Finiteness Properties of Soluble Arithmetic Groups over Global Function Fields

Kai-Uwe Bux

preliminary version (1999/01/28)
subjected to change

Abstract

A Chevalley group $\mathcal{G}$ is a semisimple algebraic group scheme defined over $\mathbb{Z}$–e.g. SL_n. A Borel subgroup $\mathcal{B} \leq \mathcal{G}$ is a maximal connected soluble subgroup scheme–e.g. the group scheme B_n^0 of all upper triangular matrices in SL_n. One can always find a Borel subgroup that is defined over $\mathbb{Z}$. In this case for each kommutative ring R with unity, the group $\mathcal{B}(R)$ of R-points of $\mathcal{B}$ exists.

Let K be a global function field–e.g. the field $k(t)$ of rational functions over a finite field. Let S be a finite non-empty set of places over K and let $\mathcal{O}_S$ be the corresponding S-arithmetic ring–e.g. the ring $k[t]$ of polynomials or the ring $k[t, t^{-1}]$ of Laurent polynomials.

In this paper, we will determine the finiteness length of the groups $\mathcal{B}(\mathcal{O}_S)$ and of some of their subgroups. A group G is of type F_m if there is an Eilenberg-MacLane space $K(G,1)$ with finite m-sceleton. It is of type FP_m if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admitts a projective resolution which is finitely generated in dimensions $\leq m$.

We will proove that $\mathcal{B}(\mathcal{O}_S)$ is of type $\mathrm{F}_{|S|-1}$ but not of type $\mathrm{FP}_{|S|}$. Moreover we will derive lower and upper bounds for the geometric invariants $\Sigma^m(\mathcal{B}(\mathcal{O}_S))$ which are sharp for

rank-1-groups. Concerning higher ranks the estimates imlpy that normal subgroups of $\mathcal{B}(\mathcal{O}_S)$ with abelian quotients generically satisfy strong finiteness conditions.

Once upon a time, all finitely generated groups were finitely presented. They were discontinous subgroups of Lie groups or admitted actions on nice geometries—you should think of finitely generated Fuchsian groups for example. Then B.H. Neuman gave the first example of a finitely generated infinitely related group in [Neum37] where he even showed that there are uncountably many 2 generator groups. Among these, of couse, only countably many are finitely presented. Hence finite presentabilty must be a stronger propterty than finite generation. More than twenty years later G. Baumslag, W.W. Boone, and B.H. Neuman working on decission problems in [BBN59] showed that even finitely generated subgroups of finitely presented groups need not be finitely presented.

However, it took some time till fintely generated infinitely related groups have been found in a more natural setting. Here we have to mention the groups $\mathrm{SL}_2(k[t,t^{-1}])$ where k is a finite field. These groups have been prooven to be infinitely reladed by U. Stuhler in [Stuh76]. He extended these results in [Stuh80] constructing series of groups with increasing finiteness properties. R. Strebel in [Stre84] gives a historical and systematic survey with focus on soluble groups.

Because of the geometric background of finite generation and finite presentability there are two generalizations to higher dimensions: A group G is <u>of type F_m</u> if there is an Eilenberg MacLane space $K(G,1)$ with finite m-sceleton. G is <u>of type FP_m</u> if the trivial $\mathbb{Z}G$-modul $\mathbb{Z}$ admitts a projective resolution that is finitely generated in dimensions $\leq m$. This is a homological variant of the homotopical finiteness property F_m. By the way, C.T.C. Wall dicussed in [Wall65] what it means for a CW-complex to have a finite m-sceleton.

Finiteness properties are somewhat mysterious. Nobody can tell how our knowledge about a group increases when we learn that it is of type F_{18} but not of type FP_{23}. Concerning metabelian groups,

the Bieri-Strebel theory of geometric invariants leads to nice conjectures which are confirmed by a lot of examples and partitial results. Finite presentabilty is well understood within this contex. E.g. these groups are finitely presented if and only if they are of type FP_2. In general however, this equivalence does not hold as shown in [BeBr97].

In a situation like this, one can only hope to get a feeling for the relationsship between finiteness conditions and group structure within small classes of groups. We will consider the class of S-arithmetic groups. These matrix groups are give by means of an algebraic group scheme $\mathcal{G}$ and a set S of primes over a global field K which determines an S-arithmetic ring $\mathcal{O}_S \subseteq K$. Since these two parameters can be varied independently this class of groups is well addapted for the programme outlined. Moreover, these groups are natural generalizations of lattices in Lie groups where finiteness properties have a geometric meaning by nature.

The theory of S-arithmetic groups is dominated by two fundamental distinctions. The field K can be a global number field or a global function field. With respect to the group scheme, there are two extremes the first of which is given by reductive groups (e.g. SL_n which is even a Chevalley group, and GL_n). Soluble groups (e.g. groups of upper triangular matrices) form the other extreme. We collect the most important results:

$\mathcal{G}$: **reductive.**

- K: **number field.** $\mathcal{G}(\mathcal{O}_S)$ is of type F_∞ (i.e. of type F_m for all $m \in \mathbb{N}$). [BoSe76]
- K: **function field.** Finite generation and finite presentability are completely understood. The latter in [Behr92].

  There are series of examples that support the conjecture that more and more finiteness conditions hold when $|S|$ tends to infinity. The most important results are:

  - $\mathrm{SL}_2(\mathcal{O}_S)$ is of type $\mathrm{F}_{|S|-1}$ but not of type $\mathrm{FP}_{|S|}$. [Stuh80]
  - If $\mathcal{G}$ is a Chevalley group of rank n not of exceptional type, then $\mathcal{G}(k[t])$ is of type F_{n-1} but not of type FP_n

provided the finite field k is large enough. [Abra96, Corollary 20, Seite 113]

$\mathcal{B}$: **soluble.**

K: **number field.** Finite generation and finite presentabilty are completely understood. Finite presentability is treated in [Abel87].

In [Tiem97, Theorem 3.1], a Hasse principle is derived: $\mathcal{B}(\mathcal{O}_S)$ is of type FP_m if and only if for each place $p \in S$ the group $\mathcal{B}(\mathcal{O}_p)$ satisfies the compactness property CP_m which is defined in [AbTi97]. In case that $\mathcal{B}$ is a Borel subgroup of a Chevalled group, $\mathcal{B}(\mathcal{O}_S)$ is of type F_∞. [Tiem97, Corollary 4.5].

Beyond these there are only some series of examples e.g. in [AbBr87].

K: **function field.** No Hasse principle holds in this case. This follows from the series of metabelian groups which is examined using Bieri-Strebel theory in [Bux97]. Here we will generalize the results of that paper to higher ranks.

Our main result is the following

Main Theorem. *Let $\mathcal{G}$ be a Chevalley group, $\mathcal{B} \leq \mathcal{G}$ a Borel subgroup, K a global function field, S a non-empty set of places over K, and $\mathcal{O}_S$ the corresponding S-arithmetic ring. Then $\mathcal{B}(\mathcal{O}_S)$ is of type $F_{|S|-1}$ but not of type $FP_{|S|}$.*

We will have to fix notions and notations in the first two sections. Then we will deal with the rank-1-case. Sections 4 to 7 constitue the proof of the Main Theorem: in Theorem 5.1 the upper bound is established whereas the lower bound is given in Theorem 7.5. The last section presents Theorem 8.4 which provides bounds for the "geometric invariants" which are defined at the beginning of Section 8.

1 Preliminaries on adeles and unipotent groups

General references about global fields and adeles are [CaFr67] or [Weil73]. In this paper

- K is a global function field. We call its elements functions. The constants of K form the finite field
- k. Further
- P denotes the set of all places of K. We regard a place as a normalized discrete valuation $p : K \to \mathbb{Z} \cup \{\infty\}$. For each place p, there is a local function field
- K_p which is the completion of K at p. Extending p continously we obtain a normalized discrete valuation on K_p which we denote by p, too.
- $\mathcal{O}_p := \{f \in K_p \mid p(f) \geq 0\}$ denotes the ring of functions holomorphic at p. This is a local ring with maximal ideal
- $\mathfrak{m}_p = \{f \in K_p \mid p(f) > 0\}$. The residue field
- $k_p := \mathcal{O}_p/\mathfrak{m}_p$ is a finite extension of the field k of degree
- $d_p := [k_p : k]$. We define a norm on K_p by
- $|f|_p := e^{d_p p(f)}$ which is proportional to the modulus of K_p.

Adeles provide a formalism to treat all places at once. For a finite non-empty set $S \subseteq P$ of places,

- $\mathcal{O}_S := \{f \in K \mid p(f) \geq 0 \ \ \forall p \in P \setminus S\}$ is the ring of functions holomorphic outside S and
- $\boldsymbol{A}_S := \times_{p \in S} K_p \times \times_{p \notin S} \mathcal{O}_p$ is the ring of S-adeles.

The ring of adeles of K is the direct limit

- $\boldsymbol{A} := \varinjlim_S \boldsymbol{A}_S$. For an adel $\boldsymbol{a} = (f_p)_{p \in P} \in \boldsymbol{A}$, let
- $|\boldsymbol{a}| := \prod_p |f_p|_p$ be the idelic norm.

There is an inclusion $K \subseteq \boldsymbol{A}$ because every function in K is holomorphic at almost all places. Hence we regard K as a discrete subring of $\boldsymbol{A}$ and set

$$\mathcal{O}_S = \boldsymbol{A}_S \cap K.$$

A fundamental theorem (confere [Weil73, Theorem 2, page 64]) says that the quotient $\boldsymbol{A}/K$ is compact. We will need a slight generalization.

We regard a linear algebraic group defined $\mathcal{G}$ over a commutative ring R as a functor from the category of (topological) commutative R-algebras into the category of (topological) groups.

Lemma 1.1. *Let U be a unipotent linear algebraic group defined over K. Then $U(K)$ is a discrete subgroup of $U(\boldsymbol{A})$ and the quotients $U(\boldsymbol{A})/U(K)$ and $U(K)\backslash U(\boldsymbol{A})$ are compact.*

Proof. Let $\widehat{U}$ denote the group of character of U. That is the linear algebraic group of K-morphism from U to $\mathfrak{Mult} = \mathrm{GL}_1$. Put $U(\boldsymbol{A})^\circ := \left\{ g \in U(\boldsymbol{A}) \mid |\chi(g)| = 1 \; \forall \chi \in \widehat{U}(K) \right\}$.

According to [Bore91, Theorem 4.8], U can be triagonalized over K. Hence [Behr62, Satz 3] implies that the quotient $U(\boldsymbol{A})^\circ/U(K)$ is compact. Therefore we only have to proove that $U(\boldsymbol{A})^\circ = U(\boldsymbol{A})$.

Note that a character $\chi : U \to \mathfrak{Mult}$ defined over K induces an action of U on the affine line K. The orbit of $1 \in K$ is Zarisky-closed by [Bore91, Proposition 4.10]. However, it does not contain $0 \in K$ whence it is not all of the line. Hence the image of χ which is nothing but the considered orbit has dimension 0 whence it is finite. So it consists of roots of unity where all valuations vanish such that $|\chi(g)| = 1$ for all elements $g \in U(\boldsymbol{A})$. **q.e.d.**

2 Chevalley groups and the associated buildings

We fix a Chevalley group

- $\mathcal{G}$. That is a semisimple linear algebraic group defined over $\mathbb{Z}$. In [Chev60], [Stei67] rr [Abra94] it is described how to build such a group scheme. Every Chevalley group comes with a root system
- Φ. We denote the group of K'-points of $\mathcal{G}$ by
- $\mathcal{G}(K')$. This group acts strong-transitively on the associated spherical building

- Δ of type Φ. Given a place p on K', there is an affine building
- $X := X_p(K')$ associated to and acted upon by $\mathcal{G}(K')$. The building Δ can be viewed in a natural way as the building at infinity of X. $\mathcal{G}(K')$ acts strong-transitively on X if K' is complete with respect to p since in this case the system of apartments in Δ induces the complete system of apartments in X. Then we have a 1-1-correspondence of spherical and affine apartments. In this case we say that an affine apartment contains a chamber, a point, or a halfapartment at infinity if the corresponding spherical apartment does.
- $X_p := X_p(K_p)$ denotes the affine building associated to $\mathcal{G}(K_p)$ whereas
- Δ_p denotes the corresponding spherical building at infinity.

The stabilizer of an apartment in Δ is a maximal torus in $\mathcal{G}$. Borel subgroups (i.e. maximal connected soluble subgroup schemes of $\mathcal{G}$) arise as stabilizers of chambers in Δ. Tori and Borel subgroups are linear algebraic groups defined and split over K. All pairs consisting of a maximal torus $\mathcal{T}$ and a Borel subgroup $\mathcal{B}$ containing it are conjugate in $\mathcal{G}$. We fix such a pair

- $\mathcal{T} \leq \mathcal{B} \leq \mathcal{G}$. We can and do arrange things such that $\mathcal{T}$ and $\mathcal{B}$ are defined over $\mathbb{Z}$. Hence we are given a standard apartment
- Σ_p in each of the buildings X_p. The group $\mathcal{T}(K_p)$ acts on it via translations. Let
- n be the dimension of $\mathcal{T}$ over K. This is by definition the rank of $\mathcal{G}$. The building X_p is a piecewise Euclidean complex of dimension n and $\mathcal{G}$ acts upon it by cell permuting isometries. Since we fixed $\mathcal{B}$ as well we are given a fundamental chamber
- C_p in Δ_p which is stabilized by $\mathcal{B}(K_p)$. This chamber is represented by a parallelity class of sectors in X_p. On the set of these sectors, $\mathcal{T}(K_p)$ acts simpe-transitively. There is a canonical projection $\mathcal{B} \to \mathcal{T}$. Let

- $\mathcal{U}$ denote its kernel which has been coined the unipotentent part of $\mathcal{B}$.

The group $\mathcal{U}(K_p)$ not only stabilizes the fundamental chamber–it fixes it. That means for each element in $\mathcal{U}(K_p)$, there is a sector representing C_p which is fixed pointwise by the element chosen. Hence $\mathcal{U}(K_p)$ acts on X_p with Σ_p as a fundamental domain. We thus obtain a projection map

- $\pi_p : X_p \to \Sigma_p$.

In order to determine the finiteness propterties of $\mathcal{B}(\mathcal{O}_S)$, we will study its action on the product

- $\boldsymbol{X} := \times_{p\in S} X_p$ of affine buildings. The projections π_p induce a map
- $\boldsymbol{\pi} : \boldsymbol{X} \to \boldsymbol{\Sigma} := \times_{p\in S} \Sigma_p$ onto the product of the standard apartments.

Lemma 2.1. *The map $\boldsymbol{\pi}$ induces a proper map $\mathcal{U}(\mathcal{O}_S)\backslash\boldsymbol{X} \to \boldsymbol{\Sigma}$.*

Proof. Let $\boldsymbol{\sigma} := \times_{p\in S} \sigma_p$ be a polysimplex in $\boldsymbol{X}$. For each place $p \in S$, the stabilizer G_p of σ_p in $\mathcal{U}(K_p)$ is an open compact subgroup. For $p \notin S$ we set $G_p := \mathcal{U}(\mathcal{O}_p)$. Then $\boldsymbol{G} := \times_p G_p$ is an open subgroup of $\mathcal{U}(\boldsymbol{A}_S)$.

There is an obvious action of $\mathcal{U}(\boldsymbol{A}_S)$ on $\boldsymbol{X}$ where components outside S act trivialy whereas a component corresponding to a place $p \in S$ acts on the factor X_p. Hence $\boldsymbol{X} = \mathcal{U}(\boldsymbol{A}_S)\boldsymbol{\Sigma}$. The stabilizer of $\boldsymbol{\sigma}$ obviously is $\boldsymbol{G}$ whence the fibre of $\boldsymbol{\pi}$ over $\boldsymbol{\sigma}$ is isomorphic to $\mathcal{U}(\boldsymbol{A}_S)/\boldsymbol{G}$ which in turn is a discrete set.

This set is acted upon by $\mathcal{U}(\mathcal{O}_S) \subseteq \mathcal{U}(\boldsymbol{A}_S)$. Since $\mathcal{O}_S = K \cap \boldsymbol{A}_S$ Lemma 1.1 implies that the double quotient

$$\mathcal{U}(\mathcal{O}_S)\backslash\mathcal{U}(\boldsymbol{A}_S)/\boldsymbol{G}$$

is not only discrete but also compact whence it is finite.

Therefore the $\boldsymbol{\pi}$-fibre over each polysimplex consists of finitely many $\mathcal{U}(\mathcal{O}_S)$-orbits of cells. Now the claim follows. **q.e.d.**

Lemma 2.2. *The group $\mathcal{G}(\mathcal{O}_S)$ acts on $\boldsymbol{X}$ with finite cell stabilizers.*

Proof. The stabilizer of a polysimplex in $\mathcal{G}(\boldsymbol{A}_S)$ is known (and easyly seen) to be compact. Hence the finiteness follows since $\mathcal{G}(\mathcal{O}_S)$ is a discrete subgroup of $\mathcal{G}(\boldsymbol{A}_S)$. **q.e.d.**

Each standard apartment Σ_p is a Euclidean space of dimension n. Within each of them, we choose a sector

- $S_p \subseteq \Sigma_p$ representing the fundamental chamber C_p of Δ_p. We regard the cone point of S_p as the origin in Σ_p turning it this way into a Euclidean vector space. Moreover, we represent all roots in Φ by linear forms on Σ_p. Following the usual convention, we call those of them negative that take negative values inside S_p. Thus we are given a system
- Φ_p^- of negative roots in Σ_p. As a subset of Φ it is independent of the place p since all fundamental chambers C_p correspond to the same Borel subgroup $\mathcal{B}$. Passing to a base we obtain a system of coordinates
- $\varphi_p := \left(\alpha_1^{(p)}, \ldots, \alpha_n^{(p)}\right)$ on Σ_p. With respect to these coordinates, the sector S_p is given by

$$S_p = \left\{ x_p \in \Sigma_p \mid \alpha_i^{(p)}(x_p) \leq 0 \ \ \forall i \in \{1, \ldots, n\} \right\}.$$

  Thus

$$\boldsymbol{x} = (x_p)_{p \in S} \mapsto \left(\left(\alpha_1^{(p)}(x_p), \ldots, \alpha_n^{(p)}(x_p) \right) \right)_{p \in S}$$

  defines coordinates on $\boldsymbol{\Sigma}$. Scaling the different p-components appropriately, we arrange things such that the action of $\mathcal{T}(\mathcal{O}_S)$ leaves the map
- $\boldsymbol{\varphi} : \boldsymbol{x} = (x_p)_{p \in S} \mapsto \left(\sum_{p \in S} \alpha_1^{(p)}(x_p), \ldots, \sum_{p \in S} \alpha_n^{(p)}(x_p) \right)$ invariant. We can do so because of the product formula [CaFr67, page 60] which forces the idelic norm to be constantly 1 on K^*. The coordinates $\alpha_i^{(p)}$ correspond to base roots and therefore to characters $\chi_i : \mathcal{T} \to \mathfrak{Mult}$. An element $\tau_p \in \mathcal{T}(K_p)$ acts on Σ_p by a translation whose i-coordinate is $d_p p(\chi_i(\tau_p))$.

$\mathcal{B}(\boldsymbol{A}_S)$ acts on $\boldsymbol{\Sigma}$ via the projection $\mathcal{B} \to \mathcal{T}$. Thus $\boldsymbol{\pi}$ becomes a $\mathcal{B}(\boldsymbol{A}_S)$-map and hence a $\mathcal{B}(\mathcal{O}_S)$-map. Since this action factors through the torus $\mathcal{T}$ it leaves the map

- $\boldsymbol{\xi} := \boldsymbol{\varphi} \circ \boldsymbol{\pi}$ invariant.

Lemma 2.3. *The $\mathcal{B}(\mathcal{O}_S)$-orbit of the preimage $\boldsymbol{\xi}^{-1}(\mathsf{K})$ of any compact subset $\mathsf{K} \subseteq \mathbb{R}^n$ is compact.*

Proof. Dirichlet's Unit Theorem [CaFr67, page 72] implies that $\mathcal{T}(\mathcal{O}_S)$ acts cocompactly on the kernel of $\boldsymbol{\varphi}$ whence it acts cocompactly on the preimage $\boldsymbol{\varphi}^{-1}(\mathsf{K})$ as well. So, let $\mathsf{K}' \subseteq \boldsymbol{\Sigma}$ be a compact set whose $\mathcal{T}(\mathcal{O}_S)$-translates cover $\boldsymbol{\varphi}^{-1}(\mathsf{K})$.

The unipotent part $\mathcal{U}(\mathcal{O}_S)$ is a normal subgroup of $\mathcal{B}(\mathcal{O}_S)$. Therefore these $\mathcal{B}(\mathcal{O}_S)$-translates of the preimage $\boldsymbol{\pi}^{-1}(\mathsf{K}')$ cover all of $\boldsymbol{X}$. Hence the claim follows from Lemma 2.1. **q.e.d.**

3 Example: Rank-1-groups and trees

It is as instructive as usefull to treat the most simple case first–this is the Chevalley group SL_2. The group D_2^0 of diagonal matrices with determinant 1 is a maximal torus and the group B_2^0 of upper triangular matrices with determinant 1 is a Borel subgroup. Its unipotent part is the group U_2 of strictly upper triangular matrices. The affine building X_p at the place p is a regular tree of order $|k_p| + 1$: points in the link of a vertex correspond to points of the projective line over k_p. The standard apartment Σ_p is a line. The projection map π_p can be regarded as a hight function on X_p by identifying the apartment Σ_p with the real line $\mathbb{R}$ via the negative base root $\alpha^{(p)} : \Sigma_p \to \mathbb{R}$. By scaling, as in the general case, we arrange things in such a way that the action of $D_2^0(\mathcal{O}_S)$ on the product $\boldsymbol{X} := \times_{p \in S} X_p$ leaves invariant the hight function $\boldsymbol{\xi} : \boldsymbol{X} \to \mathbb{R}$. This situation has been discussed already in [Bux97]. Here we will treat it without making use of Σ-theory.

Trees are important for everything that follows. Therefore we will repeatedly make use of the following lemma which, at a first glance, may look somewhat technical. In fact, however, it describes a rather perspicous geometrical situation.

Lemma 3.1. *Let $h_i : T_i \to \mathbb{R}$ $(i \in \{1, \ldots, n\})$ be a family of locally finite simplicial trees with hight functions. Suppose for all indices i,*

1. h_i maps the vertices of T_i to a discrete subset of $\mathbb{R}$,

2. there is exactly one descending end in T_i, and

3. each vertex in T_i has degree ≥ 3.

So every vertex has a unique lower neighbour and at least two higher neighbours.

Let $\boldsymbol{T}$ be the product of the trees T_i and let $\boldsymbol{h} : \boldsymbol{T} \to \mathbb{R}$ be defined by

$$\boldsymbol{h} : \boldsymbol{\tau} = (\tau_1, \ldots, \tau_n) \mapsto \sum_{i=1}^{n} h_i(\tau_i).$$

For every compact intervall $I \subset \mathbb{R}$, put $\boldsymbol{T}[I] := \boldsymbol{h}^{-1}(I)$.

Then, for each compact intervall I, the space $\boldsymbol{T}[I]$ is $(n-2)$-connected. That is, its homotopy groups vanish in dimensions $\leq n-2$.

Moreover, for two intervalls $I \subseteq J$, the obvious inclusion $\boldsymbol{T}[I] \subseteq \boldsymbol{T}[J]$ induces a trivial map $\widetilde{\mathrm{H}}_{n-1}(\boldsymbol{T}[I]) \to \widetilde{\mathrm{H}}_{n-1}(\boldsymbol{T}[J])$ in reduced homology.

Proof. The map $\boldsymbol{h}$ is a Morse function as defined in [BeBr97]. Its ascending and descending links in $\boldsymbol{T}$ are the joins of the ascending and descending links of h_i in the trees T_i respectively. Hence ascending and descending links in $\boldsymbol{T}$ are $(n-2)$-connected. Thus, [BeBr97, Corollary 2.6] implies $\boldsymbol{T}[I]$ to be $(n-2)$-connected for each intervall I since otherwise $\boldsymbol{T}$ could not be contractible.

As for the second claim, recall that each tree T_i has a unique descending end. Moving every point in T_i with unit speed towards this end defines a flow on T_i. Taking the product we obtain a flow on $\boldsymbol{T}$ that takes all points in $\boldsymbol{T}[I]$ towards $\boldsymbol{T}[\{\min(I)\}]$. This construction yields that $\boldsymbol{T}[\{\min(I)\}]$ is a strong deformation retract of $\boldsymbol{T}[I]$.

We will construct a sphere in $\boldsymbol{T}[\{\min(I)\}]$ which maps to a non-trivial embedded $(n-1)$-sphere in $\boldsymbol{T}[\{\min(J)\}]$ via the given retraction map. This prooves the claim because the latter sphere defines a non-trivial cycle that cannot be a boundary because there is no n-sceleton in $\boldsymbol{T}[\{\min(J)\}]$.

We choose a point $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n) \in \boldsymbol{T}$ all of whose coordinates are vertices such that $\boldsymbol{h}(\boldsymbol{\tau}) < \min(J)$. For each i, we moreover choose two ascending rays L_i^+ and L_i^- starting at τ_i without a common initial segment—recall that every vertex has at least two higher neighbours. The union $L_i := L_i^+ \cup L_i^-$ is a line in T_i. The distance of a point $t'_i \in L_i$ to the "split vertex" τ_i is given by $h_i(t'_i) - h_i(\tau_i)$. Hence the map $\boldsymbol{t'} \mapsto \boldsymbol{h}(\boldsymbol{t'}) - \boldsymbol{h}(\boldsymbol{\tau})$ defines a norm on the product $\boldsymbol{L} := \times_{i=1}^{n} L_i$.

The sphere we were searching for now is simply the sphere of all points in $\boldsymbol{L}$ whose distance to $\boldsymbol{\tau}$ is $\min(I) - \boldsymbol{h}(\boldsymbol{\tau})$. The retraction shrinks it to the sphere of radius $\min(J) - \boldsymbol{h}(\boldsymbol{\tau}) > 0$. **q.e.d.**

The rank-1-case is now easyly dealt with since we can quote K.S. Brown's famous criterion:

Citation 3.2 ([Brow87, Remark (2) to Theorem 2.2 and Theorem 3.2]). *Let G be a group, D a directed set, and $(X_\alpha)_{\alpha \in D}$ a directed system of CW-complexes on which G acts by cell permuting homeomorphisms such that the following hold:*

1. *For each $\alpha \in D$, the orbit space $G\backslash X_\alpha$ is compact.*
2. *The stabilizer of each i-cell in X_α is a subgroup of G of type FP_{m-i}.*
3. *The continous map $X_\alpha \to X_\beta$ indexed by $\alpha \leq \beta$ is G-equivariant.*
4. *The limit of the directed system of reduced homology groups $\left(\widetilde{\mathrm{H}}_i(X_\alpha)\right)_{\alpha \in D}$ vanishes for $i < m$.*

Then G is of type FP_m if and only if for all $i < m$, the directed system of reduced homology groups $\left(\widetilde{\mathrm{H}}_i(X_\alpha)\right)_{\alpha \in D}$ is <u>essentially trivial</u>, i.e. for each $\alpha \in D$, there is $\beta \geq \alpha$ such that the map $\widetilde{\mathrm{H}}_i(X_\alpha) \to \widetilde{\mathrm{H}}_i(X_\beta)$ induced by $X_\alpha \to X_\beta$ is trivial.

G is finitely presented if and only if the analogous system of homotopy groups is essentially trivial.

Corollary 3.3. *The group $B_2^0(\mathcal{O}_S)$ is of type $F_{|S|-1}$ but not of type $FP_{|S|}$.*

Proof. We apply Brown's Criterion. The set of all compact intervalls in $\mathbb{R}$ ordered by inclusion is a directed set. So we are looking for a family of $B_2^0(\mathcal{O}_S)$-CW-complexes over this system. $B_2^0(\mathcal{O}_S)$ acts on the product of trees $\boldsymbol{X}$ with $\boldsymbol{\xi}$ as an invariant hight function made up of hights on the distinct trees X_p in the way described in Lemma 3.1. Hence for each compact intervall I, the preimage $\boldsymbol{X}[I] := \boldsymbol{\xi}^{-1}(I)$ is a $B_2^0(\mathcal{O}_S)$-complex which defines the directed system with inclusions as continous, $B_2^0(\mathcal{O}_S)$-equivariant maps.

The conditions of Brown's Criterion are satisfied: Lemma 2.3 implies the action to be cocompact, cell stabilizers are even finite by Lemma 2.2, and condition 4 is satisfied because the limit $\boldsymbol{X}$ of all $\boldsymbol{X}[I]$ is contractible.

So by Brown's Criterion, Lemma 3.1 completes the proof. **q.e.d.**

What about GL_2? The short exact sequence $\mathrm{SL}_2 \hookrightarrow \mathrm{GL}_2 \twoheadrightarrow \mathfrak{Mult}$ with the determinant as the projection map induces by restriction a short exact sequence

$$B_2^0(\mathcal{O}_S) \hookrightarrow B_2(\mathcal{O}_S) \twoheadrightarrow \mathcal{O}_S^*$$

whence $B_2(\mathcal{O}_S)$ inherits all finiteness properties of $B_2^0(\mathcal{O}_S)$ since $\mathcal{O}_S^*$ is of type F_∞. However, it might even exhibit stronger finiteness properties. This possibilty is ruled out by

Remark 3.4. The group $B_2(\mathcal{O}_S)$ is of type $\mathrm{F}_{|S|-1}$ but not of type $\mathrm{FP}_{|S|}$.

Proof. Passing to projective groups, we obtain the following commutative diagram all of whose rows and columns are short exact sequences of groups:

$$\begin{array}{ccccc}
\{-1,1\} & \hookrightarrow & \mathcal{O}_S^* & \overset{(\cdot)^2}{\twoheadrightarrow} & \mathcal{O}_S^{*2} := \{f^2 \mid f \in \mathcal{O}_S^*\} \\
\downarrow \cdot \mathbb{I}_2 & & \downarrow \cdot \mathbb{I}_2 & & \downarrow \\
B_2^0(\mathcal{O}_S) & \hookrightarrow & B_2(\mathcal{O}_S) & \twoheadrightarrow & \mathcal{O}_S^* \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}B_2^0(\mathcal{O}_S) & \hookrightarrow & \mathbb{P}B_2(\mathcal{O}_S) & \twoheadrightarrow & \mathcal{O}_S^*/\mathcal{O}_S^{*2}
\end{array}$$

Consider the bottom line first. The factor on the right is an abelian torsion group which is finitely generated by Dirichlet's

Unit Theorem. Hence it is finite. Therefore the groups on the left fulfill identical finiteness conditions. Then, this also holds for the extensions in the middel row because the kernels on top are of type F_∞. **q.e.d.**

Remark 3.5. Of course, the same argument implies that $B_n^0(\mathcal{O}_S)$ and $B_n(\mathcal{O}_S)$ have identical finiteness properties.

4 Higher ranks–an algebraic prelude

Perhaps the most striking consequence of our main theorem is that the rank of the Chevalley group $\mathcal{G}$ has no positive influence on the finiteness properties of $\mathcal{B}(\mathcal{O}_S)$. Within this section we shall give a simple algebraic explanation dealing with $B_n^0 \subseteq \mathrm{SL}_n$.

Observation 4.1. *Suppose $G \rightleftharpoons H$ is a retract diagram of groups, i.e. the composition of arrows is the identity on H. Then H enjoys all finiteness properties of G.*

Proof. Finite generation is trivial, finite presentability is easy and dealt with in [Wall65, Lemma 1.3] where it is attributed to J.R. Stallings. Thus we will only treat the homological finiteness properties starting with FP_2. We could refer the reader to [Åber86, (ζ), page 280] but the argument is merely outlined there.

The point is, that functors and cofunctors both presereve retract diagrams. So for each index set J, consider the functor that assigns to a group G the pair $(G, \times_J \mathbb{Z}G)$ where we regard $\times_J \mathbb{Z}G$ as a $\mathbb{Z}G$-modul. Applying the homology functor $\mathrm{H}_i(-,-)$ yields the following retract diagram:

$$\mathrm{H}_i(G, \bigtimes_J \mathbb{Z}G) \rightleftharpoons \mathrm{H}_i(H, \bigtimes_J \mathbb{Z}H)$$

Hence $\mathrm{H}_i(H, \times_J \mathbb{Z}H)$ vanishes whenever $\mathrm{H}_i(G, \times_J \mathbb{Z}G)$ does.

The claim now follows by means of the Bieri-Eckmann Criterion which says that a finitely generated group G is of type FP_m if and only if $\mathrm{H}_i(G, \times_J \mathbb{Z}G) = 0$ for all index sets J and all $i \in \{1, \ldots, m-1\}$. The reader may consult [BiEc74, Proposition 1.2 and the equation above Theorem 2.3]. **q.e.d.**

Corollary 4.2. *The groups $B_n(\mathcal{O}_S)$ and $B_n^0(\mathcal{O}_S)$ are not of type $FP_{|S|}$.*

Proof. We confine ourselves to $n = 3$. The general case is settled the same way. The group B_2 embeds into B_3 like this:

$$\begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \cong B_2$$

So it is obviously a retract whence the preceeding Observation 4.1 applies. Therefore the claim follows from the Remarks 3.4 and 3.5. **q.e.d.**

5 A geometric version of the argument

Theorem 5.1. *The group $\mathcal{B}(\mathcal{O}_S)$ is not of type $FP_{|S|}$.*

This section is completely devoted to the proof which can be viewed as a geometric interpretation of the argument presented in the preceeding section. The idea can be outlined as follows. The affine buildings X_p contain trees T_p as retracts. Hence the product $\boldsymbol{X}$ contains a product $\boldsymbol{T}$ of trees as a retract. We will find a directed system of subspaces in $\boldsymbol{X}$ satisfying the assumtions of Brown's Criterion. So we only have to proove that the induced system of reduced homology groups is not essentially trivial. Using the retraction map we pass to a corresponding system of subspaces in $\boldsymbol{T}$ where Lemma 3.1 applies.

Let us call an affine aparmtent in X_p a <u>layer</u> when it contains the fundamental chamber at infinity C_p. The base root $\alpha^{(p)} := \alpha_1^{(p)}$ defines half aparmtents in X_p by inequalities of the form

$$\alpha^{(p)}(x_p) \leq t \qquad (*)$$

Call an apartment in X_p <u>special</u> if it contains such a half apartment. To put it in a slightly different way: The base root defines a half apartment $\alpha_p^\infty \subseteq \Delta_p$ which contains the fundamental chamber C_p. An affine apartment in X_p is special if and only if it contains α_p^∞. Obviously, every special aparment is a layer.

The map $h_p := \alpha^{(p)} \circ \pi_p$ restricts to an affine map on every layer—hence on every special apartment. Two special apartments Σ_p^1 and Σ_p^2 intersect in a convex set which contains a subset of the form $(*)$. Hence

$$\Sigma_p^1 \cap \Sigma_p^2 = \left\{ x \in \Sigma_p^1 \mid h_p(x) \leq t \right\} = \left\{ x \in \Sigma_p^2 \mid h_p(x) \leq t \right\}$$

with $t = \max h_p(\Sigma_p^1 \cap \Sigma_p^2)$ which implies immediately

Observation 5.2. *The union of all special apartments in X_p is a subcomplex isometric to a product $T_p \times \mathbb{R}^{n-1}$ where T_p is a tree. The projection onto the second factor $\mathbb{R}^{n-1}$ is defined by $\left(\alpha_2^{(p)} \circ \pi_p, \ldots, \alpha_n^{(p)} \circ \pi_p\right)$. In particular, the fibre over each tupel $(t_2, \ldots, t_n)$ is a tree on which h_p induces a hight function $h_p^* : T_p \to \mathbb{R}$.* **q.e.d.**

Remark 5.3. In [Rona89, Chapter 10.2], M. Ronan gives two constructions for the tree T_p. The fact that both constructions are equivalent is the background of the following argument.

We first describe the projection from X_p onto the tree T_p.

Lemma 5.4. *For each layer $\Sigma_p^{'}$, there is a special apartment Σ_p^{s} such that $h_p(\Sigma_p^{'} \cap \Sigma_p^{\mathrm{s}})$ is unbounded.*

Proof. We argue within the spherical building Δ_p. The apartment at infinity ${\Sigma^{\infty}}_p^{'}$ corresponding to $\Sigma_p^{'}$ contains C_p. The root α_p^{∞} determines a codimension 1 face of the fundamental chamber. Let $C_p^{'}$ be the neighbouring chamber in ${\Sigma^{\infty}}_p^{'}$. Then there is a unique apartment containing this chabmer and the half apartment α_p^{∞}. The corresponding affine apartment Σ_p^{s} satisfies our needs. **q.e.d.**

Lemma 5.5. *Let Σ_p^1 and Σ_p^2 be special apartments and $\Sigma_p^{'}$ be a layer. Moreover let t be real number and let $x_p^i \in \Sigma_p^i \cap \Sigma_p^{'}$ be two points such that $t = h_p(x_p^i)$. Then $t \in h_p(\Sigma_p^1 \cap \Sigma_p^2)$.*

Proof. Both intersections $\Sigma_p^i \cap \Sigma_p^{'}$ are convex and contain a sector representing C_p. Hence they contain the analogous sector with cone point x_p^i. Both sectors are simplicial cones with a

codimension 1 face where h_p equals constantly t. Hence both faces intersect within $\Sigma_p^{'}$ and their intersection is a simplicial cone of codimension 1 which is contained in $\Sigma_p^1 \cap \Sigma_p^2$. **q.e.d.**

Lemma 5.4 and 5.5 together imply:

Lemma 5.6. *There is a continous projection map $\rho_p : X_p \to T_p$ compatible with h_p, i.e. the diagram*

$$\begin{array}{ccc} X_p & \to & T_p \\ \downarrow h_p & & \downarrow h_p^* \\ \mathbb{R} & = & \mathbb{R} \end{array}$$

commutes where h_p^* is as in Observation 5.2.

Proof. There is a 1-1-correspondence from lines in T_p with respect to h_p^* and special apartments in X_p. According to Lemma 5.4, for each layer there is a line onto which the layer can be projected in a way compatible with h_p. These projection maps agree where layers intersect by Lemma 5.5. Hence we have defined a projection map on all of X_p since the affine building is the union of all layers.

This projection map is continous when restricted to a layer since the tree T_p carries the weak topology with respect to its lines. So the projection map is continous because the buidling X_p carries the weak topology with respect to the layers. **q.e.d.**

To turn this projection into a retraction we need to find a section. However, these exist in abundance due to Observation 5.2. Consider the product

- $\boldsymbol{T} := \times_{p \in S} T_p$ on which the hight function
- $\boldsymbol{h}^* : \boldsymbol{T} \to \mathbb{R}$ is defnied by

$$\boldsymbol{\tau} = (\tau_p)_{p \in S} \mapsto \sum_{p \in S} h_p^*(\tau_p).$$

  Moreover, there is the projection map

- $\boldsymbol{\rho} : \boldsymbol{X} \to \boldsymbol{T}$ that admitts a lot of sections parameterized by tupels $(t_{p,i} \mid p \in S,\ i \in \{2, \dots, n\})$ whence $\boldsymbol{T}$ is a retract of $\boldsymbol{X}$.

Observation 5.7. *The following diagram commutes:*

$$\begin{array}{ccc} \boldsymbol{X} & \rightleftharpoons & \boldsymbol{T} \\ \downarrow \xi & & \downarrow h^* \\ \mathbb{R}^n & \to & \mathbb{R} \end{array}$$

Here the arrow in the bottom row is the projection onto the first coordinate. **q.e.d.**

In order to apply Brown's Criterion again we need to specify a directed system of cocompact $\mathcal{B}(\mathcal{O}_S)$-complexes. A block is a product $\boldsymbol{I} := I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ of compact intervalls. The set of blocks ordered by inclusion is a directed set. The family of preimages $\boldsymbol{X}[\boldsymbol{I}] := \boldsymbol{\xi}^{-1}(\boldsymbol{I})$ of blocks, therefore, yields a directed system of $\mathcal{B}(\mathcal{O}_S)$-complexes. It resembles closely the one used in Section 3 whence we are reduced to proove the following

Lemma 5.8. *The system* $\widetilde{\mathrm{H}}_{|S|-1}(\boldsymbol{X}[\boldsymbol{I}])$ *of reduced homology groups is not essentially trivial.*

Proof. Given a block $\boldsymbol{I}$, let $\boldsymbol{J} := J_1 \times \cdots \times J_n$ be a block containing $\boldsymbol{I}$, i.e. $I_i \subseteq J_i$ for all i. Choose a tupel $(t_{p,i} \mid p \in S,\ i \in \{2, \ldots, n\})$ of real numbers such that $\sum_{p \in S} t_{p,i} \in I_i$ for each i. This at once yields a section of $\boldsymbol{\rho}$. What we observed in 5.7 implies that the preimage $\boldsymbol{T}[I_1] := \boldsymbol{h}^{*-1}(I_1)$ is a retract of $\boldsymbol{X}[\boldsymbol{I}]$ and that the diagram

$$\begin{array}{ccc} \boldsymbol{X}[\boldsymbol{I}] & \rightleftharpoons & \boldsymbol{T}[I_1] \\ \downarrow & & \downarrow \\ \boldsymbol{X}[\boldsymbol{J}] & \rightleftharpoons & \boldsymbol{T}[J_1] \end{array}$$

commutes. Taking homology we obtain:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_{|S|-1}(\boldsymbol{X}[\boldsymbol{I}]) & \rightleftharpoons & \widetilde{\mathrm{H}}_{|S|-1}(\boldsymbol{T}[I_1]) \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{H}}_{|S|-1}(\boldsymbol{X}[\boldsymbol{J}]) & \rightleftharpoons & \widetilde{\mathrm{H}}_{|S|-1}(\boldsymbol{T}[J_1]) \end{array}$$

The right vertical arrow is non-trivial by Lemma 3.1 whence the left one cannot be trivial either since the right hand side is a retract of the left hand side. **q.e.d.**

This completes the proof of Theorem 5.1. **q.e.d.**

6 The Moufang property

Fix a building X and therein an apartment Σ. Two half apartments α und β in Σ are prä(j if the intersections $\alpha \cap \beta$ and $-\alpha \cap -\beta$ have at least one chamber in common. In this case set

- $[\alpha, \beta] := \{\gamma \mid \alpha \cap \beta \subseteq \gamma, -\alpha \cap -\beta \subseteq -\gamma\}$ and
- $(\alpha, \beta) := [\alpha, \beta] \setminus \{\alpha, \beta\}$.

X is called Moufang if one can associate root groups

- U_α of automorphisms of the building X to the half apartments $\alpha \subseteq \Sigma$ such that the following axioms hold:

(M.1) U_α fixes every chamber in α and, for every panel (that is a codimension 1 face of a chamber) $\pi \in \partial\alpha$, it acts simple-transitively on the set of chambers in $\mathrm{St}(\pi)$ outside α.

(M.2) For each prenilpotent pair $\{\alpha, \beta\}$, we have $[\mathrm{U}_\alpha, \mathrm{U}_\beta] \leq \mathrm{U}_{(\alpha,\beta)}$. Here, $\mathrm{U}_{(\alpha,\beta)}$ denotes the group generated by all U_γ with $\gamma \in (\alpha, \beta)$.

(M.3) For each $u \in \mathrm{U}_\alpha \setminus \{1\}$, there is an element $m(u) \in \mathrm{U}_{-\alpha}\, u\, \mathrm{U}_{-\alpha}$ stabilizing Σ.

(M.4) For $n = m(u)$ as in M.3, we have $n\, \mathrm{U}_\beta\, n^{-1} = \mathrm{U}_{n\beta}$ for all half apartments β.

Fact 6.1. *The affine building associated to a Chevalley group $\mathcal{G}$ over a local field is Moufang. In particular, this hold for the building X_p.*

This is "well known to those who well know it". However, there seems to be no ready made reference for this fact in the literature whence we will outline the argument.

Sketch of proof. First, Hensel's Lemma or [Weil73, Theorem 8, page 20] implies that a local function field is isomorphic as a vield with valuation to a field of Laurent series over a finite field which is the completion of the field of rational functions over the same finite field. Hence we may confine ourselves to this case. As P. Abramenko observes in [Abra96, page 19], the corresponding building is isomorphic to the positive partner within

the twin building of $\mathcal{G}$ over the ring of Laurent polynomials. He shows that this group has an RGD-system (confere [Abra96, Definition 2, pages 14f]).

Finally one can derive the Moufang axioms from the RGD-axioms. This is not that difficult since the Moufang axioms M.2 to M.4 can be read as geometric interpretations of analogous RGD-axioms. The transitivity of the action in the Moufang axiom M.1 follows from [Tits87, 5.6 Proposition 3, page 564] whereas (RGD 3) immediately implies the action to be simple-transitive. **q.e.d.**

The following lemma is crucial to apply the method of Åberg [Åber86] to affine buildings.

Lemma 6.2. *Let X be an affine building and C a chamber at infinity. We call a (not necessaryly infinite) sequence of apartments $(\Sigma_i)_{i\in\mathbb{N}}$ directed if for each index j the set*

$$\overline{\Sigma_j \setminus \bigcup_{i<j} \Sigma_i}$$

is an intersection of half apartments in Σ_j not containing C.

If X is locally finite and Moufang then there is an infinite directed sequence $(\Sigma_i)_{i\in\mathbb{N}}$ of aparments which cover X.

The proof will take the rest of this section. So let us fix X and C as in the lemma. Further let us fix a geodesic gallery

- $g := (\dots, c_i, c_{i+1}, \dots)$ within a chosen standard apartment
- Σ. Let
- β_i be the half apartment of Σ containing c_{i-1} but not c_i. We will write shorthand
- $\mathrm{U}_i := \mathrm{U}_{\beta_i}$.

The following propositions are quoted from [Rona89, pages 74ff] or are prooved by similar arguments without difficulty.

1. U_α is fixing the star of every panel in $\alpha \setminus \partial\alpha$.
2. $\mathrm{U}^{(r,s)} := \langle \mathrm{U}_i \mid i \in \{r, \dots, s\}\rangle = \mathrm{U}_r \cdots \mathrm{U}_s$. Moreover, the corresponding factorization $u = u_r \cdots u_s$ is unique for every element $u \in \mathrm{U}^{(r,s)}$.

3. U_{r-1} and U_{s+1} normalize $\mathrm{U}^{(r,s)}$.

4. $\mathrm{U}^{(r,s)}$ is finite since the building X is locally finte.

Lemma 6.3. *An element $u = u_r \cdots u_s \in \mathrm{U}^{(r,s)}$ fixes exactly those chambers of Σ that lie in the intersection of all β_i with $u_i \neq 1$. Moreover, this is the intersection $u\Sigma \cap \Sigma$.*

Proof. Put $u^{(i)} := u_r \cdots u_i$. We proceed by induction on i. For $i = r$, we have $u^{(i)} \in \mathrm{U}_r$. If this element is trivial it fixes all of Σ which we regard, formally, as the intersection of an empty family of half apartments. Now suppose $u_r \neq 1$ and we were given a chamber c in $\Sigma \setminus \beta_r$ fixed by $u^{(r)}$ then the element $u^{(r)}$ fixed every minimal gallery connecting c to a chamber in β_r. In this case there was a panel $\pi \in \partial\beta_r$ whose star contained two chambers fixed by $u^{(r)}$ which cannot happen.

Now suppose $i > r$. By induction the lemma holds for $u^{(i-1)}$. Put $D := u^{(i-1)}\Sigma \cap \Sigma$. We have to show that for $u_i \neq 1$

$$L := D \cap \beta_i = u^{(i)}\Sigma \cap \Sigma =: R$$

holds. By induction we see that $c_r \in \beta_r \cap \cdots \cap \beta_i \subseteq L \subseteq R$ since u_i fixes the half apartment β_i.

Assume there is a chamber $c \in R \setminus L$. Then $u^{(i)}c = c$, because $u^{(i)}$ stabilizes c_r and preserves the type of minimal galleries. However, c is uniquely determined by the type of a minimal gallery connecting it to c_r in Σ. Hence $u^{(i)}$ fixes c whence all of the gallery connecting it to c_r. Consider the panel $\pi \in \partial L$ within this gallery. If $\pi \in \partial\beta_i$ then both of its neighbouring chambers in Σ lie in D and u_i fixes both of them what cannot happen. Therefore π lies in the interior of β_i. In this case, however, u_i fixes the star of the panel π whence $u^{(i-1)}$ has to fix both neighbouring chambers which it does not because one of them by assumption does not lie in D. Hence $L \subsetneqq R$ leads to a contradiction. **q.e.d.**

We will construct a sequence of automorphisms $\gamma_j \in \langle \mathrm{U}_i \mid i \in \mathbb{Z} \rangle \subseteq \mathrm{Aut}\,(X)$ that will give rise to a directed sequence

$\Sigma_j := \gamma_j(\Sigma)$. So we call a possibly finite sequence $\gamma_1, \dots, \gamma_i, \dots$ of automorphisms of X directed if, for each index j, the set

$$\bigcup_{i<j} \gamma_j^{-1}\gamma_i(\Sigma) \cap \Sigma$$

is coconvex, i.e. complement of a convex set. If $(\gamma_i)_{i\in\mathbb{N}}$ is a directed sequence of automorphisms of X then $(\gamma_i(\Sigma))_{i\in\mathbb{N}}$ is a directed sequence of apartments provided the gallery g is chosen such that every half apartment β_i contains the chamber C. So it suggests itself to search for a directed sequence of automorphisms.

Lemma 6.4. *Every directed enumeration of* $\mathrm{U}^{(r,s)}$ *can be extended to a directed enumeration of* $\mathrm{U}^{(r-1,s+1)}$.

Proof. We will only show that we can extend to $\mathrm{U}^{(r,s+1)}$. The argument for the lower index is similar.

Let $\mathrm{U}^{(r,s)} = \{\gamma_1, \dots, \gamma_q\}$ be a directed sequence enumerating $\mathrm{U}^{(r,s)}$ and choose an enumeration $\mathrm{U}_{s+1} = \{u_1, \dots, u_k\}$ starting with the identity element. Then

$$\{u_1\gamma_1, \dots, u_1\gamma_q, \dots, u_2\gamma_1, \dots, u_2\gamma_q, \dots, u_k\gamma_1, \dots, u_k\gamma_q\}$$

is an enumeration of $\mathrm{U}^{(r,s+1)}$ which splits into blocks of length q. The first block is the given enumeration of $\mathrm{U}^{(r,s)}$. So we already extended the enumeration and have to proove the extension to be directed. Within a block, quotients of elements agree with quotients in the first block. Thus it suffices to proove that sets of the form

$$\left(\bigcup_{\substack{\mu<\nu \\ i\le q}} \gamma_j^{-1}u_\nu^{-1}u_\mu\gamma_i(\Sigma) \cap \Sigma\right) \cup \left(\bigcup_{i<j} \gamma_j^{-1}\gamma_i(\Sigma) \cap \Sigma\right)$$

are coconvex. The right hand term is coconvex because we extended a directed sequence. If the left hand side is not empty it simply is the half apartment β_{s+1}: We have

$$\gamma_j^{-1}u_\nu^{-1}u_\mu\gamma_i = (\gamma_j^{-1}u_\nu^{-1}u_\mu\gamma_i u_\mu^{-1}u_\nu)(u_\nu^{-1}u_\mu).$$

As $i \in \{1, \dots, q\}$ varies, the first factor runs through $\mathrm{U}^{(r,s)}$. The second factor is a non-trivial element of U_{s+1}. By Lemma 6.3

therefore, β_{s+1} contains all sets $\gamma_j^{-1}u_\nu^{-1}u_\mu\gamma_i(\Sigma)\cap\Sigma$ and at least one of them is all of β_{s+1}. **q.e.d.**

On order to apply this lemma, we will choose an infinite geodesic gallery g such that all half apartments β_i contain the chamber C and such that the apartment Σ is the convex hull of g. Such a gallery does exist since we can find a straight line in Σ that connects an interior point of C to its antipodal point and that does not meet the codimension 2 sceleton of Σ. This can be seen e.g. by means of measure theory–countable unions of measure 0 sets have measure 0. This line meets a sequence of chambers buidling a gallery of the kind we are searching for. By Lemma 6.4, we can find a directed sequence enumerating the group $\mathrm{U}^{(-\infty,\infty)} := \langle \mathrm{U}_i \mid i\in\mathbb{Z}\rangle = \bigcup_{r,s,r\le s}\mathrm{U}_r\cdots\mathrm{U}_s = \{\gamma_i \mid i\in\mathbb{N}\}$. This sequence induces a directed sequence $\Sigma_i := \gamma_i(\Sigma)$ of apartments. Since the convex hull of g is all of Σ, these apartments cover X which prooves Lemma 6.2. **q.e.d.**

7 Tamness, connectivity, and finiteness properties

Let X be a metric CW-complex, $\pi : X\to\mathbb{E}$ be a projection onto a Euclidean space $\mathbb{E}$. Furthermore, let us fix a set Ψ of linear forms on $\mathbb{E}$. A sheet is a subcomplex of X which π maps isometrically onto $\mathbb{E}$. We follow the terminology of [Bux97] and call the tripel $(X,\mathbb{E},\pi)$ a Ψ-complex, if every compact subset of X is covered by an increasing sequence of sheets (hence X is covered by sheets). Here we call a sequence $(B_i)_i$ increasing if, at every stage, the piece $\overline{B_s\setminus\bigcup_{r<s}B_r}$ just added is identified via π with a convex subset of $\mathbb{E}$ subordinate to Ψ, i.e. it is the intersection of closed half spaces of the form $\{e\in\mathbb{E}\mid\lambda(e)\ge c_\lambda\}$ with $\lambda\in\Psi$.

Lemma 6.2 applies in particular to the building X_p and the chamber C_p. We obtain

Consequence 7.1. *(X_p,Σ_p,π_p) is a Φ_p^--complex.* **q.e.d.**

Using [Bux97, Example 7.2] we may infer

Consequence 7.2. *Each linear form in Φ_p^- induces by composition with the projection $\boldsymbol{\Sigma} \to \Sigma_p$ a linear form on $\boldsymbol{\Sigma}$. In this sense, put*

- $\boldsymbol{\Phi}^- := \bigcup\!\!\cdot_{p\in S} \Phi_p^-$.

Then $(\boldsymbol{X}, \boldsymbol{\Sigma}, \boldsymbol{\pi} : \boldsymbol{X} \to \boldsymbol{\Sigma})$ is a $\boldsymbol{\Phi}^-$-complex. **q.e.d.**

We will study the action of $\mathcal{B}(\mathcal{O}_S)$ on the complex

- $\boldsymbol{Y} := \boldsymbol{\xi}^{-1}(0) = \boldsymbol{\pi}^{-1}(\boldsymbol{H})$ whith
- $\boldsymbol{H} := \ker \boldsymbol{\varphi}$. $\boldsymbol{\pi} : \boldsymbol{X} \to \boldsymbol{\Sigma}$ obviously restricts to $\boldsymbol{Y}$ yielding a projection $\boldsymbol{Y} \to \boldsymbol{H}$ which we will denote by $\boldsymbol{\pi}$, as well.
  The linear forms on $\boldsymbol{\Sigma}$ in $\boldsymbol{\Phi}^-$ restrict to $\boldsymbol{H}$. Let
- $\widetilde{\boldsymbol{\Phi}^-} := \boldsymbol{\Phi}^-|_{\boldsymbol{H}}$ be the system of linear forms obtained this way.

Observation 7.3. *Let $(X, \mathbb{E}, \pi)$ be a Ψ-complex and $\mathbb{E}'$ be a subspace of $\mathbb{E}$. Then $(\pi^{-1}(\mathbb{E}'), \mathbb{E}', \pi|_{\mathbb{E}'})$ is a $\Psi|_{\mathbb{E}'}$-complex. Here, $\Psi|_{\mathbb{E}'}$ denotes the set of those linearforms in $\mathbb{E}'$ obtained by restricting a linear form in Ψ to $\mathbb{E}'$. Note that the metric on $\pi^{-1}(\mathbb{E}')$ is not the path metric but the restriction of the metric on X. Note that, contrary to [Bux97], elements of Ψ may vanish.*

In particular, $(\boldsymbol{Y}, \boldsymbol{H}, \boldsymbol{\pi})$ is a $\widetilde{\boldsymbol{\Phi}^-}$-complex. **q.e.d.**

Consequence 7.4. *$\boldsymbol{Y}$ is $(|S|-2)$-connected.*

Proof. By [Bux97, Lemma 7.3] $\boldsymbol{Y} = \boldsymbol{\xi}^{-1}(0) = \boldsymbol{\pi}^{-1}(\boldsymbol{H})$ is $(|S|-2)$-connected provided $\widetilde{\boldsymbol{\Phi}^-}$ is $(|S|-1)$-tame, i.e. no positive linear combination of up to $|S|-1$ elements in $\widetilde{\boldsymbol{\Phi}^-}$ vanishes.

Thus, it suffices to show that a positive linear combination of linear form in $\boldsymbol{\Phi}^- = \bigcup\!\!\cdot_{p\in S} \Phi_p^-$ vanishes on $\boldsymbol{H}$ only if it contains for each p an element in Φ_p^- with positive weight–then it has at least $|S|$ elements.

This follows for combinations $\sum_{i,p} \mu_{i,p} \alpha_i^{(p)}$ of base roots $\alpha_i^{(p)} \in \boldsymbol{\Phi}^-$ from the very definition of $\boldsymbol{H}$ since in this case, for each i, the coefficients $\mu_{i,p}$ agree for all $p \in S$ if the combination vanishes on $\boldsymbol{H}$.

Since each element of Φ_p^- is a positive combination of base roots $\alpha_1^{(p)}, \ldots, \alpha_n^{(p)}$ we can reduce any positive combination of elements of $\boldsymbol{\Phi}^-$ to a positive combination of base roots. Thus we are reduced to the case previously considered whence we are done. **q.e.d.**

We are now in a position to tackle the counter part of Theorem 5.1.

Theorem 7.5. *The group $\mathcal{B}(\mathcal{O}_S)$ is of type $F_{|S|-1}$.*

Proof. $\mathcal{B}(\mathcal{O}_S)$ acts cocompactly on $\boldsymbol{Y}$ by Lemma 2.3. The stabilizers are finite by Lemma 2.2. Hence the claim follows from Lemma 7.4 by means of Brown's simple criterion [Brow87, Proposition 1.1] or the more involved quoted in Citation 3.2–here we take a directed set consisting of a single element only. **q.e.d.**

Remark 7.6. The proof of Theorem 7.5 mimics the argument given in [Bux97] and therefore it makes use of global properties of the affine buildings X_p–namely the Moufang property. Hence this method does not apply to arithmetic groups over number fields. However, the discussion in Section 3 indicates that it should be possible to find an argument for Theorem 7.5 somewhat in the spirit [BeBr97] that exploits only local properties of the buildings X_p.

Theorems 5.1 and 7.5 together imply the Main Theorem formulated in the introduction. **q.e.d.**

8 The geometric invariants of Bieri, Neumann, Strebel and Renz

"Geometric invariants", born in [BiSt80], are by now connected with four names. It would be impossible to draw a complete picture of the theory. The reader may consult [Renz88] where high dimensional invariants have been introduced. A more recent source is [Bier97].

The invariant $\Sigma^m(G)$ is defined for every group G of type F_m. It is an open cone in $\mathrm{Hom}(G,\mathbb{R})$. For each group G, the inclusions $\Sigma^1(G) \supseteq \Sigma^2(G) \supseteq \cdots \supseteq \Sigma^m(G)$ hold.

Theorem and Definition 8.1. Let G be a group of type F_m and Y be a $(m-1)$-connected CW-complex upon which G acts by cell permuting homeomorphisms such that the stabilizer of each i-cell is of type F_{m-i} and such that the m-sceleton of Y is finite $\mod G$. For each non-trivial homomorphism $\chi \in \mathrm{Hom}(G,\mathbb{R})$, there exists a G-equivariant

hight function on Y. That is a continous map $h_\chi : Y \to \mathbb{R}$ with $h_\chi(gy) = \chi(g) + h_\chi(y)$ for all points $y \in Y$ and all elements $g \in G$.

Let h_χ be a hight function associated to χ. By definition $\chi \in \Sigma^m(G)$ holds if and only if Y is essentially $(m-1)$-connnected with respect to h_χ i.e. the directed systems $\left(\pi_i(h_\chi^{-1}([t,\infty)))\right)_{t\in\mathbb{R}}$ of homotopy groups are essentially trivial for all $i \leq m-1$ (confere Citation 3.2). This definition is independent of the complex Y and the hight function h_χ chosen to represent χ.

Remark 8.2. This definition differs from the tradition in two aspects. First, we only require that, for each $b \in \mathbb{R}$, there exists an $a \leq b$ such that $\pi_i(h_\chi^{-1}([b,\infty))) \to \pi_i(h_\chi^{-1}([a,\infty)))$ is trivial. Usually one requires the existence of a global bound for $b-a$. However, both definitions are equivalent even to the weaker condition that there is at least one pair $a \leq b$ with the induced map vanishing.

Second and more severe, we do not require the action of G on Y to be free. The weaker conditions on cell stabilizers was prooven to be equivalent to the original definition first by H. Meinert in his PhD-thesis. Its contents is published in his papers [Mein96] and [Mein97]. However, we refer the reader to [BiGe98, Theorem 12.1] since this approach avoids the route via homological invariants.

Generally, the geometric invariants of a group G are hard to compute—they convey a lot of information. E.g. if one knows $\Sigma^m(G)$ then one can tell for each normal subgroup of G with abelian factor whether it is of type F_m:

Citation 8.3 (confere [Renz88, Satz C, page 17]). *Let G be a group of type F_m and $N \leq G$ be a normal subgroup with abelian factor group. Then, N is of type F_m if and only if all non-trivial homomorphisms in $\mathrm{Hom}(G,\mathbb{R})$ that vanish on N belong to $\Sigma^m(G)$.*

We already know that $\mathcal{B}(\mathcal{O}_S)$ is of type $\mathrm{F}_{|S|-1}$. Hence the geometric invariants $\Sigma^m(\mathcal{B}(\mathcal{O}_S))$ are defined for $m < |S|$. Every homomorphism from $\mathcal{B}(\mathcal{O}_S)$ to $\mathbb{R}$ factors via $\mathcal{T}(\mathcal{O}_S)$ which spans a maximal lattice in the vector space $\boldsymbol{H}$ that can, therefore, be described by $\boldsymbol{H} \cong \mathcal{T}(\mathcal{O}_S) \otimes_{\mathbb{Z}} \mathbb{R}$. Hence

- $\boldsymbol{H}^* := \mathrm{Hom}_{\mathbb{R}}(\boldsymbol{H}, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{T}(\mathcal{O}_S), \mathbb{R}) \cong \mathrm{Hom}(\mathcal{B}(\mathcal{O}_S), \mathbb{R}) \supseteq \Sigma^m(\mathcal{B}(\mathcal{O}_S))$. To avoid cumbersome notation, we will frequently use the complement
- $\Sigma^m(\mathcal{B}(\mathcal{O}_S))^{\mathrm{c}} := \boldsymbol{H}^* \setminus \Sigma^m(\mathcal{B}(\mathcal{O}_S))$.

In order to compute the geometric invariants of $\mathcal{B}(\mathcal{O}_S)$, we need a complex to act upon. We will use $\boldsymbol{Y}$. A homomorphism from $\mathcal{T}(\mathcal{O}_S)$ to the reals $\mathbb{R}$ can be represented by a linear form on $\boldsymbol{H}$. Composition with $\boldsymbol{\pi} : \boldsymbol{Y} \to \boldsymbol{H}$ yields a $\mathcal{B}(\mathcal{O}_S)$-equivariant hight function on $\boldsymbol{Y}$.

We will derive a lower and an upper bound for $\Sigma^m(\mathcal{B}(\mathcal{O}_S))^{\mathrm{c}}$. These bounds will be sharp only in the rank-1-case. To avoid wordy formulations we have to introduce some shorthand notations. Let

- $\boldsymbol{\Delta}^- :=\subseteq \boldsymbol{\Phi}^-$ denote the set of all linear forms on $\boldsymbol{\Sigma}$ induced by base roots and let
- $\widetilde{\boldsymbol{\Delta}^-} := \boldsymbol{\Delta}^-|_{\boldsymbol{H}} \subseteq \widetilde{\boldsymbol{\Phi}^-}$ be the restriction of $\boldsymbol{\Delta}^-$ to $\boldsymbol{H}$.
- $\mathrm{conv}_{m,S}(\widetilde{\boldsymbol{\Delta}^-})$ is the set of all linear forms on $\boldsymbol{H}$ that can be written as a positive linear combination $\sum_{p\in S} \mu_p \alpha_{i_p}^{(p)}$ wherein at most m coefficents do not vanish. Note that all base roots occuring in the combination belong to distinct places.

  For an arbitrary set Ψ of linear forms, let
- $\mathrm{conv}_m(\Psi)$ be the set of all linear forms that are positive combinations of up to m elements of Ψ.

Theorem 8.4. $\mathrm{conv}_{m,S}(\widetilde{\boldsymbol{\Delta}^-}) \subseteq \Sigma^m(\mathcal{B}(\mathcal{O}_S))^{\mathrm{c}} \subseteq \mathrm{conv}_m(\widetilde{\boldsymbol{\Phi}^-})$ *for* $m < |S|$.

Remark 8.5. In the rank-1-case, the upper and the lower bound coincide. Hence they determine $\Sigma^m(\mathcal{B}(\mathcal{O}_S))^{\mathrm{c}}$. In this case, $\mathcal{B}(\mathcal{O}_S)$ is metabelian and Theorem 8.4 establishes the Σ^m-conjecture for this group which D. Kochloukova has done before in [Koch97].

Remark 8.6. It is easy to pin down the consequence of Theorem 8.4 according to the result of B. Renz quoted in Citation 8.3. Let A be a normal subgroup $\mathcal{B}(\mathcal{O}_S)$ containing $\mathcal{U}(\mathcal{O}_S)$. Such a subgroup always arrises as a preimage of a subgroup $T' \leq \mathcal{T}(\mathcal{O}_S)$. A homomorphism in $\mathrm{Hom}(\mathcal{B}(\mathcal{O}_S), \mathbb{R})$ vanishes on A if and only if the corresponding linear

form in $\boldsymbol{H}^*$ vanishes on $\boldsymbol{H}_{T'} := T' \otimes_{\mathbb{Z}} \mathbb{R} \leq \boldsymbol{H} \cong \mathcal{T}(\mathcal{O}_S) \otimes_{\mathbb{Z}} \mathbb{R}$. Thus, we consider $\boldsymbol{H}^*_{T'} := \{\lambda \in \boldsymbol{H}^* \mid \lambda(\boldsymbol{H}_{T'}) = 0\} \leq \boldsymbol{H}^*$. The codimension of $\boldsymbol{H}^*_{T'}$ is nothing but the $\mathbb{Z}$-rank of T'. According to Citation 8.3, A is of type F_m provided the intersection $\Sigma^m(\mathcal{B}(\mathcal{O}_S))^{\mathrm{c}} \cap \boldsymbol{H}^*_{T'}$ is trivial.

By the upper bound of Theorem 8.4, $\Sigma^m(\mathcal{B}(\mathcal{O}_S))^{\mathrm{c}}$ is a subset of $\mathrm{conv}_m(\widetilde{\boldsymbol{\Phi}^-})$ which is a finite union of m-dimensional subspaces of $\boldsymbol{H}^*$. Hence $\Sigma^m(\mathcal{B}(\mathcal{O}_S))^{\mathrm{c}}$ has a trivial intersection with $\boldsymbol{H}^*_{T'}$ provided the $\mathbb{Z}$-rank of T' is $\geq m$ and $\boldsymbol{H}_{T'}$ (and hence $\boldsymbol{H}^*_{T'}$) is "in general position".

The lower bound analogously implies that the $\mathbb{Z}$-rank of T' is at least m if A is of type F_m. Hence in the "generic case", the finiteness length of A is given by the minimum of $|S|-1$ and the $\mathbb{Z}$-rank of T'.

Remark 8.7. On the other side, it is not difficult to find subgroups of $\mathcal{B}(\mathcal{O}_S)$ above $\mathcal{U}(\mathcal{O}_S)$ to which our result does not apply. Consider e.g. the group

$$\left\{ \begin{pmatrix} t & * & * \\ & t^{-2} & * \\ & & t \end{pmatrix} \in \mathrm{SL}_3(\mathcal{O}_S) \mid t \in \mathcal{O}_S^* \right\}.$$

In the number field case, a related group is discussed in [AbBr87].

For $|S| \geq 2$, this group is finitely generated: We can get the elements in the upper right corner by commutators. This implies that the upper bound of Theorem 8.4 is not sharp.

However, this group, independently of $|S|$, is not finitely related : It is step-2-nilpotent-by-abelian. If it was finitely presented then, by [BiSt80, Corollary 5.8], the group in the upper right corner was finitely generated as a normal subgroup which is impossible since the torus acts trivially on it. From this it follows that the lower bound of Theorem 8.4 is not sharp, as well.

We now embark on the proof of Theorem 8.4. Both inclusions will be prooven seperately. The first in Lemma 8.8 and the second one in Consequence 8.11.

Lemma 8.8. $\mathrm{conv}_{m,S}(\widetilde{\boldsymbol{\Delta}^-}) \subseteq \Sigma^m(\mathcal{B}(\mathcal{O}_S))^{\mathrm{c}}$ *for* $m < |S|$.

Proof. Let $\lambda = \sum_{p\in S} \mu_p \alpha_p^{(i_p)}$ be a linear form on $\boldsymbol{H}$. Suppose m coefficients are greater than 0 whereas the others vanish. We will proove the system $(\pi_{m-1}((\lambda \circ \boldsymbol{\pi}|_{\boldsymbol{Y}})^{-1}[t,\infty)))_{t\in\mathbb{R}}$ to be not essentially trivial applying the technique of Section 5.

There we have seen how to construct a tree T_p associated to a base root $\alpha_p^{(i_p)}$ that is a rectract of the building X_p and that carries a hight function $h_p^* : T_p \to \mathbb{R}$ related to the hight function $h_p = \alpha_p^{(i_p)} \circ \pi_p : X_p \to \mathbb{R}$ via the retraction map–confere Lemma 5.6 and Observation 5.2.

Put $\boldsymbol{T'} := \times_{\mu_p\neq 0} T_p$. The induced retraction $\times_{\mu_p\neq 0} X_p \to \boldsymbol{T'}$ has a section $\sigma : \boldsymbol{T'} \to \times_{\mu_p\neq 0} X_p$. Since $m < |S|$ there is at least one place $p \in S$ with $\mu_p = 0$. Hence we find a continous map $\rho : \boldsymbol{T'} \to \times_{\mu_p=0} \Sigma_p$ such that, for each point $\boldsymbol{\tau'} \in \boldsymbol{T'}$, the tupel $(\rho(\boldsymbol{\tau'}), \sigma(\boldsymbol{\tau'})) \in \times_{\mu_p=0} \Sigma_p \times \times_{\mu_p\neq 0} X_p \subseteq \boldsymbol{X}$ belongs to $\boldsymbol{Y} = \boldsymbol{\pi}^{-1}(\boldsymbol{H})$. Thus we see that $\boldsymbol{T'}$ is a retract of $\boldsymbol{Y}$.

Hence $\sum_{p\in S} \mu_p h_p^*$ is a hight function on $\boldsymbol{T'}$ compatible with the hight function induced by λ on $\boldsymbol{Y}$:

$$\begin{array}{ccc} \boldsymbol{Y} & \rightleftharpoons & \boldsymbol{T'} \\ \downarrow {\scriptstyle \lambda\circ\boldsymbol{\pi}|_{\boldsymbol{Y}}} & & \downarrow {\scriptstyle \sum_{p\in S} \mu_p h_p^*} \\ \mathbb{R} & = & \mathbb{R} \end{array}$$

Therefore the system $(\pi_{m-1}((\lambda \circ \boldsymbol{\pi}|_{\boldsymbol{Y}})^{-1}[t,\infty)))_{t\in\mathbb{R}}$ of homotopy groups is not essentially trivial since the corresponding system on the retract $\boldsymbol{T'}$ is not essentially trivial by Lemma 3.1. **q.e.d.**

We turn to the upper bound.

Observation 8.9. *Let Ψ be an m-tame set of linear forms on $\boldsymbol{H}$ and λ be a linear form on $\boldsymbol{H}$. A positive combination of elements of Ψ vanishes on $\ker\lambda$ if and only if it belongs to the span of λ in $\boldsymbol{H}^*$. Hence Ψ restricts to an m-tame set of linear forms on $\ker\lambda$ if and only if $\lambda \notin \mathrm{conv}_m(\Psi)$ and $-\lambda \notin \mathrm{conv}_m(\Psi)$.* **q.e.d.**

The following lemma which is the key observation for the upper bound of Theorem 8.4 is a modification of [Bux97, Lemma 7.3].

Lemma 8.10. *Let $(X, \mathbb{E}, \pi)$ be a Ψ-complex and λ be a linear form on $\mathbb{E}$. Suppose Ψ is m-tame and $\lambda \notin \mathrm{conv}_m(\Psi)$. Then the preimage $X' := (\lambda \circ \pi)^{-1}[0,\infty)$ is $(m-1)$-connected.*

Proof. Let $(B_i)_{i\in\mathbb{N}}$ be an increasing sequence in X. We have to show that for each $r \in \mathbb{N}$ the space

$$W^r := \bigcup_{i\le r} X' \cap B_i$$

is $(m-1)$-connected.

Proceeding by induction we observe that the case $r = 1$ is trivial. So for $r > 1$, we assume by induction that W^{r-1} is $(m-1)$-connected. We obtain W^r from W^{r-1} by gluing a closed convex area $V' \subseteq B_r$ into W^{r-1}.

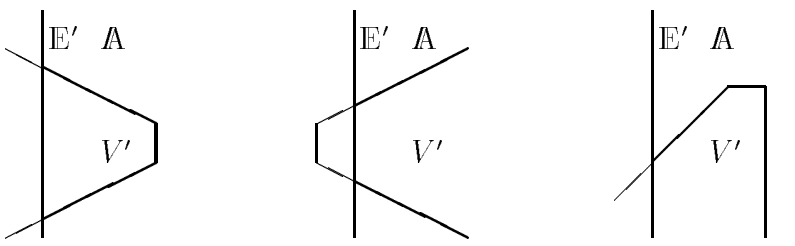
We put $\mathbb{A} := \lambda^{-1}[0,\infty)$, $\mathbb{E}' := \ker\lambda = \partial\mathbb{A}$, and identify the sheet B_r with $\mathbb{E}$ via the projection π. This way, $V' = \mathbb{A} \cap V$ for a convex set $V \subseteq \mathbb{E}$ subordinate to Ψ. The "new piece" V' is glued in along the set $\mathbb{A} \cap \partial V$.

This process only adds something if $\mathbb{E}'$ contains an interior point of V which we may therefore assume. In this case, $\mathbb{E}' \cap \partial V = \partial(\mathbb{A} \cap V)$.

According to [Bux97, Theorem 6.4] we distinguish two cases:

V' is decomposable: $V' = V \cap \mathbb{A}$ is a product of a convex compact set and a Euclidean space which is therefore a factor of $\mathbb{A}$ as well whence we may ignore it. So suppose V' is already compact. Then its intersection with $\mathbb{E}'$ is a "free face" which we can push into. Hence V' can be collapsed to $\partial V \cap \mathbb{A}$ which is therefore a retract. Since this is the part along which V' is glued in, the homotopy type does not change.

V' is rectractable: In this case, gluing in V' amounts to glue in $\partial V'$ allong $\mathbb{A} \cap \partial V$. However, everything that really matters



$-\lambda \notin \mathrm{conv}_m(\Psi')$ $\quad$ $-\lambda \in \mathrm{conv}_m(\Psi')$

V' decomposable $\quad$ V' retractable

happens withing $\mathbb{E}'$. Thus we could equivalently glue in $\partial V' \cap \mathbb{E}' = V \cap \mathbb{E}'$ along $\partial V \cap \mathbb{E}' = \partial(\mathbb{E}' \cap V)$.

Again, we have to distinguish two cases. Let $\Psi' \subseteq \Psi$ be the set of all linear forms in Ψ that correspond to hyperplanes supporting V.

$\underline{-\lambda \notin \mathrm{conv}_m(\Psi')}$: We have seen in Observation 8.9 that in this case the system Ψ' induces an m-tame system of linear forms on $\mathbb{E}'$. Hence $V \cap \mathbb{E}'$ is either retractable or decomposible with a compact factor of dimension $\geq m$. This follows from [Bux97, Corollary 6.7]. In either case, the gluing does not change homotopy groups in low dimensions.

$\underline{-\lambda \in \mathrm{conv}_m(\Psi')}$: Since the set V' is retractable it contains an infinte ray without its antipode. This ray cannot move away from $\mathbb{E}'$ since otherwise λ was unbounded on V'. Hence the ray is paralell to $\mathbb{E}'$. Thus the set $V \cap \mathbb{E}'$ is rectractable as a subset of $\mathbb{E}'$ for it contains a parallel ray. Hence gluing in this set does not change the homotopy type. **q.e.d.**

Observation 7.3 and Lemma 8.10 immediately imply:

Consequence 8.11. $\Sigma^m(\mathcal{B}(\mathcal{O}_S))^{\mathrm{c}} \subseteq \mathrm{conv}_m(\widetilde{\mathbf{\Phi}^-})$ *for* $m < |S|$. **q.e.d.**

Together with Lemma 8.8 this prooves Theorem 8.4 **q.e.d.**.

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