped to the vertex x_0. Intersections of these open stars are open stars of simplices $e \in E$ with $\pi(e) = x_0$. So we are reduced to consider restrictions of π to open stars of simplices. Here the hypotheses on relative links come into play.

However, it turns out that this approach does not yield Theorem A. The homotopy lifting property is somewhat to restrictive. The hypotheses of Theorem A roughly say that l-simplices in E are no obstacles to the lifting process for an $(m-l)$-ball. However, for using the homotopy lifting property, we have to require that there are no obstacles to the lifting of m-balls at all. Using the homotopy lifting approach, we could prove a weaker version of Theorem A where all relative links are supposed to be m-connected.

So the strategy for proving Theorem A will be as follows:

1. We define a certain weakening of the m-lifting property that still implies the existence of an exact sequence of homotopy groups. We will call maps satisfying this conditions <u>weakly m-lifting</u>.

2. We prove analogoues of Proposition 2.1 and Proposition 2.2 for weak m-liftings.

3. Using these analogoues, we will reduce the weak m-lifting property to relative links.

The following section, however, is devoted to some technical lemmas yielding a slightly more general version of Theorem A.

3 Posets and simplicial complexes

A weak simplicial complex is a poset P such that for each element $p \in P$ the subposet of faces $p^{\leq} := \{p' \mid p' \leq p\}$ is isomorphic to the poset of non-empty subsets of a finite set $\{0, \ldots, \dim p\}$. Note that a weak simplicial complex is a simplicial complex if and only if for each two elements having a common face there exists a unique maximal common face. Hence, a weak simplicial complex can be realized geometrically such that each element $p \in P$ is represented by a $\dim p$-simplex. However, two simplices may intersect in more than one face. The notion of relative links carries over in an obvious way to order-preserving maps of posets.

For each poset P there is a corresponding "complex of flags" $\overline{P} := \{p_0 < \cdots < p_l \mid p_i \in P\}$ which is a poset where a flag is a face of another flag if the latter refines the former. This simplicial complex is the barycentric subdivision of P. In this section we shall prove

Proposition 3.1. *Let $\pi : E \to X$ be a morphism of weak simplicial complexes. Suppose that for each element $e \in E$ and each coface $x \geq \pi(e)$ the relative link $\mathrm{Lk}^{\pi}_{x}(e)$ is $(m - 1 - \dim e)$-connected. Then the same condition holds for the induced map $\overline{\pi} : \overline{E} \to \overline{X}$ between the corresponding barycentric subdivisions.*

This implies that Theorem A carries over to weak simplicial complexes in the following slightly generalized form.

Theorem 3.2. *Let $\pi : E \to X$ be a morphism of weak simplicial complexes. Suppose that for every element $e \in E$ and for every strict coface x of $\pi(e)$ the relative link $\mathrm{Lk}^{\pi}_{x}(e)$ is $(m - 1 - \dim e)$-connected. Let P be a contractible subposet of X and let $F := \pi^{-1}(P)$ be the fibre over P. Then there is an exact sequence*

$$\pi_m F \to \pi_m E \to \pi_m X \to \pi_{m-1} F \to \cdots \to \pi_l F \to \pi_l E \to \pi_l X \to \cdots \to \pi_0 X$$

of homotopy groups.

Proof. Passing to the barycentric subdivision $\overline{\pi} : \overline{E} \to \overline{X}$ and applying Proposition 3.1, we see that for $p \in P$ the preimage $F_p := \pi^{-1}(p)$ fits into the exact sequence.

On the other hand, we consider the map $\pi\,|_F : F \to P$. This in turn satisfies the assumptions of our theorem as well, whence we have – applying only what we have shown in the preceding paragraph – an exact sequence

$$\pi_m F_p \to \pi_m F \to \pi_m P \to \pi_{m-1} F_p \to \cdots \to \pi_l F_p \to \pi_l F \to \pi_l P \to \cdots \to \pi_0 P$$

wherein all $\pi_l P$ vanish. Putting together both exact sequences, we obtain

$$\begin{tikzcd}
\pi_m F_p \arrow[r] \arrow[dr] & \pi_m F \arrow[r] \arrow[d] & 0 \arrow[r] & \pi_{m-1} F_p \arrow[r, equal] & \pi_{m-1} F \arrow[r] \arrow[d] & \cdots \\
 & \pi_m E \arrow[rr] & & \pi_m X \arrow[u] & \pi_{m-1} E \arrow[r] & \cdots
\end{tikzcd}$$

which proves the theorem. **q.e.d.**

In order to prove Lemma 3.1, we have to deal at first with maps between simplices. Hence, for a finite set N, we denote by $\mathbb{P}(N)$ the subset of all non-empty subsets of N and put $\mathbb{P}^*(N) := \mathbb{P}(N) \cup \{\emptyset\}$. Note that both posets are contractible since the first is just a barycentrically subdivided simplex whereas the second is a cone over such a subdivided simplex.

Lemma 3.3. *Let $\pi : N \twoheadrightarrow M$ be a surjection of finite maps. Let $N' \subseteq N$ and $\pi(N') \subseteq \tilde{M} \subseteq M$. Then the poset*

$$\mathbb{P}(N', \pi; \tilde{M}) := \left\{ \tilde{N} \,\middle|\, N' \subseteq \tilde{N} \subseteq N,\ \pi(\tilde{N}) = \tilde{M} \right\}$$

is contractible.

Proof. We put $M' := \pi(N')$ and $N'' := \pi^{-1}(M')$. If $M' = \tilde{M}$, we have

$$\mathbb{P}(N', \pi; \tilde{M}) \cong \mathbb{P}(N'' \setminus N')$$

whence the claim follows. Otherwise the map

$$\left(\mathbb{P}^*(N'' \setminus N') \times \bigtimes_{m \in \tilde{M} \setminus M'} \pi^{-1}(m)\right) \to \mathbb{P}(N', N; \tilde{M})$$

given by

$$(N_\emptyset, (N_m)) \mapsto N' \cup N_\emptyset \cup \bigcup_{m \in \tilde{M} \setminus M'} N_m$$

is an isomorphism. **q.e.d.**

Corollary 3.4. *Let $\pi : N \to M$ be a surjection. Fix two subsets $N' \subseteq N$ and $\tilde{M} \subseteq M$ with $\pi(N') \subseteq \tilde{M}$. Then the poset of flags*

$$\mathbb{F}_{N'}(\pi; \tilde{M}) := \left\{ N_0 \subset \cdots \subset N_l \,\middle|\, N' \subseteq N_0,\ N_i \subseteq N,\ \pi(N_i) = \tilde{M} \right\}$$

is isomorphic to the barycentric subdivision of $\mathbb{P}(N', \pi; \tilde{M})$ whence it is contractible.

Let $\pi : P \to P'$ be a morphism of posets, $p_- \leq p_+$ elements of P, and a flag $\mathcal{F}' := p'_0 < \cdots < p'_l$, we define the posets

$$\mathbb{F}_{p_-}(\pi; \mathcal{F}') := \left\{ p_{0,0} < \cdots < p_{0,i_0} < \cdots < p_{l,0} < \cdots < p_{l,i_l} \,\middle|\, p_{j,k} \in P,\ p_- \leq p_{0,0},\ \pi(p_{j,k}) = p'_j \right\},$$

$$\mathbb{F}_{=p_-}(\pi; \mathcal{F}') := \left\{ p_{0,0} < \cdots < p_{0,i_0} < \cdots < p_{l,0} < \cdots < p_{l,i_l} \,\middle|\, p_{j,k} \in P,\ p_- = p_{0,0},\ \pi(p_{j,k}) = p'_j \right\},$$

and

$$\mathbb{F}_{=p_-}^{=p_+}(\pi; \mathcal{F}') := \left\{ p_{0,0} < \cdots < p_{0,i_0} < \cdots < p_{l,0} < \cdots < p_{l,i_l} \,\middle|\, p_- = p_{0,0},\ p_+ = p_{l,i_l},\ \pi(p_{j,k}) = p'_j \right\}.$$

Lemma 3.5. *Let $\pi : N \twoheadrightarrow M$ be a surjection. Fix a subset $N' \subseteq N$ and a flag $M_0 \subset \cdots \subset M_l$ with $\pi(N') \subseteq M_0 \subset \cdots \subset M_l \subseteq M$. Then the poset $\mathbb{F}_{N'}(\pi; M_0 \subset \cdots \subset M_l)$ is contractible.*

Proof. We use induction on l. There is a natural embedding

$$\mathbb{F}_{N'}(\pi; M_0 \subset \cdots \subset M_l) \hookrightarrow \mathbb{F}_{N'}(\pi; M_0 \subset \cdots \subset M_{l-1}) \times \mathbb{F}_{N'}(\pi; M_l)$$

For a given flag $N_{l,0} \subset \cdots \subset N_{l,i_l} \in \mathbb{F}(\pi; M_l)$ the set of partners in $\mathbb{F}_{N'}(\pi; M_0 \subset \cdots \subset M_{l-1})$ is isomorphic to $\mathbb{F}_{N'}(\pi\,|_{N_{l,0}}; M_0 \subset \cdots \subset M_{l-1})$ which is contractible by induction. Hence, by [2, Proposition 1.7], $\mathbb{F}_{N'}(\pi; M_0 \subset \cdots \subset M_l)$ is homotopy equivalent to $\mathbb{F}_{N'}(\pi; M_l)$ which is contractible by Lemma 3.4. Note that this settles the start of the induction $l = 0$ as well. **q.e.d.**

Corollary 3.6. *Let $\pi : N \twoheadrightarrow M$, N' and $M_0 \subset \cdots \subset M_l$ as above. Suppose that $\pi(N') = M_0 \subset \cdots \subset M_l = M$. Then the poset $\mathbb{F}^{=N}_{=N'}(\pi; M_0 \subset \cdots \subset M_l)$ is contractible.*

Proof. The map

$$\mathbb{F}_{N'}(\pi; M_0 \subset \cdots \subset M_l) \to \mathbb{F}_{N'}(\pi; M_0 \subset \cdots \subset M_l)$$

that appends, if necessary, N' in front of and N at the end of a given flag satisfies the assumptions of [2, 1.3] whence it is homotopic to the identity. Hence it defines a retraction of $\mathbb{F}_{N'}(\pi; M_0 \subset \cdots \subset M_l)$ onto $\mathbb{F}^{=N}_{=N'}(\pi; M_0 \subset \cdots \subset M_l)$. **q.e.d.**

Lemma 3.7. *Let $\pi : E \to X$ be a morphism of weak simplicial complexes. We fix an element $e \in E$ and a flag $\pi(e) = x_0 < \cdots < x_l$. Then the poset $\mathbb{F}_{=e}(\pi; x_0 < \cdots < x_l)$ is homotopy equivalent to $\mathrm{Lk}^{\pi}_{x_l}(e)$.*

Proof. If $l = 0$, we have

$$\mathbb{F}_{=e}(\pi; x_0 < \cdots < x_l) = \overline{\mathrm{Lk}^{\pi}_{x_l}(e)}.$$

For $l > 0$, there is a natural embedding

$$\mathbb{F}_{=e}(\pi; x_0 < \cdots < x_l) \hookrightarrow \mathbb{F}_{=e}(\pi; x_0 < \cdots < x_{l-1}) \times \overline{\mathrm{Lk}^{\pi}_{x_l}(e)}.$$

For a given flag in $\overline{\mathrm{Lk}^{\pi}_{x_l}(e)}$, the set of partners in $\mathbb{F}_{=e}(\pi; x_0 < \cdots < x_{l-1})$ is $\mathbb{F}_{N'}(\pi\,|_{e_{l,0}}; x_0 < \cdots < x_{l-1})$ which is isomorphic to a poset discussed in Lemma 3.5. **q.e.d.**

Another consequence of Lemma 3.5 is

Lemma 3.8. *Let $\pi : N \twoheadrightarrow M$ be a surjection. Fix a flag $\mathcal{E} = (N_0 \subset \cdots \subset N_l)$ with $N_l = N$ and a superflag $\mathcal{B}$ of $\pi(\mathcal{E})$. Then the poset of flags*

$$\mathbb{F}_{\mathcal{E}}(\pi; \mathcal{B})$$

of all superflags of $\mathcal{E}$ whose $\overline{\pi}$-image is $\mathcal{B}$ is contractible.

Proof. $\mathbb{F}_{\mathcal{E}}(\pi; \mathcal{B})$ is the product of posets we discussed so far.

For each “edge” $N_i \subset N_{i+1}$ we have a corresponding subflag $\mathcal{B}'$ of $\mathcal{B}$ from $\pi(N_i)$ to $\pi(N_{i+1})$. Each segment contributes a factor $\mathbb{F}^{=N_{i+1}}_{=N_i}(\pi; \mathcal{B}')$. These factors are contractible by Corollary 3.6. **q.e.d.**

Proof of Proposition 3.1. Let $\pi : E \to X$ be a morphism of weak simplicial complexes. We fix a flag $\mathcal{E}$ in E and a strict superflag $\mathcal{B}$ of $\pi(\mathcal{E})$. Let e be the maximal cell in $\mathcal{E}$. The preimage of $\mathcal{B}$ in $\overline{E}$ decomposes into a product of two factors. The first corresponds to the segment of $\mathcal{B}$ up to $\pi(e)$ and is contractible by the preceding Lemma 3.8. The segment of $\mathcal{B}$ beyond $\pi(e)$ contributes the second factor which, by Lemma 3.7, is homotopy equivalent to a relative link of e. Hence the claim follows since $\dim \mathcal{E} \leq \dim e$. **q.e.d.**

4 Weakly lifting maps

From now on, we will consider only piecewise linear maps between simplicial complexes. All the balls we consider are supposed to be simplicially subdivided. However, since the class of piecewise linear maps on a simplicial complex does not depend on the given decomposition, we will not bother to take care of it in notation.

Recall that we have $\mathbb{B}^l \subset \mathbb{S}^l = \partial \mathbb{B}^{l+1} \subset \mathbb{B}^{l+1}$. Let us denote by $\bar{\mathbb{B}}^l$ the subset of antipodes of $\mathbb{B}^l$ in $\mathbb{S}^l$. Hence, $\mathbb{S}^l = \mathbb{B}^l \cup \bar{\mathbb{B}}^l$, and for $l \geq 0$, $\mathbb{B}^l \cap \bar{\mathbb{B}}^l = \mathbb{S}^{l-1}$. Let us call a map $\pi : E \to X$ <u>weakly m-lifting</u> if, for every $l \leq m$ and every pair of maps $(\psi, \tilde{\psi})$ making the outer square commute, the <u>weak lifting problem</u>

$$\begin{tikzcd}
\mathbb{B}^l \arrow[rr, "\tilde{\psi}"] \arrow[dd, hook] \arrow[dr] & & E \arrow[dd, "\pi"] \\
& \mathbb{B}^{l+1} \arrow[ur, dotted, "\tilde{\Psi}"] \arrow[dl, dotted, "\Psi"] & \\
\mathbb{B}^{l+1} \arrow[rr, "\psi"] & & X
\end{tikzcd} \tag{2}$$

has a <u>weak solution</u>, i.e. a pair of maps $(\Psi, \tilde{\Psi})$ such that the diagram commutes and $\Psi(\bar{\mathbb{B}}^l) = \bar{\mathbb{B}}^l$.

Let Z be a simplicial complex with base point $\bullet_Z$. By simplicial approximation, we can represent each element of $\pi_l Z$ by a piecewise linear map on $\mathbb{B}^l$ taking $\partial\mathbb{B}^l$ to $\bullet_Z$. This is the main ingredient to prove the following easy but fundamental

Proposition 4.1. *Let $\pi : E \to X$ be a map of pointed simplicial complexes with base points $\bullet_E$ and $\bullet_X = \pi(\bullet_E)$ and with fibre $F = \pi^{-1}(\bullet_X)$ and suppose that π is weakly m-lifting. Then there is an exact sequence*

$$\pi_m F \to \pi_m E \to \pi_m X \to \pi_{m-1} F \to \cdots \to \pi_l F \to \pi_l E \to \pi_l X \to \cdots \to \pi_0 X$$

of homotopy groups.

Proof. This is completely analogous to the ordinary case. So let us first describe the map $\pi_{l+1} X \to \pi_l F$. Representing a given element of $\pi_{l+1} X$ by a map on $\mathbb{B}^{l+1}$ sending $\partial\mathbb{B}^{l+1}$ to $\bullet_X$, we take this map to be ψ in the diagram 2. We take $\tilde{\psi}$ to map all of $\mathbb{B}^l$ to $\bullet_E$. By hypothesis the lifting problem has a weak solution $(\Psi, \tilde{\Psi})$. Since $\psi \circ \Psi$ sends $\bar{\mathbb{B}}^l$ to $\bullet_X$, $\tilde{\Psi}$ maps $\bar{\mathbb{B}}^l$ to F sending its boundary $\partial\bar{\mathbb{B}}^l$ to the base point $\bullet_E$ since that part of $\bar{\mathbb{B}}^l$ also belongs to $\mathbb{B}^l$. Hence $\tilde{\Psi}\mid_{\bar{\mathbb{B}}^l}$ defines an element of $\pi_l F$.

The fact that the lifting problem in dimension l admits a weak solution also implies that the map $\pi_l X \to \pi_{l-1} F$ is well defined: Suppose we were given two maps ψ_0 and ψ_1 from $\mathbb{B}^l$ to X both of which take $\partial\mathbb{B}^l$ to $\bullet_X$, and suppose they determine the same element in $\pi_l X$. Then they are homotopic relative to $\partial\mathbb{B}^l$. Let $(\Psi_0, \tilde{\Psi}_0)$ and $(\Psi_1, \tilde{\Psi}_1)$ be weak solutions of the corresponding lifting problem defining two elements of $\pi_{l-1} F$ as above We have to show that $\tilde{\Psi}_0\mid_{\bar{\mathbb{B}}^{l-1}}$ and $\tilde{\Psi}_1\mid_{\bar{\mathbb{B}}^{l-1}}$ are homotopic relative to $\partial\bar{\mathbb{B}}^{l-1}$. Consider the following lifting problem

$$\begin{tikzcd}
\mathbb{B}^l \times \{0\} \cup \mathbb{B}^{l-1} \times [0,1] \cup \mathbb{B}^l \times \{1\} \arrow[rr, "\tilde{\psi}"] \arrow[dd, hook] \arrow[dr] & & E \arrow[dd, "\pi"] \\
& \mathbb{B}^l \times [0,1] \arrow[ur, dotted, "\tilde{\Psi}"'] \arrow[dl, dotted, "\Psi"] & \\
\mathbb{B}^l \times [0,1] \arrow[rr, "\psi"] & & X
\end{tikzcd}$$

where the map ψ on $\mathbb{B}^l \times [0,1]$ restricts to $\psi_\epsilon \circ \Psi_\epsilon$ on $\mathbb{B}^l \times \{\epsilon\}$ for $\epsilon \in \{0,1\}$ sending all of $\partial\mathbb{B}^l \times [0,1]$ to $\bullet_X$. So ψ models the homotopy of $\psi_0 \circ \Psi_0$ and $\psi_1 \circ \Psi_1$ which are homotopic since Ψ_0 and Ψ_1 both restrict to homeomorphisms on $\partial\mathbb{B}^l$. The map $\tilde{\psi}$ on $\mathbb{B}^l \times \{0\} \cup \mathbb{B}^{l-1} \times [0,1] \cup \mathbb{B}^l \times \{1\}$ sends $\mathbb{B}^{l-1} \times [0,1]$ to $\bullet_E$ and restricts to $\tilde{\Psi}_\epsilon$ on $\mathbb{B}^l \times \{\epsilon\}$ for $\epsilon \in \{0,1\}$. By hypothesis, the lifting problem has a weak solution $(\Psi, \tilde{\Psi})$.

Note that $\bar{\mathbb{B}}^{l-1} \times [0,1]$ is the set of antipodes of the ball $\mathbb{B}^l \times \{0\} \cup \mathbb{B}^{l-1} \times [0,1] \cup \mathbb{B}^l \times \{1\}$ in the boundary sphere of $\mathbb{B}^l \times [0,1]$. Restricting $\tilde{\Psi}$ to $\bar{\mathbb{B}}^{l-1} \times [0,1]$ leads to a homotopy of $\tilde{\Psi}_0 \mid_{\bar{\mathbb{B}}^{l-1}}$ to $\tilde{\Psi}_1 \mid_{\bar{\mathbb{B}}^{l-1}}$ relative to to $\partial\bar{\mathbb{B}}^{l-1}$ whence the connecting homomorphism $\pi_l X \to \pi_{l-1} F$ is well defined.

Now we show that the composition of two adjacent arrows is trivial. Note that $\tilde{\Psi} \mid_{\bar{\mathbb{B}}^l}$ defines the trivial element in $\pi_l E$ since $\tilde{\Psi}$ defines a contraction of this sphere to $\bullet_E$. This shows, that $\pi_{l+1} X$ becomes trivial in $\pi_l E$. On the other hand, consider an element in $\pi_{l+1} E$ and represent it as a map $\tilde{\Psi} : \mathbb{B}^{l+1} \to E$ taking $\partial\mathbb{B}^{l+1}$ to $\bullet_E$. Then $\pi \circ \tilde{\Psi}$ represents its image in $\pi_{l+1} X$. Taking $\Psi : \mathbb{B}^{l+1} \to \mathbb{B}^{l+1}$ to be the identity, we see that the lifting problem for $\pi \circ \tilde{\Psi}$ has a trivial solution $(\Psi, \tilde{\Psi})$ inducing the trivial l-sphere in F for $\tilde{\Psi}(\bar{\mathbb{B}}^{l+1}) = \bullet_E$ whence the image of $\pi_{l+1} E$ in $\pi_l F$ is trivial—we are assuming here, that the map $\pi_{l+1} X \to \pi_l F$ is well defined which will be shown in a minute. Finally, $\pi_l F$ has trivial image in $\pi_l X$ since F itself maps just to the base point $\bullet_X$.

Now we turn to exactness. So we start with an element of $\pi_l E$ vanishing in $\pi_l X$. We represent it by a map $\tilde{\psi} : \mathbb{B}^l \to E$ sending the boundary $\partial\mathbb{B}^l$ to $\bullet_E$. Since the composition with π induces a trivial element of $\pi_l X$, there is a map $\psi : \mathbb{B}^{l+1} \to X$ that restricts to $\pi \circ \tilde{\psi}$ on $\mathbb{B}^l \subset \partial\mathbb{B}^{l+1}$ and takes $\bar{\mathbb{B}}^l$ to $\bullet_X$. The lifting problem defined by $(\psi, \tilde{\psi})$ has a weak solution $(\Psi, \tilde{\Psi})$ wherein $\tilde{\Psi}$ defines a homotopy of the sphere represented by $\tilde{\psi}$ to a sphere in F represented by $\tilde{\Psi}\mid_{\bar{\mathbb{B}}^l}$. Hence, the element we started with, already comes from $\pi_l F$. This proves exactness at $\pi_l E$.

Exactness at $\pi_l F$ can be proven as follows. We represent an element of $\pi_l F$ vanishing in $\pi_l E$ by a map $\tilde{\psi} : \mathbb{B}^l \to F \subset E$ that takes $\partial\mathbb{B}^l$ to $\bullet_E$. Since it represents an element trivial in $\pi_l E$, there is a map $\tilde{\Psi} : \mathbb{B}^{l+1} \to E$ extending $\tilde{\psi}$ and sending all of $\bar{\mathbb{B}}^l$ to $\bullet_E$. Put $\psi := \pi \circ \tilde{\Psi}$. This map describes an element of $\pi_{l+1} X$ since it sends $\partial\mathbb{B}^{l+1}$ to $\bullet_X$. This element is easily seen to be the preimage we were looking for.

To prove exactness at $\pi_{l+1} X$, we consider a an element vanishing in $\pi_l F$ and represent it by a ma $\psi : \mathbb{B}^{l+1} \to X$ taking the boundary $\partial\mathbb{B}^{l+1}$ to $\bullet_X$. We construct the induced element in $\pi_l F$ as above from a weak solution $(\Psi, \tilde{\Psi})$ of the corresponding lifting problem by restricting $\tilde{\Psi}$ to $\bar{\mathbb{B}}^l \subset \partial\mathbb{B}^{l+1}$. Let us take a second $(l+1)$-ball $\bar{\mathbb{B}}^{l+1}$ and identify half of its boundary with $\bar{\mathbb{B}}^l$. Since $\tilde{\Psi}\mid_{\bar{\mathbb{B}}^l}$ represents the trivial element in $\pi_l F$, there is a map from $\bar{\mathbb{B}}^{l+1} \supseteq \bar{\mathbb{B}}^l$ to F extending $\tilde{\Psi}$ such that the complement $\partial\bar{\mathbb{B}}^{l+1} \setminus \bar{\mathbb{B}}^l$ is taken completely to $\bullet_E$. Gluing together $\mathbb{B}^{l+1}$ and $\bar{\mathbb{B}}^{l+1}$ along $\bar{\mathbb{B}}^l$ we obtain another $(l+1)$-ball together with a map to E now taking all of the boundary to $\bullet_E$. It is a routine matter to recognize the corresponding element of $\pi_{l+1} E$ as a preimage of the element in $\pi_{l+1} X$ taken at the beginning. **q.e.d.**

Being weakly m-lifting is a slight generalization of enjoying the weak homotopy lifting property with respect to m-simplices. Hence the following lemma is not surprising.

Lemma 4.2. *Given a map $\pi : E \to X$, let $\Phi : \mathbb{B}^l \times [-1, 0] \to E$ be a piecewise linear map such that, for each $t \in [-1, 0]$, the map*

$[\tilde{\psi}, t] := \Phi(\cdot, t) : \mathbb{B}^l \to E$ *does not depend on* t*. Furthermore, let* $\psi : \mathbb{B}^l \times [0,1] \to X$ *be a map such that* $\psi\mid_{\mathbb{B}^l \times \{0\}} = \pi \circ \tilde{\psi}_{-1} = \pi \circ \tilde{\psi}_0$*. Then the lifting problem* $(\psi, \tilde{\psi}_{-1})$ *has a weak solution if the lifting problem* $(\psi, \tilde{\psi}_0)$ *has.*

Proof. Let $(\Psi_0, \tilde{\Psi}_0)$ be a weak solution for the lifting problem posed by $(\psi, \tilde{\psi}_0)$. Then Ψ_0 restricts to the identity on $\mathbb{B}^l \times \{0\}$. Hence we can extend it to a map $\Psi : \mathbb{B}^l \times [-1,1] \to \mathbb{B}^l \times [0,1]$ via the projection onto the first coordinate on $\mathbb{B}^l \times [-1,0]$. On the other hand, Φ and $\tilde{\Psi}_0$ coincide on $\mathbb{B}^l \times \{0\}$. Hence gluing together these maps yields a map $\tilde{\Psi} : \mathbb{B}^l \times [-1,1] \to E$ making

$$\begin{tikzcd}
\mathbb{B}^l \arrow[rr, "\tilde{\psi}_{-1}"] \arrow[dd, hook] \arrow[dr] & & E \arrow[dd, "\pi"] \\
& \mathbb{B}^l \times [-1,1] \arrow[ur, "\tilde{\Psi}"'] \arrow[dl, "\Psi"] & \\
\mathbb{B}^l \times [0,1] \arrow[rr, "\psi"] & & X
\end{tikzcd}$$

commute. It is obvious that $(\Psi, \tilde{\Psi})$ is a weak solution for the lifting problem posed by $(\psi, \tilde{\psi}_{-1})$. **q.e.d.**

There is another kind of freedom to vary a weak lifting problem.

Lemma 4.3. *Let* $\varphi : \mathbb{B}^l \to \mathbb{B}^l \subset \partial\mathbb{B}^{l+1}$ *be a map such that* $\varphi(\partial\mathbb{B}^l) = \partial\mathbb{B}^l$*, and let* $\pi : E \to X$ *be weakly* l*-lifting. Then, for any pair* $(\psi : \mathbb{B}^{l+1} \to X, \tilde{\psi} : \mathbb{B}^l \to E)$ *satisfying* $\pi \circ \tilde{\psi} = \psi \circ \varphi$ *the* *generalized lifting problem*

$$\begin{tikzcd}
\mathbb{B}^l \arrow[rr, "\tilde{\psi}"] \arrow[dd, "\varphi"'] \arrow[dr] & & E \arrow[dd, "\pi"] \\
& \mathbb{B}^{l+1} \arrow[ur, dotted, "\tilde{\Psi}"'] \arrow[dl, dotted, "\Psi"'] & \\
\mathbb{B}^{l+1} \arrow[rr, "\psi"] & & X
\end{tikzcd}$$

has a weak solution $(\Psi, \tilde{\Psi})$*, i.e.* $\tilde{\Psi}(\bar{\mathbb{B}}^l) = \bar{\mathbb{B}}^l$*. Note that* $\varphi = \Psi \mid_{\mathbb{B}^l}$*.*

Proof. Chose a map $\Phi : \mathbb{B}^{l+1} \to \mathbb{B}^{l+1}$ that restricts to φ on $\mathbb{B}^l$ and that takes $\bar{\mathbb{B}}^l$ to $\bar{\mathbb{B}}^l$. Consider the diagram

$$\begin{tikzcd}
\mathbb{B}^l \arrow[rr, "\tilde{\psi}"] \arrow[d, hook] \arrow[dr] & & E \arrow[dd, "\pi"] \\
\mathbb{B}^{l+1} \arrow[d, "\Phi"'] & \mathbb{B}^{l+1} \arrow[l, dotted, "\Psi'"] \arrow[ur, dotted, "\tilde{\Psi}"'] & \\
\mathbb{B}^{l+1} \arrow[rr, "\psi"] & & X
\end{tikzcd}$$

where the maps Ψ' and $\tilde{\Psi}$ exist because of π being weakly l-lifting. Then the pair $(\Phi \circ \Psi', \tilde{\Psi})$ fits to our needs. **q.e.d.**

Modifying slightly the proof of Lemma 4.2, we can show also sort of a converse.

Lemma 4.4. *Suppose* $\varphi : \Delta^l \to \Delta^l$ *maps every face of the* l*-simplex* Δ^l *onto itself. Let* $(\psi : \Delta^l \times [0,1] \to X, \tilde{\psi} : \Delta^l \to E)$ *be a pair of maps such* *that the generalized lifting problem*

$$\begin{tikzcd}
\Delta^l \arrow[rr, "\tilde{\psi}\circ\varphi"] \arrow[dd, "\varphi"'] \arrow[dr] & & E \arrow[dd, "\pi"] \\
& \Delta^l \times [0,1] \arrow[ur, dotted, "\tilde{\Psi}'"'] \arrow[dl, dotted, "\Psi'"] & \\
\Delta^l \times [0,1] \arrow[rr, "\psi"] & & X
\end{tikzcd}$$

has a weak solution $(\Psi', \tilde{\Psi}')$. *Then the ordinary lifting problem posed by* $(\psi, \tilde{\psi})$ *has a weak solution as well.*

Proof. Suppose we already have maps $\tilde{\Psi}' : \Delta^l \times [0,1] \to E$ restricting to $\tilde{\psi} \circ \varphi$ on $\Delta^l \times \{0\}$ and $\Psi' : \Delta^l \times [0,1] \to \Delta^l \times [0,1]$ restricting to φ on $\Delta^l \times \{0\}$.

Consider the map $\Phi : \Delta^l \times [-1,0] \to \Delta^l$ given by $(x,t) \mapsto -tx + (1+t)\varphi(x)$ which defines a homotopy of the identity at $t=-1$ to φ at $t=0$. Then $\tilde{\Psi}'' := \tilde{\psi} \circ \Phi : \Delta^l \times [-1,0] \to E$ defines a homotopy from $\tilde{\psi}$ to $\tilde{\psi} \circ \varphi$.

We glue together the maps $\tilde{\Psi}'$ and $\tilde{\Psi}''$ to obtain a map $\tilde{\Psi} : \Delta^l \times [-1,1] \to E$. This is well defined since $\tilde{\Psi}'$ and $\tilde{\Psi}''$ agree on $\Delta^l \times \{0\} \cong \Delta^l$ where they yield $\tilde{\psi} \circ \varphi$. On the other hand, we extend Ψ' to a map $\Psi : \Delta^l \times [-1,1] \to \Delta^l \times [0,1]$ by sending all of $\Delta^l \times [-1,0]$ to $\Delta^l \times \{0\}$ via Φ.

This way we obtain a pair $(\Psi, \tilde{\Psi})$ such that

$$\begin{tikzcd}
\Delta^l \arrow[rr, "\tilde{\psi}"] \arrow[dd, hook] \arrow[dr, leftrightarrow] & & E \arrow[dd, "\pi"] \\
& \Delta^l \times [-1,1] \arrow[ur, "\tilde{\Psi}"'] \arrow[dl, "\Psi"] & \\
\Delta^{l+} \times [0,1] \arrow[rr, "\psi"] & & X
\end{tikzcd}$$

commutes. The hypothesis on φ ensure that Φ takes $\partial\Delta^l \times [-1,0]$ to $\partial\Delta^l$ whence our construction yields a weak solution of the lifting problem. **q.e.d.**

5 Weak liftings and covers

Since we are dealing with piecewise linear maps in the category of simplicial complexes it is not surprising that we consider covers by subcomplexes rather that open covers. So the analogoue of Proposition 2.1 reads

Proposition 5.1. *Let $\pi : E \to X$ be a piecewise linear map between simplicial complexes and suppose that $X = \bigcup_{j \in I} X_j$ is a cover of X by subcomplexes such that, for each $j \in I$, the restriction of π to $E_j := \pi^{-1}(X_j)$ is weakly m-lifting. Then π is weakly m-lifting.*

Proof. Given a lifting problem $(\psi : \mathbb{B}^{l+1} \to X, \tilde{\psi} : \mathbb{B}^l \to E)$, choose a simplicial subdivision of $\mathbb{B}^{l+1}$ each of whose simplices has an image that is contained entirely in one of the subcomplexes X_j. By passing, if necessary, to an even finer subdivision K we arrange things such that K contains $\mathbb{B}^l$ as a subcomplex onto which it can be collapsed. That is, we can find a finite sequence $\mathbb{B}^l = K_0, \dots, K_n = K$ such that for $i \geq 1$, K_i contains a free face τ_i with coface σ_i satisfying $K_i = K_{i-1} \cup \sigma_i$. We put $\bar{\tau}_i := K_{i-1} \cap \sigma_i$. This is just the union of of faces of σ_i distinct from τ_i.

We will construct the weak solution to the lifting problem inductively on the subcomplexes K_i. That is, we will construct maps $\Psi_i : K_i \to K_i$ and $\tilde{\Psi}_i : K_i \to K_i$ such that each of the Ψ_i maps every simplex of K_i to itself and such that the diagrams

$$\begin{tikzcd}
K_0 \arrow[rr, "\tilde{\psi}"] \arrow[dd, hook] \arrow[dr] & & E \arrow[dd, "\pi"] \\
& K_i \arrow[ur, "\tilde{\Psi}_i"'] \arrow[dl, "\Psi_i"] & \\
K_i \arrow[rr, "\psi"] & & X
\end{tikzcd}$$

all commute. Note that Ψ_0 is the identity whence $\tilde{\Psi}_0$ is $\tilde{\psi}$.

Suppose now we already constructed the maps Ψ_{i-1} and $\tilde{\Psi}_{i-1}$. Consider the lifting problem

$$\begin{tikzcd}
\bar{\tau}_i \arrow[rr, "\tilde{\Psi}_{i-1} |_{\bar{\tau}_i}"] \arrow[dd, "\Psi_{i-1} |_{\bar{\tau}_i}"'] \arrow[dr] & & E \arrow[dd, "\pi"] \\
& \sigma_i \arrow[ur, dotted, "\tilde{\Psi}_{\sigma_i}"'] \arrow[dl, dotted, "\Psi_{\sigma_i}"] & \\
\sigma_i \arrow[rr, "\psi |_{\sigma_i}"] & & X
\end{tikzcd}$$

Since the bottom arrow in fact maps σ_i to one of the covering subcomplexes X_j, by Lemma 4.3, there is a weak solution $(\Psi_{\sigma_i}, \tilde{\Psi}_{\sigma_i})$ where Ψ_{σ_i} takes τ_i to itself. Since Ψ_{σ_i} and Ψ_{i-1} coincide on $\bar{\tau}_i$, we can glue them together to obtain the map $\Psi_i : K_i \to K_i$ which, by construction, takes σ_i and τ_i to themselves and does so for every simplex in K_{i-1} by induction. On the other hand, $\tilde{\Psi}_{\sigma_i}$ and $\tilde{\Psi}_{i-1}$ also agree on $\bar{\tau}_i$ whence they define the map $\tilde{\Psi}_i : K_i \to E$ which restricts to $\tilde{\psi}$ on K_0 as $\tilde{\Psi}_{i-1}$ does.

The pair $(\Psi_n, \tilde{\Psi}_n)$ is a weak solution of the lifting problem posed by $(\psi, \tilde{\psi})$ since Ψ_n sends every simplex to itself whence it does the same to the subcomplex in K_n corresponding to $\bar{\mathbb{B}}^l \subset \partial\mathbb{B}^{l+1}$. **q.e.d.**

If $\pi : E \to X$ is a weakly m-lifting map between simplicial complexes and X' is a subcomplex of X, then the restriction of π to $\pi^{-1}(X')$ is obviously weakly m-lifting as well. Hence from the preceding lemma we have

Corollary 5.2. *A piecewise linear map $\pi : E \to X$ between simplicial complexes is weakly m-lifting if and only if, for each closed simplex $\bar{\sigma} \subseteq X$, its restriction to the fibre $\pi^{-1}(\bar{\sigma})$ in X is weakly m-lifting.* **q.e.d.**

We now turn to the analogoue of Proposition 2.2. Let us call a decomposition of an m-simplex Δ^m into convex polyhedral cells <u>cosimplicial</u> if every k-cell c_k is a codimension-1-face of at most $1 + l - k$ cofaces where l is the dimension of the minimal face of Δ^m containing c_k. Decompositions of this type can be obtained, for example, by dualizing simplicial decompositions.

Observation 5.3. *Let Δ^m be a simplicially decomposed m-simplex. In its barycentric subdivision, vertices correspond to simplices in Δ^m. Taking the dual, we obtain a cosimplicial decomposition of Δ^m where m-cells are in 1-1-correspondence to simplices of the original decomposition. We call this the <u>derived cosimplicial decomposition</u>. The correspondence gives rise to a piecewise linear map $\Delta^m \to \Delta^m$ mapping each convex m-cell of the derived decomposition onto its corresponding simplex. Note that this map which we call the <u>derived map</u> maps every face of Δ^m onto itself.* **q.e.d.**

We are now prepared to state the analogoue of Proposition 2.2.

Proposition 5.4. *Let $\pi : E \to X$ be a piecewise linear map between simplicial complexes and suppose that $E = \bigcup_{j \in I} E_j$ is a cover of E by subcomplexes such that, for each non-empty finite subset $J \subseteq I$, the restriction of π to the intersection $E_J := \bigcap_{j \in J} E_j$ is weakly $(m + 1 - |J|)$-lifting. Then π is weakly m-lifting.*

Proof. Let $(\psi : \Delta^l \times [0,1] \to X, \tilde{\psi} : \Delta^l \to E)$ pose a weak lifting problem for π, i.e. the diagram

$$\begin{tikzcd}
\Delta^l \times \{0\} \arrow[r, "\tilde{\psi}"] \arrow[d, hook] & E \arrow[d, "\pi"] \\
\Delta^l \times [0,1] \arrow[r, "\psi"] & X
\end{tikzcd}$$

commutes. We have to show that it has a weak solution.

Consider a simplicial subdivision of Δ^l such that any subsimplex is sent to one of the E_j of the cover. Passing to the derived subdivision, we consider the derived map $\varphi : \Delta^l \to \Delta^l$. This map satisfies the assumptions of Lemma 4.4 whence we may assume without loss of generality, that $\tilde{\psi}$ sends every cell of the derived subdivision to one of the E_j in the cover. Since this decomposition is cosimplicial, $\tilde{\psi}$ takes each k-cell into a subcomplex $\bigcap_{j\in J} E_j$ where $|J| \leq 1 + l - k$. Note that π restricts to a weakly k-lifting map on this intersection.

Given the derived decomposition of Δ^l into convex polyhedral cells, let us denote by $\Delta^{(k)}$ the k-skeleton of that decomposition. Suppose we are given a pair of maps $(\Psi_{k-1} : \Delta^{(k-1)} \times [0,1] \to \Delta^{(k-1)} \times [0,1], \tilde{\Psi}_{k-1} : \Delta^{(k-1)} \times [0,1] \to E)$ making

$$\begin{tikzcd}
\Delta^{(k-1)} \times \{0\} \arrow[rr, "\tilde{\psi}\,|_{\Delta^{(k-1)}\times\{0\}}"] \arrow[dd, hook] \arrow[dr, leftrightarrow] & & E \arrow[dd, "\pi"] \\
& \Delta^{(k-1)} \times [0,1] \arrow[ur, "\tilde{\Psi}_{k-1}"'] \arrow[dl, "\Psi_{k-1}"] & \\
\Delta^{(k-1)} \times [0,1] \arrow[rr, "\psi\,|_{\Delta^{(k-1)}\times[0,1]}"] & & X
\end{tikzcd}$$

commute where we formally put $\Delta^{(-1)} = \emptyset$. Hence, Ψ_0 and $\tilde{\Psi}_0$ both are to be taken as the empty map.

For each k-cell c_k the lifting problem

$$\begin{tikzcd}
c_k \times \{0\} \cup \partial c_k \times [0,1] \arrow[rr, "(\tilde{\psi} \mid_{c_k \times \{0\}}) \cup_{\partial c_k \times \{0\}} (\tilde{\Psi}_{k-1} \mid_{\partial c_k \times [0,1]})"] \arrow[d, hook] \arrow[dr] & & E \arrow[d, "\pi"] \\
& c_k \times [0,1] \arrow[ur, dotted, "\tilde{\Psi}_{c_k}"'] \arrow[dl, dotted, "\Psi_{c_k}"] & \\
c_k \times [0,1] \arrow[rr, "\psi \mid_{c_k \times [0,1]}"] & & X
\end{tikzcd}$$

is weakly solvable. Hence the maps $\Psi_k := \bigcup_{c_k \in \Delta^{(k)}} \Psi_{c_k}$ and $\tilde{\Psi}_k := \bigcup_{c_k \in \Delta^{(k)}} \tilde{\Psi}_{c_k}$ make the diagram

$$\begin{tikzcd}
\Delta^{(k)} \times \{0\} \arrow[rr, "\tilde{\psi} \mid_{\Delta^{(k)} \times \{0\}}"] \arrow[dd, hook] \arrow[dr] & & E \arrow[dd, "\pi"] \\
& \Delta^{(k)} \times [0,1] \arrow[ur, "\tilde{\Psi}_k"'] \arrow[dl, "\Psi_k"] & \\
\Delta^{(k)} \times [0,1] \arrow[rr, "\psi \mid_{\Delta^{(k)} \times [0,1]}"] & & X
\end{tikzcd}$$

commute.

Moreover, by induction

$$\Psi_{k-1}(((\Delta^{(k-2)} \cap \partial \Delta^l) \times [0,1]) \cup (\Delta^{(k-1)} \times \{1\})) = ((\Delta^{(k-2)} \cap \partial \Delta^l) \times [0,1]) \cup (\Delta^{(k-1)} \times \{1\})$$

implies that also

$$\Psi_k(((\Delta^{(k-1)} \cap \partial \Delta^l) \times [0,1]) \cup (\Delta^{(k)} \times \{1\})) = ((\Delta^{(k-1)} \cap \partial \Delta^l) \times [0,1]) \cup (\Delta^{(k)} \times \{1\}\,.)$$

Hence $(\Psi_l, \tilde{\Psi}_l)$ is a weak solution to the lifting problem posed by $(\psi, \tilde{\psi})$. **q.e.d.**

6 Sufficient local conditions for weak liftings

In this section we will prove the following

Theorem 6.1. *Let $\pi : E \to X$ be a map of pointed simplicial complexes with fibre F and suppose that for every simplex $e \in E$ and for every strict coface x of $\pi(e)$ the relative link $\mathrm{Lk}^{\pi}_{x}(e)$ is $(m-1-\dim e)$-connected. Then π is weakly m-lifting.*

Together with Proposition 4.1, this immediately implies Theorem A.

The proof of Theorem 6.1 proceeds by induction based on the following

Lemma 6.2. *Given a weakly m-lifting simplicial map $\pi' : E' \to X'$ with E' being $(m-1)$-connected, let E be the cone over E' with cone point $\bullet_E$ and let X be the cone over X' with cone point $\bullet_X$. Then the map $\pi : E \to X$ interpolating linearly π' and $(\bullet_E \mapsto \bullet_X)$ is weakly m-lifting.*

Proof. For $l \leq m$, we have to solve a weak lifting problem $(\psi : \Delta^{l+1} \to X, \tilde{\psi} : \bar{\tau} \to E)$ where $\bar{\tau}$ consist of all but one codimension-1-faces of Δ^{l+1}. Let τ be the face we left out, and let $\bullet_\Delta$ be the opposite vertex of Δ^{l+1}.

The maps ψ and $\tilde{\psi}$ are piecewise linear, i.e. linear on a certain subdivision of Δ^{l+1}. By further subdivision, we may assume that Δ^{l+1} can be collapsed onto $\bar{\tau}$ with respect to this subdivision. Using the same inductive procedure as in the proof of Propositions 5.1 we may therefore even assume without loss of generality, that ψ is in fact linear.

Since $E \setminus \{\bullet_E\}$ has the structure of the product $E' \times [0, 1)$ and since the map $\pi' : E' \to X'$ is weakly m-lifting, there is an obvious way to solve the lifting problem unless $\bullet_\Delta$ is mapped to $\bullet_X$ by ψ whence it is sent to $\bullet_E$ by $\tilde{\psi}$. However, if ψ squeezes a whole edge to $\bullet_X$ then the lifting problem also has a simple solution. In this case a lift can be obtained by precomposing $\tilde{\psi}$ with the projection of Δ^{l+1} onto the codimension-1-face not containing the squeezed edge.

So we assume, that $\bullet_\Delta$ is the only vertex of Δ^{l+1} sent to $\bullet_X$. By rescaling we even may assume that all the other vertices are mapped to X'. Then $\partial\tau$ lifts via $\tilde{\psi}$ to a sphere in E'. Since E' is $(m-1)$-connected, this sphere bounds a PL-disc. The rays joining this disc to $\bullet_E$ then define a lifting $\tilde{\Psi} : \Delta^{l+1} \to E$ such that $\psi = \pi \circ \tilde{\Psi}$. **q.e.d.**

Corollary 6.3. *Let X be the cone over X' with cone point $\bullet_X$, and let $\pi : E \to X$ be a simplicial map taking the base vertex $\bullet_E$ to $\bullet_X$. Suppose $E' := \pi^{-1}(X')$ is $(m-1)$-connected, the restriction $\pi' : E' \to X'$ is weakly m-lifting, and E is the closed star of $\bullet_E$. Then π is weakly m-lifting.*

Proof. The cone over E' with cone point $\bullet_E$ is a subcomplex $\tilde{E} \leq E$. Moreover, shrinking the subcomplex $\pi^{-1}(\bullet_X)$ to $\bullet_E$ induces by linear interpolation a deformation retraction of E onto $\tilde{E}$ that preserves the fibres of π. Hence every weak lifting problem can be fibre-wise homotoped to a weak lifting problem for the map $\pi\mid_{\tilde{E}}: \tilde{E} \to X$. However, Lemma 6.2 does apply to this map whence the latter lifting problem has a weak solutions. Hence by Lemma 4.2, the former has a weak solution, as well. **q.e.d.**

Observation 6.4. *By Corollary 5.2 a map $\pi : E \to X$ is weakly m-lifting if its restriction to simplex preimages is. Since relative links can be seen within these preimages, it suffices to prove Theorem 6.1 for simplices. Moreover, if all simplices in X have dimension $\leq n$, Theorem 6.1 holds for maps to X provided it holds for maps to simplices of dimension $\leq n$.* **q.e.d.**

Lemma 6.5. *Theorem 6.1 holds provided $X = \Delta^n$ is a simplex.*

Proof. We use induction on n where the case $n = 0$ is trivial since any map onto a point is a fibration whence it is weakly m-lifting a fortiori.

Let $\overline{X}$ be the barycentric subdivision of X. In each simplex of E, we choose a point in the preimage of each vertex of $\overline{X}$. These points become the vertices of a barycentric subdivision $\overline{E}$ of E such that the map π induces a simplicial map $\overline{\pi}: \overline{E} \to \overline{X}$ that is the same map on the geometric realizations.

We refer to the simplices of the barycentric subdivision as the simplices in $\overline{E}$ whereas we refer to the undivided simplices as simplices of E. Given a vertex in $\overline{E}$ being the barycentre of a certain simplex in E, let us call the dimension of that simplex the degree of the corresponding vertex. Note that the vertices of a simplex in $\overline{E}$ have pairwise distinct degrees. We consider this as a height function and speak therefore of the highest or lowest vertex of a simplex meaning the vertex of maximum or minimum degree respectively. Let us call the ascending star of a vertex the union of those simplices in $\overline{E}$ containing it as their vertex of lowest degree. The same notion of degree applies obviously to vertices in $\overline{X}$.

The closed stars of degree 0 vertices in $\overline{X}$ cover $\overline{X}$. Let $\overline{X}_0$ be one of these closed stars and let $\overline{E}_0$ be its $\overline{\pi}$-preimage. Let $\bullet_X$ be the unique degree 0 vertex in $\overline{X}_0$. By Proposition 5.1 it suffices to show that $\overline{\pi}\,|_{\overline{E}_0}: \overline{E}_0 \to \overline{X}_0$ is weakly m-lifting.

Check the following statements:

- Vertices in $\overline{E}_0$ of degree 0 are vertices in E. They are mapped to $\bullet_X$ by $\overline{\pi}$.
- Ascending stars of vertices of degree 0 cover all of $\overline{E}_0$.
- Given a collection of vertices of degree 0, their ascending stars have a non-empty intersection if and only if these vertices form a simplex in E in which case the intersection of their ascending stars is the ascending star of the barycentre of that simplex which is a vertex in $\overline{E}$ of higher degree. This barycentre is mapped to $\bullet_X$ by $\overline{\pi}$.

Therefore, considering the cover of $\overline{E}_0$ by ascending stars in virtue of Proposition 5.4, we only have to show, that the restriction of $\overline{\pi}$

to an ascending star A of a degree-i-vertex $\bullet_A$ in $\pi^{-1}(\bullet_X)$ is weakly $(m-i)$-lifting.

We write $\overline{X}_0$ as a cone over $\overline{X}'_0$ with cone point $\bullet_X$ and let A' be the $\overline{\pi}\,|_A$-preimage of $\overline{X}'_0$ in A. By Proposition 3.1, the relative links in A' satisfy the hypotheses of Theorem 6.1. Hence, by induction, Observation 6.4 implies that $\overline{\pi}\,|_{A'}\colon A' \to \overline{X}'_0$ is weakly $(m-i)$-lifting.

So in order to apply the Corollary 6.3, we only have to show that A' is $(m-1-i)$-connected. However, the fibres of $\overline{\pi}\,|_{A'}\colon A' \to \overline{X}'_0$ are nothing else but the relative links of $\bullet_A$ with respect to the various simplices of $\overline{X}_0$. These links are relative links of the simplex in E represented by $\bullet_A$. Hence these fibres are $(m-1-i)$-connected by hypothesis. Applying the fact that $\overline{\pi}\,|_{A'}\colon A' \to \overline{X}'_0$ is weakly $(m-i)$-lifting and that $\overline{X}'_0$ is contractible, we infer from Proposition 4.1, that A' itself is $(m-1-i)$-connected. Hence Corollary 6.3 yields that $\overline{\pi}\,|_A\colon A \to \overline{X}_0$ is weakly $(m-i)$-lifting whence we are done. **q.e.d.**

Proof of Theorem 6.1. Combine the argument of Observation 6.4 with Lemma 6.5. **q.e.d.**

References

1. M. Bestvina, N. Brady, 'Morse theory and finiteness properties of groups', *Inventiones mathematicae* 129 (1997) p. 445 – 470.

2. D. Quillen, 'Homotopy Properties of the Poset of Nontrivial p-Subgroups of a Group', *Advances in Mathematics* 28 (1978) p. 101 – 128.