# Finiteness Properties of Certain Metabelian Arithmetic Groups in the Function Field Case

Kai-Uwe Bux

J.W. Goethe Universit"at Frankfurt am Main, Germany

– 6th November 1996 –

Recall that by definition a group $G$ is of type $\mathrm{FP}_n$ if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a projective resolution which is finitely generated in dimensions $\leq n$. Consider the group scheme

$$R \mapsto G(R) := R \rtimes R^* = \begin{pmatrix} 1 & R \\ 0 & R^* \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \cap \mathrm{GL}_2(R)$$

where $R$ is an arbitrary commutative ring with $1 \neq 0$ and a unit $x \in R^*$ acts on $R$ by multiplication.

We will study the finiteness properties of subgroups of $G(\mathcal{O}_S)$ where $\mathcal{O}_S$ is an $S$-arithmetic subring of a global function field. The subgroups we are interested in are of the form $\mathcal{O}_S \rtimes Q$ where $Q$ is a subgroup of $\mathcal{O}_S^*$. We can give necessary and sufficient conditions for such groups to be of type $\mathrm{FP}_n$. They concern the geometrical $\Sigma$-invariant introduced by R. Bieri and R. Strebel in [BiSt80]. The invariant $\Sigma^c_{\mathcal{O}_S}(Q,\mathbb{Z})$ is a certain subset of the sphere $\mathbb{S}(Q)$ of rays in $\mathrm{Hom}(Q,\mathbb{R})$. We will give a precise definition at the beginning of Section 2. A subset $\Sigma^c$ of a sphere is called <u>$n$-tame</u> if every subset of $\Sigma^c$ consisting of at most $n$ points is contained in an open hemisphere. A finitely generated $\mathbb{Z}Q$-module is called $n$-tame if $\Sigma^c_{\mathcal{O}_S}(Q,\mathbb{Z})$ is $n$-tame. Our main result is the following

---

1991 Mathematics Subject Classification: 20E08, 20F16, 20G30, 52A20.

Key words and phrases: $\mathrm{FP}_n$-conjecture, metabelian groups, arithmetic groups, trees.

**Theorem.** *Let $S$ be a finite set of places of a global function field $K$ and $\mathcal{O}_S$ the corresponding $S$-arithmetic ring. Let $Q$ be a subgroup of $\mathcal{O}_S^*$. Then $Q$ is finitely generated and for all integers $n \geq 1$ the following are equivalent:*

1. *$\mathcal{O}_S \rtimes Q$ is of type FP$_n$.*
2. *$\mathcal{O}_S$ is $n$-tame as a $\mathbb{Z}Q$-module.*
3. *Each $p \in S$ restricts to a non-trivial homomorphism $p|_Q : Q \to \mathbb{R}$ and the set $\{[p|_Q] \mid p \in S\} \subseteq \mathbb{S}(Q)$ is $n$-tame.*

*If these conditions hold for at least one $n \geq 1$ then the identity*

$$\Sigma^c_{\mathcal{O}_S}(Q, \mathbb{Z}) = \{[p|_Q] \mid p \in S\}$$

*holds.*

Consider first the equivalence $1 \Longleftrightarrow 2$. There is a conjecture of R. Bieri that this equivalence should hold in quite a general context. Namely the

**FP$_n$-Conjecture.** *Let $Q$ be a finitely generated Abelian group, $A$ a finitely generated $\mathbb{Z}Q$-module and $A \hookrightarrow G \twoheadrightarrow Q$ a short exact sequence of groups. Then the following are equivalent:*

1. *$G$ is of type FP$_n$.*
2. *$\Sigma^c_A(Q, \mathbb{Z})$ is $n$-tame.*

This, he tells me, started as a "wild speculation" with its roots in [BiSt80] whose main result shows it to be true for $n = 2$. It mellowed to a serious conjecture only after it was supported by the results of [BiGr82]. H. Åberg [Åber86] has proved the conjecture provided that $G$ is of finite Prüfer rank. In particular it follows from his results that the theorem remains true in the number field case. G.A. Noskov [Nosk93] has proved that the implication $1 \Longrightarrow 2$ holds if $A$ is torsion free and the extension splits. In our example neither is $G(\mathcal{O}_S)$ of finite Prüfer rank nor is $\mathcal{O}_S$ torsion free. Thus it provides further evidence in favor of the FP$_n$-Conjecture. Recently D.H. Kochloukova [Koch95] has proved the FP$_n$-Conjecture for extensions of $Q$ by $\mathbb{Z}Q$-modules $A$ of Krull-dimension $1$ with $p$-torsion for some prime number $p$. Thus her results of which I became aware after having finished my work include this part of mine.

Another reason why our result may be of interest is the equivalence 1 $\Longleftrightarrow$ 3 which implies the following

**Theorem.** *Let $S$, $K$, $\mathcal{O}_S$ and $Q$ be as above and let $r$ denote the rank of $Q$. Then the following holds:*

1. *The group $\mathcal{O}_S \rtimes Q$ is not of type FP$_{r+1}$.*
2. *If $Q$ has maximum rank $r = |S| - 1$ then the group $\mathcal{O}_S \rtimes Q$ is of type FP$_r$.*

*In particular $G(\mathcal{O}_S) = \mathcal{O}_S \rtimes \mathcal{O}_S^*$ has finiteness length $|S| - 1$, i.e., it is of type FP$_{|S|-1}$ but not of type FP$_{|S|}$.*

This contributes to the theory of $S$-arithmetic groups. The theory is dominated by two independent fundamental distinctions: the distinction between number fields and functions fields on the one hand and the distinction between reductive groups and solvable groups on the other hand. The four corresponding pairs mark different regions of the theory. Concerning the finiteness properties FP$_n$ the state of the art varies from complete classification to rather special series of examples.

First let us turn to reductive groups for their being better known. A. Borel and J.-P. Serre [BoSe74] have shown that $S$-arithmetic subgroups of reductive groups over number fields are of type FP$_n$ for all $n \in \mathbb{N}$. In the function field case H. Behr [Behr92] has obtained a complete classification of all $S$-arithmetic subgroups of the types FP$_1$ and FP$_2$. Concerning higher finiteness properties there are the examples $\mathrm{SL}_2(\mathcal{O}_S)$ having finiteness length $|S| - 1$ due to U. Stuhler [Stuh80] and $\mathrm{SL}_n(\mathbb{F}_q[t])$ having finiteness length $n - 2$ provided that $q$ is large enough compared to $n$. The last result is independently due to H. Abels [Abel91] and P. Abramenko [Abra87] whose bounds on $q$ are slightly better. Abramenko has given similar results for all classical groups in [Abra96].

Less is known in the case of $S$-arithmetic subgroups of solvable groups. In [Abel87] H. Abels gave a complete description of all these groups of type $\mathrm{FP}_2$ in the number field case. A series of groups with increasing finiteness length due to H. Abels and K.S. Brown can be found in [AbBr87]. The case of $S$-arithmetic subgroups of solvable groups in the function field case had not previously been studied. Thus ours is the first series in this area.

The proof proceeds as follows. The implications $1 \Longrightarrow 2 \Longrightarrow 3$ are proven using $\Sigma$-theory. The invariant $\Sigma^c_{\mathcal{O}_S}(Q,\mathbb{Z})$ is computed in Theorem 2.3. In order to establish the implication $3 \Longrightarrow 1$ we study the action of $G(\mathcal{O}_S)$ on the product $X = \prod_{p\in S} X_p$ of the Serre trees $X_p$ associated to $\mathrm{SL}_2(K)$ and the place $p$. The most important part of that analysis is to show a certain subspace of $X$ to be highly connected. To do that we will use a geometric argument due to H. Åberg [Åber86]. It should be mentioned that our approach to this technique can be generalized to prove that the group $B_n(\mathcal{O}_S)$ has finiteness length $|S|-1$ where $B_n$ is the group scheme of upper triangular matrices in $\mathrm{SL}_n$. In this case the trees have to be replaced by Bruhat-Tits buildings which are often used in the context of arithmetic groups. But that is work in progress.

In the case $Q = \mathcal{O}_S^*$ the results presented here date back to my diploma thesis of 1993 which I prepared as a student of Prof. R. Bieri. I want to thank him and Dr. P. Abramenko for a lot of encouragement and very helpful criticism of an earlier version of this paper.

## 1 Preliminaries on global fields

First we have to introduce a lot of notation. Let us denote by

- $K$ a global function field whose elements are called <u>functions</u>,
- $k$ its finite field of constants,
- $R^*$ the group of units of any unital ring $R$, and
- $\mathcal{P}$ the set of non-trivial places of $K$. Since all valuations on $K$ are discrete we may regard a place $p \in \mathcal{P}$ as a normalized

discrete valuation $p : K^* \twoheadrightarrow \mathbb{Z}$. Formally we set $p(0) := \infty$.

For each place $p \in \mathcal{P}$,

- $\mathcal{O}_p := \{x \in K \mid p(x) \geq 0\}$ denotes the valuation ring of the place $p$. This is a local ring with maximal ideal
- $\mathfrak{m}_p = \{x \in K \mid p(x) > 0\}$ generated by an element
- $\pi_p$ with $p(\pi_p) = 1$. The residue field
- $k_p := \mathcal{O}_p/\mathfrak{m}_p$ is a finite extension of $k$ of degree
- $d_p := [k_p : k]$.
- $K_p$ denotes the completion of $K$ with respect to the valuation $p$. $K_p$ is a local field. The place $p$ extends by continuity to a place $\bar{p} : K_p^* \twoheadrightarrow \mathbb{Z}$. The corresponding valuation ring of $\bar{p}$ is the closure
- $\overline{\mathcal{O}_p}$ of $\mathcal{O}_p$ in $K_p$. This is an open, compact, local subring of $K_p$. Its maximal ideal is the closure
- $\overline{\mathfrak{m}_p}$. The residue field yields nothing new for $\overline{\mathcal{O}_p}/\overline{\mathfrak{m}_p} = k_p = \mathcal{O}_p/\mathfrak{m}_p$.

Then we have $k = \bigcap_{p \in \mathcal{P}} \mathcal{O}_p$. We write

- $\mathrm{Div}(K)$ for the free Abelian group generated by $\mathcal{P}$. Its elements are called <u>divisors</u>, and are written multiplicatively as products $D = \prod_{p \in \mathcal{P}} p^{D_p} \in \mathrm{Div}(K)$. For each function $x \in K^*$ there are only finitely many places $p \in \mathcal{P}$ with $p(x) \neq 0$. Therefore we have a well defined group homomorphism

$$\begin{aligned} (-) : K^* &\longrightarrow \mathrm{Div}(K) \\ x &\mapsto (x) := \prod_{p \in \mathcal{P}} p^{p(x)}. \end{aligned}$$

Its image consists of the <u>principal divisors</u>. The degree map $d : p \mapsto d_p$ extends to the homomorphism

$$\begin{aligned} \deg : \mathrm{Div}(K) &\longrightarrow \mathbb{Z} \\ D = \prod_{p \in \mathcal{P}} p^{D_p} &\mapsto \deg(D) := \sum_{p \in \mathcal{P}} d_p D_p. \end{aligned}$$

To each divisor $D = \prod_{p \in \mathcal{P}} p^{D_p}$ we assign the set

- $L(D)$ of all functions $x \in K$, such that $D_p \leq p(x)$ for all $p \in \mathcal{P}$. This is a $k$-vector space of finite dimension
- $l(D)$.

- $S$ denotes a finite non-empty set of places. For each such set let
  - $\mathcal{O}_S := \bigcap_{p \in \mathcal{P} \setminus S} \mathcal{O}_p = \{x \in K \mid p(x) \geq 0 \ \ \forall p \in \mathcal{P} \setminus S\}$ be the corresponding $S$-arithmetic ring of all functions holomorphic outside $S$, and let
  - $A_S := \prod_{p \in S} K_p \times \prod_{p \in \mathcal{P} \setminus S} \overline{\mathcal{O}_p}$ be the topological ring of $S$-adeles.

  The ring of adeles of $K$ is

- $A := \varinjlim A_S$. This is a topological ring containing $K$ by the diagonal embedding $K \hookrightarrow A$. $A_S$ is open in $A$, and an arbitrary subset of $A$ is open if and only if its intersection with each $A_S$ is open in $A_S$. We have

$$\begin{aligned} A &= \bigcup_S A_S, \\ \mathcal{O}_S &= A_S \cap K. \end{aligned}$$

Now we recall several results we shall use.

**Riemann's Theorem (**Compare [Weil67, VI. Theorem 2]**)** *There is a non-negative integer $g$, such that*

$$1 - g \leq l(D) + \deg(D)$$

*holds for each divisor $D \in \mathrm{Div}(K)$. The least $g$ that satisfies this condition is the genus of $K$.*

**Product formula for principal divisors (**Compare [O'Me73, 33:1]**)** *For $x \in K$,*

$$1 = \prod_{p \in \mathcal{P}} \mathrm{e}^{-d_p p(x)}$$

*holds. Thus the degree $\deg((x))$ of any principal divisor vanishes.*

**Dirichlet's unit theorem** (Compare [O'Me73, 33:10]) *The group* $\mathcal{O}_S^*$ *is a finitely generated Abelian group of rank* $|S|-1$.

**Compactness of** $K\backslash A$ (Compare [Weil67, IV.2. Theorem 2]) $K$ *is a discrete subgroup of* $A$. *The quotient* $K\backslash A$ *is compact. Moreover there is a compact subset of* $A$, *that maps onto* $K\backslash A$.

## 2 Computation of $\Sigma^c_{\mathcal{O}_S}(Q,\mathbb{Z})$

We have to recall some notions of $\Sigma$-theory. In the following,

- $Q$ denotes a finitely generated Abelian group,
- $R$ a commutative ring with unit $1\neq 0$, and
- $M$ a finitely generated $RQ$-module.

The set $\operatorname{Hom}(Q,\mathbb{R})$ of all <u>characters</u> of $Q$ is an $\mathbb{R}$ vector space of finite dimension. We call two characters <u>equivalent</u> if they are positive scalar multiples of each other. The set of equivalence classes of non-trivial characters can be identified with a sphere

- $\mathbb{S}(Q)$ which we call the <u>character sphere</u> of $Q$. To each non-trivial character $\chi\in\operatorname{Hom}(Q,\mathbb{R})\setminus\{0\}$ we assign the <u>character monoid</u>
- $Q_\chi:=\{q\in Q\mid \chi(q)\geq 0\}$. This monoid depends only on the equivalence class
- $[\chi]$ of $\chi$. Then the subset
- $\Sigma_M(Q,R):=\{[\chi]\in\mathbb{S}(Q)\mid M$ is finitely generated over $RQ_\chi\}$ is by definition the <u>Bieri-Strebel-invariant</u>. But it is more convenient to use the complement
- $\Sigma^c_M(Q,R):=\mathbb{S}(Q)\setminus\Sigma_M(Q,R)$.

**Proposition 2.1** (Compare [BiGr84]). *Let* $Q$, $R$ *and* $M$ *be as above. Let further* $A=RQ/\operatorname{Ann}(M)$ *where* $\operatorname{Ann}(M)$ *denotes the annihilator of* $M$.

1. *$\Sigma^c_M(Q,R)$ is the set of all $[\chi] \in \mathbb{S}(Q)$ for which $\chi$ is induced by some valuation $v : A \to \mathbb{R}^\infty := \mathbb{R} \cup \{\infty\}$ with $v(R) \geq 0$.* [loc. cit. Theorem 8.1]

2. *If $K$ is a field extension of the field $k$ and $Q$ is a subgroup of $K^*$ let $k[Q]$ denote the $k$-subalgebra of $K$ generated by $Q$. For any given valuation $v : k \to \mathbb{R}^\infty$ a character $\chi$ of $Q$ is induced by an extension of $v$ to $k[Q]$ if and only if it is induced by an extension of $v$ to $K$.* [loc. cit. Corollary 6.3]

Now we specialize $Q$ to be a subgroup of $\mathcal{O}_S^*$. We will use 2.1 to compute $\Sigma^c_{\mathcal{O}_S}(Q,\mathbb{Z})$. The first step is trivial.

**Observation 2.2.** *The action of $\mathbb{Z}\mathcal{O}_S^*$ on $\mathcal{O}_S$ factors via $k_0\mathcal{O}_S^*$, where $k_0$ is the prime field of $K$. Therefore we have for any subgroup $Q$ of $\mathcal{O}_S^*$*

$$\Sigma^c_{\mathcal{O}_S}(Q,\mathbb{Z}) = \Sigma^c_{\mathcal{O}_S}(Q,k_0).$$

Thus we are reduced to computing $\Sigma^c_{\mathcal{O}_S}(Q,k_0)$ in order to establish

**Theorem 2.3.** *Let $Q$ be a subgroup of $\mathcal{O}_S^*$. If $\mathcal{O}_S$ is finitely generated over $\mathbb{Z}Q$ then each place $p \in S$ induces by restriction a non-trivial character on $Q$ and we have*

$$\Sigma^c_{\mathcal{O}_S}(Q,\mathbb{Z}) = \{[p|_Q] \in \mathbb{S}(Q) \mid p \in S\}.$$

**Proof.** Let $R := k_0$ and $A := RQ/\operatorname{Ann}(\mathcal{O}_S)$. The $k_0$-algebra $RQ = k_0Q$ consists of finite formal sums $a_1g_1 \oplus \cdots \oplus a_rg_r$ with $a_i \in k_0 \subseteq K$ and $g_i \in Q \subseteq K$. Therefore we have an obvious evaluation map $\operatorname{ev} : RQ \to K$. The action of $RQ$ on $\mathcal{O}_S$ satisfies

$$\begin{aligned} x \cdot (a_1g_1 \oplus \cdots \oplus a_rg_r) &= x(a_1g_1 + \cdots + a_rg_r) \\ &= x\operatorname{ev}(a_1g_1 \oplus \cdots \oplus a_rg_r). \end{aligned}$$

Hence $\operatorname{Ann}(\mathcal{O}_S) = \ker(\operatorname{ev})$. Thus we may identify $A$ with $\operatorname{im}(\operatorname{ev}) = k_0[Q] \leq K$.

Since $R$ is a finite field it admits only the trivial valuation. Hence 2.1-1 implies that $\Sigma^c_{\mathcal{O}_S}(Q,\mathbb{Z}) = \Sigma^c_{\mathcal{O}_S}(Q,k_0)$ is the set of all $[\chi] \in \mathbb{S}(Q)$ for which $\chi$ is induced by a valuation on $A = k_0[Q]$.

By 2.1-2 a non-trivial character of $Q$ is induced by a valuation on $k_0[Q]$ if and only if it is induced by a valuation on $K$. Since all these valuations are discrete for $K$ a global function field we may choose them to be normalized changing $\chi$ only within its equivalence class. Hence $\Sigma^c_{\mathcal{O}_S}(Q,\mathbb{Z})$ consists precisely of the equivalence classes of non-trivial characters of $Q$ induced by places $p : K^* \twoheadrightarrow \mathbb{Z}$.

Since $Q \leq \mathcal{O}_S^* = \{x \in K^* \mid p(x) = 0 \ \ \forall p \in \mathcal{P} \setminus S\}$ a non-trivial character on $Q$ which is a restriction of a place has to be induced by a place in $S$.

Thus it remains to show that for each $p \in S$ we have $p(Q) \neq \{0\}$. This follows from Riemann's Theorem as it implies that there exists no lower bound for $p(\mathcal{O}_S)$. We assumed that $\mathcal{O}_S$ is finitely generated over $Q$. But the $p$-image of a finite generating subset is bounded. Therefore $p(Q)$ has to be non-trivial. **q.e.d.**

## 3 Tameness properties of $\mathcal{O}_S$

We keep the $\Sigma$-theoretic notational conventions. Let $\mathbb{S}$ be a sphere. A subset $\Sigma^c \subseteq \mathbb{S}$ is called <u>fully tame</u> if it is contained in an open hemisphere of $\mathbb{S}$. It is called <u>$n$-tame</u> if each subset of $\Sigma^c$ consisting of at most $n$ points is contained in an open hemisphere of $\mathbb{S}$. A finitely generated $RQ$-module $A$ is called <u>$n$-tame</u> if $\Sigma^c_A(Q,R)$ is $n$-tame as a subset of $\mathbb{S}(Q)$.

**Theorem 3.1 (**Compare [BiGr82, Theorem D]**).** *Let $A \hookrightarrow G \twoheadrightarrow Q$ be a short exact sequence of groups with both $A$ and $Q$ Abelian and $G$ finitely generated. If $G$ is of type FP$_n$ then $A \otimes_{\mathbb{Z}} k$ is $n$-tame as a $kQ$-module for every field $k$.*

Now we can show the first part of our main result.

**Lemma 3.2.** *Let $G := \mathcal{O}_S \rtimes Q$ where $Q \leq \mathcal{O}_S^*$. The following implications hold for any integer $n \geq 1$:*

*$G$ is of type FP$_n$.*

*$\Longrightarrow$ $\mathcal{O}_S$ is $n$-tame as a $\mathbb{Z}Q$-module.*

$\Longrightarrow$ *Each* $p \in S$ *restricts to a non-trivial character on* $Q$ *and* $\{[p|_Q] \mid p \in S\}$ *is an* $n$*-tame subset of* $\mathbb{S}(Q)$.

**Proof.** Assume that $G$ is of type $\mathrm{FP}_n$. Then $\mathcal{O}_S = \mathcal{O}_S \otimes_{\mathbb{Z}} k_0$ is $n$-tame as a $k_0Q$-module according to 3.1. Now it follows from Observation 2.2 that $\mathcal{O}_S$ is $n$-tame as a $\mathbb{Z}Q$-module.

Now assume that $\mathcal{O}_S$ is $n$-tame. As this implies by definition that $\mathcal{O}_S$ is a finitely generated $\mathbb{Z}Q$-module we can apply Theorem 2.3. Therefore we have

$$\Sigma^c_{\mathcal{O}_S}(Q,\mathbb{Z}) = \{[p|_Q] \mid p \in S\}$$

whence the second implication holds. **q.e.d.**

## 4 Serre's tree

The remaining implication will be proved by a different method. We study the action of $\mathcal{O}_S \rtimes Q \leq G(\mathcal{O}_S)$ on the product $X := \prod_{p \in S} X_p$ where $X_p$ is the Serre tree corresponding to $\mathrm{SL}_2(K_p)$. We will describe the case $n = 2$ of a non-standard model for the Bruhat-Tits building associated to $\mathrm{SL}_n(K_p)$ that also works for higher dimensions. In [Åber86, p. 295] H. Åberg attributes this construction to K. Brown. The fact that it is equivalent to Serre's construction in [Serr80, II.1] is neither obvious nor will we make any use of it. An ambitious reader may verify it on his own.

Let $p \in S$ be a fixed place. For each $i \in \mathbb{Z}$ let

- $L_p^i := \{x + \pi_p^i \overline{\mathcal{O}_p} \mid x \in K_p\} = K_p / \pi_p^i \overline{\mathcal{O}_p}$ and
- $X_p := \bigcup_{i \in \mathbb{Z}} L_p^i$ be the set of affine rank-1 $\overline{\mathcal{O}_p}$-submodules of $K_p$. We call them <u>vertices</u> when we are talking about the tree $X_p$. We call the elements of $X_p$ <u>lattices</u> if we want to emphasize their set structure. A <u>special</u> lattice is one of the form
- $\Lambda_i := \pi_p^i \overline{\mathcal{O}_p} \in L_p^i$.

Each lattice $\Lambda \in L_p^{i+1}$ is contained as a maximal sub-lattice in a unique lattice $\Lambda' \in X_p$ which then is an element of $L_p^i$. We call

$\Lambda'$ the parent of $\Lambda$ which is a child of $\Lambda'$. Because of the isomorphism $\pi_p^i\overline{\mathcal{O}_p}/\pi_p^{i+1}\overline{\mathcal{O}_p} \cong \mathcal{O}_p/\mathfrak{m}_p = k_p$ each lattice has exactly $|k_p|$ children. We turn $X_p$ into a locally finite graph without terminal vertices by inserting an edge connecting each lattice with its parent. This graph cannot contain loops. Furthermore we have a height function

- $\lambda_p : X_p \longrightarrow \mathbb{Z}$ sending all $\Lambda \in L_p^i$ to $i$. This map is given by

$$\lambda_p(\Lambda) = \min\{p(x-y) \mid x,\, y \in \Lambda\}.$$

We do not distinguish between $X_p$ and its geometric realization. So we extend $\lambda_p$ to a real-valued function by linear interpolation.

There is a unique descending end $\mathcal{L}^\infty$ of $X_p$ represented by $\Lambda - \Lambda' - \Lambda'' - \cdots$ because for all $n \geq j - p(x)$ the $n$th ancestor of $x + \pi_p^j\overline{\mathcal{O}_p}$ is a special lattice.

This shows that $X_p$ is connected whence it is a tree. One easily checks that

$$\begin{pmatrix} 1 & t \\ 0 & x \end{pmatrix} \Lambda := x^{-1}(t + \Lambda)$$

defines an action of $K_p \rtimes K_p^*$ on $X_p$ by automorphisms preserving the parent-child relation.

**Observation 4.1.** *The induced action on $X_p$ of the subgroup $(K_p, +)$ consisting of matrices $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ preserves height and is transitive on each $L_p^i \subseteq X_p$. The stabilizer of a special lattice $\Lambda_i$ in $K_p$ is $\pi_p^i\overline{\mathcal{O}_p} = \{x \in K_p \mid p(x) \geq i\}$.*

*For all $x \in K_p^*$ and all $\Lambda \in X_p$ we have*

$$\lambda_p\left(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \Lambda\right) = \lambda_p(\Lambda) - p(x).$$

**Corollary 4.2.** *The stabilizer of any vertex $\Lambda \in X_p$ in $K_p \rtimes K_p^*$ is a subset of $\pi_p^i\overline{\mathcal{O}_p} \rtimes \overline{\mathcal{O}_p}^*$ for some $i \in \mathbb{Z}$.*

**Proof.** The stabilizer of $\Lambda$ has to fix the path $\Lambda - \Lambda' - \Lambda'' - \cdots$ representing $\mathcal{L}^\infty$ pointwise. Therefore it is contained in the stabilizer of a special lattice $\Lambda_i$ which is $\pi_p^i\overline{\mathcal{O}_p} \rtimes \overline{\mathcal{O}_p}^*$. **q.e.d.**

# 5 Finiteness properties of $\mathcal{O}_S \rtimes Q$

We want to study the action of $\mathcal{O}_S \rtimes Q$ on the product

- $X := \prod_{p\in S} X_p$. We have an obvious projection
- $\lambda := \prod_{p\in S} \lambda_p : X \twoheadrightarrow \mathbb{R}^S$.

  In order to describe the places simultaneously let
- $\varphi : K \to \mathbb{R}^S$ be given by $\varphi : x \mapsto (p(x))_{p\in S}$.

From Observation 4.1 we conclude

**Proposition 5.1.** *The group $G(K) = K \rtimes K^*$ acts diagonally on $X$.*

1. *The action of $(K,+)$ on $X$ preserves $\lambda$.*
2. *For all $x \in K^*$ and all $\underline{\Lambda} \in X$*

$$\lambda\left(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\underline{\Lambda}\right) = \lambda(\underline{\Lambda}) - \varphi(x)$$

*holds.* **q.e.d.**

Since $\mathcal{O}_S$ is normal in $\mathcal{O}_S \rtimes Q$ we can build the quotient space $(\mathcal{O}_S \rtimes Q)\backslash X$ in two steps. So we first consider the quotient

- $X' := \mathcal{O}_S\backslash X$ where $(\mathcal{O}_S,+)$ acts as a subgroup of $(K,+) \leq G(K)$. Proposition 5.1-1 implies that $\lambda$ induces a projection
- $\lambda' : X' \twoheadrightarrow \mathbb{R}^S$.

**Lemma 5.2.** *The projection $\lambda' : X' \twoheadrightarrow \mathbb{R}^S$ is a proper map.*

**Proof.** As any compact subset in $\mathbb{R}^S$ is contained in a cuboid with corners in $\mathbb{Z}^S$ we only have to show that $\lambda'$-preimages of such cuboids are compact. So consider the cuboid $C_{\underline{z},\underline{n}} := \prod_{p\in S}[z_p - n_p, z_p]$ with upper corner $\underline{z} = (z_p)_{p\in S}$. Since

$$\lambda'^{-1}(C_{\underline{z},\underline{n}}) = \bigcup_{(\Lambda_p)_{p\in S}\in\lambda'^{-1}(\underline{z})} \prod_{p\in S} \Lambda_p - \Lambda'_p - \cdots - \Lambda_p^{(n_p)}$$

we are reduced to proving the preimage $\lambda'^{-1}(\underline{z})$ to be finite for any $S$-tuple $\underline{z} = (z_p)_{p\in S} \in \mathbb{Z}^S \subseteq \mathbb{R}^S$.

To see this note first that by definition of $\lambda$ there is an 1-1-correspondence

$$\lambda^{-1}(\underline{z}) \longleftrightarrow \prod_{p\in S} L_p^{z_p} = \prod_{p\in S} K_p/\pi_p^{z_p}\overline{\mathcal{O}_p} \cong A_S/A_{\underline{z}}$$

where $A_{\underline{z}}$ is the open subgroup

$$A_{\underline{z}} := \prod_{p\in S} \pi_p^{z_p}\overline{\mathcal{O}_p} \times \prod_{p\in\mathcal{P}\setminus S} \overline{\mathcal{O}_p}$$

of $A_S$. Therefore we only have to show that the set

$$\mathcal{O}_S\backslash A_S/A_{\underline{z}} \subseteq K\backslash A/A_{\underline{z}}$$

of double cosets is finite. But the right hand side is discrete since $A_{\underline{z}}$ is open in $A$ and it is compact since $K\backslash A$ is compact. Hence the right hand is finite whence so is the left hand. **q.e.d.**

Now we turn to the space $X_Q$ that will enable us to compute the finiteness properties of $\mathcal{O}_S \rtimes Q$. Let $x \in \mathcal{O}_S^*$ act on $\mathbb{R}^S$ by $\mathbb{R}^S \ni v \mapsto v - \varphi(x) \in \mathbb{R}^S$. Then $\lambda : X \to \mathbb{R}^S$ is a $\mathcal{O}_S^*$-map by Proposition 5.1-2. So we are led to consider the image $\varphi(Q)$. Let

- $H_Q \leq \mathbb{R}^S$ be the $\mathbb{R}$-subspace of $\mathbb{R}^S$ generated by $\varphi(Q) \subseteq \mathbb{R}^S$ and
- $X_Q := \lambda^{-1}(H_Q) \subseteq X$ be the preimage of $H_Q$. Since the cubic CW-structure on $\mathbb{R}^S$ induces a CW-decomposition of $H_Q$ we see that $X_Q$ is a CW-complex.

By Proposition 5.1 the operation of $K \rtimes K^*$ restricts to an action of $\mathcal{O}_S \rtimes Q$ on $X_Q$.

**Proposition 5.3.** *The quotient $\widetilde{X}_Q := (\mathcal{O}_S \rtimes Q)\backslash X_Q$ is compact. Furthermore the stabilizers of all vertices in $X_Q$ are finite.*

**Proof.** First we consider $(\mathcal{O}_S, +)\backslash X_Q$ which is the preimage $\lambda'^{-1}(H_Q) \subseteq X'$.

By definition $\varphi(Q)$ is a lattice of maximal rank in $H_Q$. Therefore there is a compact subset $C_0 \subset H_Q$ whose $Q$-translates cover $H_Q$.

Lemma 5.2 says that ${\lambda'}^{-1}(C_0)$ is compact. Thus we have found a compact subset of $X'$ whose $Q$-translates cover ${\lambda'}^{-1}(H_Q)$. But then the quotient $\widetilde{X}_Q = Q\backslash{\lambda'}^{-1}(H_Q)$ is compact, being the image of a compact set.

Now we turn to the stabilizers. We will show that the stabilizer of each point in the geometric realization of $X$ in the group $G(\mathcal{O}_S) \geq \mathcal{O}_S \rtimes Q$ is finite. It suffices to prove this for vertex tuples $\underline{\Lambda} = (\Lambda_p)_{p\in S} \in X$. The stabilizer of $\underline{\Lambda}$ is the intersection of all the stabilizers of the single components $\Lambda_p$. Hence using Corollary 4.2 we see that

$$\begin{aligned} \mathrm{Stab}_{G(\mathcal{O}_S)}(\underline{\Lambda}) &\subseteq \left\{ \begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} \in G(K) \mid p(x) \geq D_p,\, p(u) = 0 \;\; \forall p \in S \right\} \cap G(\mathcal{O}_S) \\ &= L(D) \rtimes k^* \end{aligned}$$

for an appropriate divisor $D := \prod_{p\in S} p^{D_p}$. But now the claim follows from finiteness of $L(D)$ which holds since $L(D)$ is a vector space of finite dimension over the finite field $k$. **q.e.d.**

**Positive FP$_n$-Criterion (**Compare [Brow87, Proposition 1.1]**)** *Let $G$ be a group and $X_0$ a CW-complex on which $G$ acts by homeomorphisms that permute the cells of $X_0$. Let us further suppose that the $n$-skeleton of $X_0$ is finite $\bmod G$. Then $G$ is of type FP$_n$ if the following two conditions are satisfied:*

1. *$X_0$ is acyclic in dimensions $< n$, i.e., the reduced homology $\widetilde{\mathrm{H}}_i(X_0) = 0$ for $i < n$.*
2. *For $0 \leq i \leq n$ the stabilizer of any $i$-cell of $X_0$ is of type FP$_{n-i}$.*

Now we are prepared to prove our main result.

**Theorem 5.4.** *For each $n \in \mathbb{N}$ the following three conditions are equivalent:*

1. *$\mathcal{O}_S \rtimes Q$ is of type FP$_n$.*
2. *$\mathcal{O}_S$ is $n$-tame as a $\mathbb{Z}Q$-module.*
3. *Each $p \in S$ restricts to a non-trivial homomorphism $p|_Q : Q \to \mathbb{R}$ and the set $\{[p|_Q] \mid p \in S\} \subseteq \mathbb{S}(Q)$ is $n$-tame.*

*If these conditions hold for at least one $n \geq 1$ then the identity*

$$\Sigma^c_{\mathcal{O}_S}(Q, \mathbb{Z}) = \{[p|_Q] \mid p \in S\}$$

*holds.*

**Proof.** The implications $1 \Longrightarrow 2 \Longrightarrow 3$ were proved in Lemma 3.2. The identity was derived in Theorem 2.3.

So we only have to prove the implication $3 \Longrightarrow 1$. Using the positive FP$_n$-Criterion and Proposition 5.3 we are reduced to showing that $X_Q$ is acyclic in dimensions $< n$. This will be done in Section 7. In 7.4 we will prove the slightly stronger claim that $X_Q$ is $\underline{(n-1)\text{-connected}}$. This means by definition that the homotopy groups $\pi_i(X_Q)$ are trivial for all $i \in \{0, \ldots, n-1\}$. **q.e.d.**

## 6 Preliminaries on convex sets

We have to develop some basic facts about convex sets. Let

- $\mathbb{E}$ be a Euclidean space of finite positive dimension,
- $\mathbb{T}$ the $\mathbb{R}$-vector space of translations of $\mathbb{E}$,
- $\mathbb{S} := \mathbb{S}(\mathbb{T})$ the sphere of rays in $\mathbb{T}$, and
- $\mathbb{S}^* := \mathbb{S}^*(\mathbb{T})$ where for an Euclidean vector space $V$,
- $\mathbb{S}^*(V)$ denotes the unit sphere of the dual $V^*$. By duality the points of $\mathbb{S}$ correspond to the hemispheres of $\mathbb{S}^*$ and vice versa. For any convex subset $C \subseteq \mathbb{E}$ let
- $\widehat{C}$ denote its convex hull. For any convex set we will study the set
- $\widetilde{C} := \{t \in \mathbb{S} \mid t + C \subseteq C\}$ of <u>directions</u> of $C$. Here we use the notation $t + C := \{\tau + x \mid \tau \in t,\, x \in C\}$.

We start by recalling two well known propositions for later use.

**Carathéodory's Theorem** (Compare [Eggl58, 2.2. Theorem 18]) *For any $C \subseteq \mathbb{E}$ the convex hull $\widehat{C}$ is the union of the convex hulls of all finite subsets of $C$ containing at most $\dim \mathbb{E} + 1$ points.*

**Corollary 6.1** (Compare [Eggl58, p. 38, Remark (ii)]). *The convex hull of a compact set is compact.* **q.e.d.**

**Lemma 6.2.** *Any closed $(\dim \mathbb{E} + 1)$-tame subset of $\mathbb{S}^*$ is fully tame.* **q.e.d.**

**Proof.** Let $C \subseteq \mathbb{S}^*$ be closed and $(\dim \mathbb{E} + 1)$-tame. Assume for a moment that $0 \in \widehat{C}$. Then by Carathéodory's Theorem there are $\dim \mathbb{E} + 1$ points in $C$ spanning a simplex that contains $0$. But this contradicts the assumption that $C$ is $(\dim \mathbb{E} + 1)$-tame. Therefore $0 \notin \widehat{C}$ whence there is a hyperplane separating $0$ and $\widehat{C}$ because by 6.1 $\widehat{C}$ is compact and therefore closed. This implies that $C \subset \widehat{C}$ is contained in an open hemisphere of $\mathbb{S}^*$. **q.e.d.**

Since $\tau + C \subseteq C$ and $\tau' + C \subseteq C$ together imply $(\tau + \tau') + C \subseteq C$ for any $\tau, \tau' \in \mathbb{T}$ we know that for any two non-antipodal points in $\widetilde{C}$ the minimal geodesic joining them is contained in $\widetilde{C}$. So we may say that $\widetilde{C}$ is <u>spherically convex</u>.

**Observation 6.3.** *If $C \subseteq \mathbb{E}$ is closed, convex, and non-empty then $t \in \mathbb{S}$ is a direction of $C$ if and only if there is a point $x_0 \in C$ such that the ray $t + x_0$ does not leave $C$.*

**Proof.** Suppose there is $x_0 \in C$ with $t + x_0 \subseteq C$. Then we can infer $t + x \subseteq C$ for all $x \in C$ from the fact that $C$ is closed and that $t + x$ lies in the closure of the convex hull of $t + x_0$ and $x$. The converse is obvious. **q.e.d.**

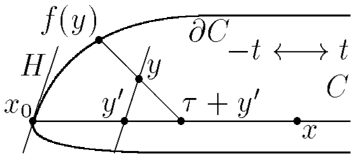
Now we can classify the convex subsets of $\mathbb{E}$ as far as we need to.

**Theorem 6.4.** *Let $C \subseteq \mathbb{E}$ be a closed convex set with non-empty interior. Then the following hold:*

1. *If there is a direction $t \in \widetilde{C}$ such that $-t \notin \widetilde{C}$ then the boundary $\partial C$ is a deformation retract of $C$. In this case we call $C$ <u>retractable</u>.*

2. *If for each direction $t \in \widetilde{C}$ the opposite $-t$ is also a direction of $C$ then there exist a unique subspace $V \leq \mathbb{T}$ and a unique compact convex subset $C' \subseteq \mathbb{E}/V$ such that $C$ is the preimage of $C'$ under the canonical projection. This $C'$ has non-empty interior. In this case we call $C$ <u>decomposable</u> since it is homeomorphic to $V \times C'$.*
   *In this case $\widetilde{C} = \emptyset$ is equivalent to $V = \{0\}$. Hence $C$ is compact if and only if $\widetilde{C} = \emptyset$.*

**Proof.** In order to prove the first claim we construct a continous map $f : C \to \partial C$ that restricts to the identity on $\partial C$. Given such a map, moving any $y \in C$ to $f(y)$ along the line segment $[y, f(y)]$ defines a retracting deformation of $C$ onto $\partial C$.




So let $t$ be a direction of $C$ with $-t \notin \widetilde{C}$. Take an interior point $x \in C$. The ray $-t + x$ intersects $\partial C$ in, say, $x_0$. There is a hyperplane $H \ni x_0$ such that $C$ is contained in a halfspace determined by $H$. Fix a non-trivial translation $\tau \in t$. For any $y \in C$, let $y'$ be the intersection of $t + x_0$ and a hyperplane parallel to $H$ containing $y$. Then the ray starting at the point $\tau + y'$ passing through $y$ intersects $\partial C$ because it intersects $H$. Define $f(y)$ to be the point of intersection which is unique since $\tau + y'$ is an interior point of $C$. The map $f$ defined this way is continous and restricts to the identity on $\partial C$.

To prove the second claim we start with a special case. If $\widetilde{C} = \emptyset$ then $C$ is compact. Assume for a moment $C$ is not compact. Then $C$ is unbounded. So choose a sequence $(x_i)_{i \in \mathbb{N}}$ of points in $C$ such that $x_0$ is an interior point of $C$ and such that the distance satisfies $|x_i - x_0| \geq i$. The points $t_i \in \mathbb{S}$ with $t_i \ni (x_i - x_0) \in \mathbb{T}$ have a cluster point since $\mathbb{S}$ is compact. Because $C$ is closed this point is a direction of $C$. This contradicts $\widetilde{C} = \emptyset$.

We turn to the general case. Assume that for each $t \in \widetilde{C}$ the antipodal point $-t$ also lies in $\widetilde{C}$. Since $\widetilde{C}$ is spherically convex we can conclude that it is a subsphere of $\mathbb{S}$ with center $0$. Hence the cone $V := \bigcup_{t \in \widetilde{C}} t$ is a subspace of $\mathbb{T}$ such that for each

$x \in C$ we have $V + x \subseteq C$. Hence $C$ is the preimage of $C' := C/V \subseteq \mathbb{E}/V$. The image $C'$ is convex because of the projection being affine. Furthermore by construction we have $\widetilde{C'} = \emptyset$ whence $C'$ is compact. **q.e.d.**

For a closed half space $A \subseteq \mathbb{E}$ the set $\widetilde{A} \subseteq \mathbb{S}$ is a closed hemisphere. Let $e(A) \in \mathbb{S}^*$ be the corresponding point. Fix a subset $\Sigma^c \subseteq \mathbb{S}^*$. A closed convex set $C \subseteq \mathbb{E}$ with non-empty interior is subordinate to $\Sigma^c$ if it can be written as an intersection of closed half spaces

$$C = \bigcap_{A \in \mathcal{A}} A$$

with pairwise different $e(A) \in \Sigma^c$. A subset $C \subseteq \mathbb{E}$ is called fully tame ($n$-tame) if it is subordinate to a fully tame ($n$-tame) finite subset of $\mathbb{S}^*$.

**Lemma 6.5.** *A fully tame subset $C \subseteq \mathbb{E}$ is retractable.*

**Proof.** Let $\Sigma^c \subseteq \mathbb{S}^*$ be fully tame such that $C$ is subordinate to it. And let $C = \bigcap_{A \in \mathcal{A}} A$ with pairwise different $e(A) \in \Sigma^c$. Then $\Sigma^c$ is contained in an open hemisphere of $\mathbb{S}^*$. Let $t \in \mathbb{S}$ be the corresponding point. Then we have $t + A \subseteq A$ for all $A \in \mathcal{A}$ whence $t \in \widetilde{C}$. But $-t \notin \widetilde{C}$ since $\Sigma^c$ is finite and $-t + A \nsubseteq A$ for all $A \in \mathcal{A}$. Hence $C$ is retractable. **q.e.d.**

**Corollary 6.6.** *No fully tame set is compact.* **q.e.d.**

**Corollary 6.7.** *Let $V \leq \mathbb{T}$ be a subspace, $C' \subseteq \mathbb{E}/V$ be a convex compact set with non-empty interior. If the preimage $C \subseteq \mathbb{E}$ of $C'$ is $n$-tame then $V$ has codimension $\geq n$.*

**Proof.** It is easy to see that $C'$ is $n$-tame. But since $C'$ is compact it cannot be fully tame. Hence $n \leq \dim(\mathbb{E}/V)$ by Lemma 6.2. **q.e.d.**

## 7 Connectivity Properties of $X_Q$

Lemma 7.3, which is the main result of this section, is due to H. Åberg (compare [Åber86, p. 299ff]) in the case of products of trees. We will use Theorem 6.4 to prove it in a slightly more general form that can be applied to products of certain Euclidean buildings as well because we think that this way provides a bit more insight into the geometric situation. However, in this paper we will make use of the Lemma only for products of trees.

A sheet complex is a triple $(X, \mathbb{E}, \pi)$, where $\mathbb{E}$ is a Euclidean space of finite positive dimension considered as a CW-complex with a given cell decomposition, $X$ is a metric CW-complex and $\pi : X \to \mathbb{E}$ is a cellular map mapping cells of $X$ isometrically onto cells of $\mathbb{E}$, such that $X$ is the union of all its sheets. These are the subcomplexes $\Delta$ of $X$, that $\pi$ maps isometrically onto $\mathbb{E}$. We will consider sheets as Euclidean spaces.

Let $\Sigma^c \subseteq \mathbb{S}^*$ be finite. A sheet complex is a $\Sigma^c$-complex if any compact subset of $X$ is contained in a finite union $\bigcup_{i=1}^r \Delta^i$ of sheets such that the closure

$$\overline{\Delta^j \setminus \bigcup_{i<j} \Delta^i}$$

of any "new piece" is subordinate to $\Sigma^c$ as a subset of $\Delta^j$ for each $j \in \{2, \ldots, r\}$. We call such a sequence of sheets increasing.

**Example 7.1.** The triples $(X_p, \mathbb{R}, \lambda_p)$ are $\{\mathrm{id}_{\mathbb{R}}\}$-complexes.

**Example 7.2.** Let $(X_1, \mathbb{E}_1, \pi_1)$ and $(X_2, \mathbb{E}_2, \pi_2)$ be $\Sigma_1^c$- respectively $\Sigma_2^c$-complexes. Each linear form on $\mathbb{T}_1$ induces a linear form on $\mathbb{T}_1 \times \mathbb{T}_2$. Hence we can embed $\mathbb{S}^*(\mathbb{T}_i)$ in $\mathbb{S}^*(\mathbb{T}_1 \times \mathbb{T}_2)$. Let $\iota_i$ be the inclusion map. Then $(X_1 \times X_2, \mathbb{E}_1 \times \mathbb{E}_2, \pi_1 \times \pi_2)$ is a $\iota_1(\Sigma_1^c) \cup \iota_2(\Sigma_2^c)$-complex.

**Proof.** The sheets in $X_1 \times X_2$ are precisely the products $\Delta_1 \times \Delta_2$ where $\Delta_i$ is a sheet of $X_i$ $(i \in \{1, 2\})$.

Now let $(\Delta_1^1, \ldots, \Delta_1^r)$ and $(\Delta_2^1, \ldots, \Delta_2^s)$ be increasing sequences of sheets in $X_1$ and $X_2$. We have to show that the finite set of products $\Delta_1^i \times \Delta_2^j$ can be arranged to form an increasing sequence. But this is simple. Consider the lexicographic ordering

$$(i, j) < (k, l) :\Longleftrightarrow [i < k] \vee [i = k \wedge j < l]$$

and the corresponding sequence

$$(\Delta_1^1 \times \Delta_2^1, \ldots, \Delta_1^1 \times \Delta_2^s, \Delta_1^2 \times \Delta_2^1, \ldots, \Delta_1^r \times \Delta_2^s).$$

A moment's thought shows that

$$\begin{aligned} \Delta_1^k \times \Delta_2^l \setminus \bigcup_{(i,j)<(k,l)} \Delta_1^i \times \Delta_2^j &= \Delta_1^k \times \Delta_2^l \setminus \bigcup_{\substack{(i,j)<(k,l) \\ j \leq l}} \Delta_1^i \times \Delta_2^j \\ &= \left( \Delta_1^k \setminus \bigcup_{i<k} \Delta_1^i \right) \times \left( \Delta_2^l \setminus \bigcup_{j<l} \Delta_2^j \right). \end{aligned}$$

But the right hand side is subordinate to $\iota_1(\Sigma_1^c) \cup \iota_2(\Sigma_2^c)$ if $(1,1) < (k,l)$. **q.e.d.**

**Lemma 7.3.** *Let $(X, \mathbb{E}, \pi)$ be a $\Sigma^c$-complex and $H \leq \mathbb{T}$ a subspace such that each form $\xi \in \Sigma^c$ does not vanish on $H$. Assume furthermore that by restriction to $H$ the set $\Sigma^c$ induces an $n$-tame subset $\Sigma_H^c \subseteq \mathbb{S}^*(H)$. Then for each $x_0 \in \mathbb{E}$ the preimage $Y := \pi^{-1}(H + x_0) \subseteq X$ is $(n-1)$-connected.*

**Proof.** It suffices to show that any compact subset $C \subseteq Y \subseteq X$ is contained in an $(n-1)$-connected subspace of $Y$. But for compact $C \subseteq Y$ there is a finite increasing sequence $\Delta^1, \ldots, \Delta^r \subseteq X$ of sheets covering $C$. We will show by induction that

$$W^r := Y \cap \bigcup_{i=1}^{r} \Delta^i = \bigcup_{i=1}^{r} Y \cap \Delta^i$$

is $(n-1)$-connected. If $r = 1$ then $Y \cap \Delta^1 \cong H$ is a contractible space. Now assume $r > 1$. Since $\Delta^1, \ldots, \Delta^{r-1}$ is an increasing sequence we may assume that $W^{r-1} = Y \cap \bigcup_{i=1}^{r-1} \Delta^i$ is $(n-1)$-connected. We have

$$W^r = W^{r-1} \cup \left( Y \cap \left( \overline{\Delta^r \setminus \bigcup_{i=1}^{r-1} \Delta^i} \right) \right) = W^{r-1} \cup (Y \cap \Delta)$$

where $\Delta \subseteq \Delta^r$ is subordinate to $\Sigma_H^c$. Moreover $W^r$ is obtained from $W^{r-1}$ by gluing in $Y \cap \Delta \cong H \cap \pi(\Delta)$ along its boundary because

$$\partial_Y(Y \cap \Delta) = Y \cap \partial_{\Delta^r}(\Delta) = Y \cap (\Delta \cap \bigcup_{i=1}^{r-1} \Delta^i) = (Y \cap \Delta) \cap W^{r-1}.$$

If $W^r \neq W^{r-1}$ we have to distinguish two cases according to Theorem 6.4.

$H \cap \pi(\Delta)$ is retractable: Then $W^{r-1}$ is a deformation retract of $W^r$.

$H \cap \pi(\Delta)$ is decomposable: Here $W^r$ is obtained from $W^{r-1}$ by gluing in something of the form $V \times K$ along its boundary where $V$ is a real vector space and $K$ is a compact convex set with non-empty interior. We know by Corollary 6.7 that $K$ cannot be embedded in $\mathbb{R}^{n-1}$. Hence $V \times K$ is homotopy equivalent to $V \times \mathbb{B}^s$ with $s \geq n$. Using the same idea as in the proof of Theorem 6.4 you can show that $V \times \partial\mathbb{B}^s \cup \{0\} \times \mathbb{B}^s$ is a deformation retract of $V \times \mathbb{B}^s$. Hence up to homotopy $W^r$ is obtained from $W^{r-1}$ by gluing in $\mathbb{B}^s$ along its boundary. This is a cellular extension that does not change the homotopy groups in dimensions $< n$ since $n \leq s$.

Therefore $W^r$ is $(n-1)$-connected in both cases. **q.e.d.**

We can now complete the proof of Theorem 5.4. Combining Examples 7.1 and 7.2 we see, that $(X = \prod_{p \in S} X_p, \mathbb{R}^S, \lambda)$ is a $\{\xi_p \mid p \in S\}$-complex where

- $\xi_p : \mathbb{R}^S \to \mathbb{R}$ is the $p$-coordinate function.

Since $\xi_p|_{H_Q}(\varphi(x)) = p(x)$ for all $p \in S$ and all $x \in Q$ we can identify $\{\xi_p|_{H_Q} \mid p \in S\}$ and $\{p|_Q \mid p \in S\}$. Hence Lemma 7.3 yields

**Corollary 7.4.** *Suppose that each $p \in S$ restricts to a non-trivial homomorphism $p|_Q : Q \to \mathbb{R}$ and that the set $\{[p|_Q] \mid p \in S\} \subseteq \mathbb{S}(Q)$ is $n$-tame. Then the subcomplex $X_Q = \lambda^{-1}(H_Q) \subseteq X$ is $(n-1)$-connected where $H_Q$ is the subspace defined in Section 5.* **q.e.d.**

## 8 Application to $S$-arithmetic groups

We now turn to the second result that was announced in the introduction.

**Theorem 8.1.** *Let $r$ denote the rank of $Q$. Then the following holds:*

1. *The group $\mathcal{O}_S \rtimes Q$ is not of type $FP_{r+1}$.*

2. *If $Q$ has maximum rank $r = |S| - 1$ then the group $\mathcal{O}_S \rtimes Q$ is of type FP$_r$.*

*In particular $G(\mathcal{O}_S) = \mathcal{O}_S \rtimes \mathcal{O}_S^*$ has finiteness length $|S| - 1$.*

**Proof.** Since the kernel of the group homomorphism $\varphi|_{\mathcal{O}_S^*} : \mathcal{O}_S^* \to \mathbb{R}^S$ is finite as

$$\begin{aligned} \ker\varphi &= \{x \in \mathcal{O}_S^* \mid p(x) = 0 \quad \forall p \in S\} \\ &= \{x \in K \mid p(x) = 0 \quad \forall p \in \mathcal{P}\} \qquad \text{(product formula!)} \\ &= k^* \end{aligned}$$

we can conclude that $\varphi(Q)$ has rank $r$. Hence $H_Q$ has dimension $r$ since $\varphi(Q)$ is discrete.

Now assume for a moment that $\mathcal{O}_S \rtimes Q$ is of type FP$_{r+1}$. Then our Main Theorem implies that each $p \in S$ restricts to a non-trivial homomorphism $p|_Q : Q \to \mathbb{R}$ and that the set $\Sigma^c_{\mathcal{O}_S}(Q, \mathbb{Z}) = \{[p|_Q] \mid p \in S\}$ is $(r+1)$-tame. Since $\dim H_Q = r$, Lemma 6.2 implies that $\Sigma^c_{\mathcal{O}_S}(Q, \mathbb{Z})$ is fully tame. But this is false because the product formula

$$\sum_{p \in S} d_p p|_Q = 0$$

implies that the points in $\Sigma^c_{\mathcal{O}_S}(Q, \mathbb{Z})$ cannot be contained in an open hemisphere of $\mathbb{S}(Q)$. Hence $\mathcal{O}_S \rtimes Q$ is not of type FP$_{r+1}$.

To prove the second claim we define the hyperplane $H \le \mathbb{R}^S$ by the linear form

$$\sum_{p \in S} d_p \xi_p : (z_p)_{p \in S} \mapsto \sum_{p \in S} d_p z_p$$

and note that by the product formula $H_Q \le H$ for all subgroups $Q \le \mathcal{O}_S^*$. Dirichlet's Unit Theorem implies that $\dim H = \dim H_Q$ if $Q$ has maximum rank $r = |S| - 1$ whence $H = H_Q$ in this case. So we only have to show that the set $\Sigma^c_H := \{\xi_p|_H \mid p \in H\}$ is $(|S| - 1)$-tame. But this follows from the fact that any $|S| - 1 = \dim H$ of the functionals $\xi_p|_H$ are independent because $d_p \ne 0$ for all $p \in S$. **q.e.d.**

It is not the case that the rank of $Q$ determines the finiteness properties of $\mathcal{O}_S \rtimes Q$. In general there are subgroups $Q \le \mathcal{O}_S^*$ of positive rank such that $\mathcal{O}_S \rtimes Q$ is not even finitely generated. For example choose a strict subset $S' \subset S$ and a place $p \in S \setminus S'$. Then $p(\mathcal{O}^*_{S'}) = 0$ implies by Theorem 2.3 or by the Main Theorem that $\mathcal{O}_S \rtimes \mathcal{O}^*_{S'}$ is not finitely generated.

## References


[Abel87] H. Abels: Finite Presentability of $S$-Arithmetic Groups – Compact Presentability of Solvable Groups. Springer LNM 1261; Berlin, Heidelberg, New York (1987)

[Abel91] H. Abels: Finiteness properties of certain arithmetic groups in the function field case. Israel J. Math. 76 (1991) p. 113 – 128

[AbBr87] H. Abels, K.S. Brown: Finiteness properties of solvable $S$-arithmetic groups; an example. Journal of Pure and Applied Algebra 44 (1987) p. 77 – 83

[Åber86] H. Åberg: Bieri-Strebel Valuations (of Finite Rank). Proc. Lond. Math. Soc. (3) 52 (1986) p. 269 – 304

[Abra87] P. Abramenko: Endlichkeitseigenschaften der Gruppen $\mathrm{SL}_n(\mathbb{F}_q[t])$. Doctoral thesis, Frankfurt am Main (1987)

[Abra96] P. Abramenko: Twin Buildings and Applications to $S$-Arithmetic Groups. Springer LNM 1641; Berlin, Heidelberg, New York (1996)

[Behr92] H. Behr: Arithmetic groups over function fields; A complete characterization of finitely generated and finitely presented arithmetic subgroups of reductive algebraic groups. Preprint 92 – 033, SFB 343; Bielefeld (1992)

[BiGr82] R. Bieri, J.R.J. Groves: Metabelian Groups of Type $(FP)_\infty$ are Virtually of Type $(FP)$. Proc. London Math. Soc. (3) 45 (1982) p. 365 – 384

[BiGr84] R. Bieri, J.R.J. Groves: The geometry of the set of characters induced by valuations. Journal für die reine und angewandte Mathematik 347 (1984) p. 168 – 195

[BiSt80] R. Bieri, R. Strebel: Valuations and Finitely Presented Metabelian Groups. Proc. Lond. Math. Soc. (3) 41 (1980) p. 439 – 464

[BoSe74] A. Borel, J.-P. Serre: Corners and arithmetic groups. Comment. Math. Helv. 48 (1974) p. 244 – 297

[Brow87] K.S. Brown: Finiteness Properties of Groups. Journal of Pure and Applied Algebra 44 (1987) p. 45 – 75

[Eggl58] H.G. Eggleston: Convexity. Cambridge Tracts in Mathematics and Mathematical Physics, Cambridge (1958)

[Koch95] D.H. Kochloukova: The $FP_m$-Conjecture for a Class of Metabelian Groups. Journal of Algebra 184 (1996) p. 1175 – 1204

[O'Me73] O.T. O'Meara: Introduction to Quadratic Forms. Berlin, Heidelberg, New York (1973)

[Nosk93] G.A. Noskov: Bieri-Strebel Invariant and Homological Finiteness Properties of Metabelian Groups. Preprint 93 – 028, SFB 343; Bielefeld (1993), to appear in: Algebra i Logica

[Serr80] J.-P. Serre: Trees. Berlin, Heidelberg, New York (1980)

[Stuh80] U. Stuhler: Homological properties of certain arithmetic groups in the function field case. Inv. math. 57 (1980) p. 263 – 281

[Weil67] A. Weil: Basic Number Theory. Berlin, Heidelberg, New York (1967)