Let \( \{ C_i \} \) be a collection of abelian groups indexed by the non-negative integers and \( \partial_i : C_i \to C_{i-1} \) homomorphisms with \( \partial_i \circ \partial_{i+1} = 0 \). For convenience we assume that \( \partial_0 = 0 \). Then \( C = \{(C_i, \partial_i)\} \) is a \textit{chain complex}. For each \( i \) we define subgroups of \( C_i \) by 
\[
Z_i(C) = \ker \partial_i \quad \text{and} \quad B_i(C) = \operatorname{im} \partial_{i+1}.
\]
The \( Z_i \) is the subgroup of \textit{cycles} and \( B_i \) is the subgroup of \textit{boundaries}. An element of \( C_i \) is a \textit{chain}. The condition that \( \partial_i \circ \partial_{i+1} = 0 \) implies that \( B_i(C) \subset Z_i(C) \). We then define the homology groups by
\[
H_i(C) = Z_i(C)/B_i(C).
\]
Two cycles \( z_0, z_1 \in Z_i(C) \) are \textit{homologous} if they differ by a boundary; that is there exists a \( b \in B_i(C) \) such that \( b = z_0 - z_1 \) or, equivalently, there exists a chain \( c \in C_{i+1} \) such that \( \partial_{i+1} c = z_0 - z_1 \). If \( z \in Z_i(C) \) is a cycle then \([z]\) will represent the homology class in \( H_i(C) \).

1. Calculate \( H_2(C) \), \( H_1(C) \) and \( H_0(C) \) if
   - (a) \( \partial_1 \) is an isomorphism;
   - (b) \( \partial_1 \) is the zero map;
   - (c) \( \partial_1 \) is multiplication by \( n \).

A \textit{chain map} \( \phi : A \to C \) be chain complexes \( A \) and \( C \) is a collection of homomorphisms \( \phi_i : A_i \to C_i \) such that \( \phi_{i-1} \circ \partial_i = \partial_i \circ \phi_i \).

2. Show that \( \phi_i(Z_i(A)) \subset Z_i(C) \).

3. Show that if \( z_0, z_1 \in Z_i(A) \) are homologous then \( \phi_i(z_0) \) and \( \phi_i(z_1) \) are homologous.

4. Show that there is a well defined homomorphism \( (\phi_i)_* : H_i(A) \to H_i(C) \) given by 
   \[
   (\phi_i)_*([z]) = [\phi_i(z)].
   \]

Now let \( A_i \) be family of abelian groups and \( \phi_i : A_i \to A_{i-1} \) homomorphisms. This sequence is \textit{exact} if \( \operatorname{im} \phi_i = \ker \phi_{i+1} \). A sequence that is indexed by non-negative integers is typically called a \textit{long exact sequence}. A sequence of length five where the starting and
ending groups are trivial is a short exact sequence. If $A$, $B$ and $C$ are chain complexes and $\phi: A \to B$ and $\psi: B \to C$ are chain maps then

$$0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

is a short exact sequence of chain complexes if for each $i$ we have that

$$0 \to A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i \to 0$$

is a short exact sequence.

A fundamental result is that a short exact sequence of chain complexes determines a long exact sequence of homology groups. This is called the “snake lemma” and the proof follows from “diagram chasing”.

5. Show that $\text{im}(\phi_i)_* \subset \ker(\psi_i)_*$.

6. Show that if $\beta_i \in B_i$ is a cycle and $\psi_i(\beta_i) = 0$ then there exists a cycle $\alpha_i \in A_i$ with $\phi_i(\alpha_i) = \beta_i$. Conclude that $\text{im}(\phi_i)_* = \ker(\psi_i)_*$.

7. Given a cycle $\gamma_i \in C_i$ show that there exists a chain $\beta_i \in B_i$ with $\psi_i(\beta_i) = \gamma_i$ and a $\alpha_i \in A_{i-1}$ such that $\phi_{i-1}(\alpha_i) = \partial_i \beta_i$.

For the below problems assume that $\alpha_i \in A_{i-1}$, $\beta_i \in B_i$ and $\gamma_i \in C_i$ with $\phi_{i-1}(\alpha_i) = \partial_i \beta_i$ and $\psi_i(\beta_i) = \gamma_i$.

8. Show that if $\gamma_i = 0$ then $\alpha_i$ is a boundary. Conclude that if $\beta_0, \beta_1 \in B_i$ with $\psi_i(\beta_i) = \gamma_i$ and $\alpha_0, \alpha_1 \in A_{i-1}$ with $\phi_{i-1}(\alpha_i) = \partial_i \beta_i$ then $\alpha_0$ and $\alpha_1$ are homologous.

9. If $\gamma_i$ is a boundary show that $\beta_i$ can be chosen to be a boundary. Use this and (8) to show that if $\gamma_0, \gamma_1 \in C_i$ are homologous, $\beta_0, \beta_1 \in B_i$ with $\psi_i(\beta_i) = \gamma_i$ and $\alpha_0, \alpha_1 \in A_{i-1}$ with $\phi_{i-1}(\alpha_i) = \partial_i \beta_i$ then $\alpha_0$ and $\alpha_1$ are homologous.

10. Conclude that there is a well defined homomorphism $\delta_i: H_i(C) \to H_{i-1}(A)$ given by $\delta_i([\gamma]) = [\alpha]$.

11. If $\beta_i$ is a cycle show that $\alpha_i = 0$ and conclude that $\text{im}(\psi_i)_* \subset \ker \delta_i$.

12. If $\alpha_i$ is a boundary show that $\beta_i$ can be chosen to be a cycle. Conclude that $\ker \delta_i \subset \text{im}(\psi_i)_*$ and therefore $\ker \delta_i = \text{im}(\psi_i)_*$.

13. By the definition of $\alpha_i$ we have that $\psi_{i-1}(\alpha_i) = \partial_i \beta_i$ is a boundary. Conclude that $\text{im}(\delta_i) \subset \ker(\psi_{i-1})_*$. 

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14. Given a cycle $\alpha' \in A_{i-1}$ such that $\phi_{i-1}(\alpha')$ is a boundary show that there exists a $\beta' \in B_i$ and a cycle $\gamma' \in C_i$ with $\psi_i(\beta') = \gamma'$ and $\phi_{i-1}(\alpha') = \partial_i \beta'$. Conclude that $\ker(\psi_{i-1})_* \subset \im \delta_i$ and therefore $\ker(\psi_{i-1})_* = \im \delta_i$.

Congratulations! You have proved the snake lemma!

There are some important examples. Let $C$ be a chain complex. If $B_i \subset C_i$ are subgroups with $\partial_i(B_i) \subset B_{i-1}$ then $B = \{ (B_i, \partial_i) \}$ is a sub-chain complex. The quotient groups $C_i/B_i$ also form a chain complex:

15. Let $c_0, c_1 \in C_i$ be chains such that $c_1 - c_0 \in B_i$. Show that $\partial_i c_0 - \partial_i c_1 \in B_{i-1}$. Conclude that $\partial_i$ descends to a map $C_i/B_i \to C_{i-1}/B_{i-1}$.

16. Show that $0 \to B \to C \to C/B \to 0$

is a short exact sequence of chain complexes.

Another natural example comes from a chain complex $C$ and two subcomplexes $A, B \subset C$ such that for each $i$, $A_i$ and $B_i$ generate $C_i$. That is every element $c$ can be written as a sum $c = a + b$ where $a \in A_i$ and $b \in B_i$:

17. Let $D_i = A_i \cap B_i$ and show that $D = \{ (D_i, \partial_i) \}$ is a subcomplex of $C$.

18. Show that $A \bigoplus B = \{ (A_i \bigoplus B_i, \partial_i \oplus \partial_i) \}$ is a chain complex.

19. Let $\iota_A$ and $\iota_B$ be the inclusion maps of $D$ in $A$ and $B$, respectively. Show that these are chain maps and the map $D \to A \bigoplus B$ given by $d \mapsto (\iota_A(d), -\iota_B(d))$ is a chain map.

20. Let $j_A$ and $j_B$ be the inclusion maps of $A$ and $B$ into $C$. Show that the map $(a, b) \mapsto j_A(a) + j_B(b)$ is a chain map from $A \bigoplus B \to C$.

21. Show that $0 \to D \xrightarrow{(\iota_A, -\iota_B)} A \bigoplus B \xrightarrow{j_A + j_B} C \to 0$

is a short exact sequence of chain complexes.

Let $\Delta$ be an ordered set with $n+1$ objects. Secretly, $\Delta$ is an $n$-dimensional simplex. Let $S_i(\Delta)$ be the set of subsets of $\Delta$ that have $i + 1$ elements. Let $C_i$ be formal sums of ordered subsets of $i + 1$ elements.

There is a natural way to associate a chain complex to a simplicial structure on a topological space. Recall that the standard $n$-simplex is given by

$$\Delta^n = \left\{ x \in \mathbb{R}^{n+1} \mid \sum x_i = 1 \text{ and } x_i \geq 0 \right\}.$$
The intersection of each coordinate axes in $\mathbb{R}^{n+1}$ with $\Delta^n$ is a vertex. Label the vertices $v_0, \ldots, v_n$. Any subset of $k+1$ vertices will span a $k$-dimensional sub-simplex of $\Delta^n$. These are faces of $\Delta^n$. Note that $\Delta^n$ is a face of itself. Given two $n$-dimensional simplices with labeled vertices there is a canonical map between them that respects the labels.

2. Let $w_0, \ldots, w_n$ be $n$-vertices in $\mathbb{R}^n$ in general position; that is assume that they are not contained in an $n-1$-dimensional hyperplane. Let $[w_0, \ldots, w_n] = \{ x \in \mathbb{R}^n | x = \sum \lambda_i w_i \text{ with } \sum \lambda_i = 1 \}$ and show that $[w_0, \ldots, w_n]$ is homeomorphic to $\Delta^n$.

3. A homeomorphism $\phi: \mathbb{R}^n \to \mathbb{R}^n$ is affine if it is of the form $\phi(x) = Ax + B$ where $A: \mathbb{R}^n \to \mathbb{R}^n$ is linear and $B \in \mathbb{R}^n$. Let $\sigma: \{0, \ldots, n\} \to \{0, \ldots, n\}$ be a permutation. Show that there exist a unique affine map that takes $[w_0, \ldots, w_n]$ to $[w_{\sigma(0)}, \ldots, w_{\sigma(n)}]$. An affine map $\phi$ is orientation preserving if $\det A > 0$. If $\det A < 0$ then $\phi$ is orientation reversing. Show that $\phi$ is orientation preserving if and only if $\sigma$ is an even permutation.

4. Let $\phi$ be the affine map that takes $[w_0, \ldots, w_n]$ to itself (not necessarily preserving labels) and takes the $n-1$-dimensional face $[w_1, \ldots, w_n]$ to $[w_0, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n]$ (preserving labels). Show that $\phi$ is orientation preserving if and only if $(-1)^i = 1$.

This motivates the following definition. An orientation of a simplex is a choice of labeling of the vertices and two labelings are equivalent if the differ by an even permutation.

A simplicial complex is a topological space that is a union of finitely many simplices such that the intersection of any two simplices is a single face.

5. Describe $S^n$ as a simplicial complex.

6. Describe $\Sigma_g$, the closed surface of genus $g$, as a simplicial complex for $g = 1, 2$.

A simplicial complex is finite if it is the union of finitely many simplices.

7. Show that a finite simplicial complex is a subcomplex of $\Delta^n$ for some $n$. (Hint: If $X$ is a simplicial complex with $k$ vertices then it is a subcomplex of $\Delta^{k-1}$.)

A simplicial complex $X$ determines a chain complex as follows. Let $k_i$ be the number of simplices in dimension $i$. It is convenient to think of $\mathbb{Z}^{k_i}$ as formal sums $a_1 \Delta^i_1 + \cdots + a_k \Delta^i_k$. To define the boundary map $\partial_i$ we will first define $\partial_i(\Delta^j_i)$ and then extend the map linearly to arbitrary chains. Roughly speaking $\partial_i(\Delta^j_i)$ is a sum of the $i-1$-dimensional sub-simplices. However we need to be careful about the sign of the simplex.

The key point is orientation.
5. Label the vertices of $\Delta^n$, $v_0, \ldots, v_n$ and let $\sigma$ be a permutation of $\{0,1,\ldots,n\}$. Find the unique linear transformation $A: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $A(v_i) = v_{\sigma(i)}$. Write $A$ as matrix with the basis given by $v_0, \ldots, v_n$. Show that $\det A = 1$ is $\sigma$ is an even permutation and $\det A = -1$ if $\sigma$ is an odd permutation.

6. Let $\Delta_{i}^{n-1}$ be the $n-1$-dimensional sub-simplex obtained by removing the $i$th vertex of $\Delta^n$. Let $A_{ij}$ be the linear transformation of $\mathbb{R}^{n+1}$ to itself that takes $\Delta^n$ to itself and $\Delta_{i}^{n-1}$ to $\Delta_{j}^{n-1}$ preserving the order of the vertices in the two sub-simplices. Show that $\det A_{ij} =$

Let $B$ and $C$ be chain complexes. A chain map $\phi: A \to C$ is collection of group homomorphisms $\phi_i: A_i \to C_i$ such that $\partial \circ \phi = \phi \circ \partial$.

1. If $\phi: A \to C$ is a chain map show that $\phi_i(Z_i(A)) \subset Z_i(C)$.

It take some work to show that
A chain complex is exact if all the homology groups are trivial.