Let \( \{ C_i \} \) be a collection of abelian groups indexed by the non-negative integers and \( \partial_i : C_i \to C_{i-1} \) homomorphisms with \( \partial_i \circ \partial_{i+1} = 0 \). For convenience we assume that \( \partial_0 = 0 \). Then \( C = \{(C_i, \partial_i)\} \) is a chain complex. For each \( i \) we define subgroups of \( C_i \) by

\[
Z_i(C) = \ker \partial_i \quad \text{and} \quad B_i(C) = \im \partial_{i+1}.
\]

The \( Z_i \) is the subgroup of cycles and \( B_i \) is the subgroup of boundaries. An element of \( C_i \) is a chain. The condition that \( \partial_i \circ \partial_{i+1} = 0 \) implies that \( B_i(C) \subset Z_i(C) \). We then define the homology groups by

\[
H_i(C) = Z_i(C)/B_i(C).
\]

Two cycles \( z_0, z_1 \in Z_i(C) \) are homologous if they differ by a boundary; that is there exists a \( b \in B_i(C) \) such that \( b = z_0 - z_1 \) or, equivalently, there exists a chain \( c \in C_{i+1} \) such that \( \partial_{i+1}c = z_0 - z_1 \). If \( z \in Z_i(C) \) is a cycle then \([z]\) will represent the homology class in \( H_i(C) \).

1. Calculate \( H_2(C) \), \( H_1(C) \) and \( H_0(C) \) if

   (a) \( \partial_2 \) is an isomorphism;

   (b) \( \partial_2 \) is the zero map;

   (c) \( \partial_2 \) is multiplication by \( n \).

A chain map \( \phi: A \to C \) be chain complexes \( A \) and \( C \) is a collection of homomorphisms \( \phi_i: A_i \to C_i \) such that \( \phi_{i-1} \circ \partial_i = \partial_i \circ \phi_i \).

2. Show that \( \phi_i(Z_i(A)) \subset Z_i(C) \).

3. Show that if \( z_0, z_1 \in Z_i(A) \) are homologous then \( \phi_i(z_0) \) and \( \phi_i(z_1) \) are homologous.

4. Show that there is a well defined homomorphism \( (\phi_i)_*: H_i(A) \to H_i(C) \) given by \( (\phi_i)_*([z]) = [\phi_i(z)] \).

Now let \( A_i \) be family of abelian groups and \( \phi_i: A_i \to A_{i-1} \) homomorphisms. This sequence is exact if \( \im \phi_i = \ker \phi_{i+1} \). A sequence that is indexed by non-negatives integers is typically called a long exact sequence. A sequence of length five where the starting and
ending groups are trivial is a \textit{short exact sequence}. If \( A, B \) and \( C \) are chain complexes and \( \phi: A \to B \) and \( \psi: B \to C \) are chain maps then

\[
0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0
\]

is a short exact sequence of chain complexes if for each \( i \) we have that

\[
0 \to A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i \to 0
\]

is a short exact sequence.

A fundamental result is that a short exact sequence of chain complexes determines a long exact sequence of homology groups. This is called the “snake lemma” and the proof follows from “diagram chasing”.

5. Show that \( \text{im}(\phi_i)_* \subset \ker(\psi_i)_* \).

6. Show that if \( \beta \in B_i \) is a cycle and \( \psi_i(\beta) = 0 \) then there exists a cycle \( \alpha \in A_i \) with \( \phi_i(\alpha) = \beta \). Conclude that \( \text{im}(\phi_i)_* = \ker(\psi_i)_* \).

7. Given a cycle \( \gamma \in C_i \) show that there exists a chain \( \beta \in B_i \) with \( \psi_i(\beta) = \gamma \) and a \( \alpha \in A_{i-1} \) such that \( \phi_{i-1}(\alpha) = \partial_i \beta \).

For the below problems assume that \( \alpha \in A_{i-1}, \beta \in B_i \) and \( \gamma \in C_i \) with \( \phi_{i-1}(\alpha) = \partial_i \beta \) and \( \psi_i(\beta) = \gamma \).

8. Show that if \( \gamma = 0 \) then \( \alpha \) is a boundary. Conclude that if \( \beta_0, \beta_1 \in B_i \) with \( \psi_i(\beta_j) = \gamma \) and \( \alpha_0, \alpha_i \in A_{i-1} \) with \( \phi_{i-1}(\alpha_j) = \partial_i \beta_j \) then \( \alpha_0 \) and \( \alpha_1 \) are homologous.

9. If \( \gamma \) is a boundary show that \( \beta \) can be chosen to be a boundary. Use this and (8) to show that if \( \gamma_0, \gamma_1 \in C_i \) are homologous, \( \beta_0, \beta_1 \in B_i \) with \( \psi_i(\beta_j) = \gamma_j \) and \( \alpha_0, \alpha_1 \in A_{i-1} \) with \( \phi_{i-1}(\alpha_j) = \partial_i \beta_j \) then \( \alpha_0 \) and \( \alpha_1 \) are homologous.

10. Conclude that there is a well defined homomorphism \( \delta_i: H_i(C) \to H_{i-1}(A) \) given by \( \delta_i([\gamma]) = [\alpha] \).

11. If \( \beta \) is a cycle show that \( \alpha = 0 \) and conclude that \( \text{im}(\psi_i)_* \subset \ker \delta_i \).

12. If \( \alpha \) is a boundary show that \( \beta \) can be chosen to be a cycle. Conclude that \( \ker \delta_i \subset \text{im}(\psi_i)_* \) and therefore \( \ker \delta_i = \text{im}(\psi_i)_* \).

13. By the definition of \( \alpha \) we have that \( \psi_{i-1}(\alpha) = \partial_i \beta \) is a boundary. Conclude that \( \text{im} \delta_i \subset \ker(\psi_{i-1})_* \).
14. Given a cycle $\alpha' \in A_{i-1}$ such that $\phi_{i-1}(\alpha')$ is a boundary show that there exists a $\beta' \in B_i$ and a cycle $\gamma' \in C_i$ with $\psi_i(\beta') = \gamma'$ and $\phi_{i-1}(\alpha') = \partial_i \beta'$. Conclude that $\ker(\psi_{i-1})_* \subset \text{im} \delta_i$ and therefore $\ker(\psi_{i-1})_* = \text{im} \delta_i$.

Congratulations! You have proved the snake lemma!

There are some important examples. Let $C$ be a chain complex. If $B_i \subset C_i$ are subgroups with $\partial_i(B_i) \subset B_{i-1}$ then $B = \{ (B_i, \partial_i) \}$ is a sub-chain complex. The quotient groups $C_i/B_i$ also form a chain complex:

15. Let $c_0, c_1 \in C_i$ be chains such that $c_1 - c_0 \in B_i$. Show that $\partial_i c_0 - \partial_i c_1 \in B_{i-1}$.

Conclude that $\partial_i$ descends to a map $C_i/B_i \to C_{i-1}/B_{i-1}$.

16. Show that

$$0 \to B \to C \to C/B \to 0$$

is a short exact sequence of chain complexes.

Another natural example comes from a chain complex $C$ and two subcomplexes $A, B \subset C$ such that for each $i$, $A_i$ and $B_i$ generate $C_i$. That is every element $c$ can be written as a sum $c = a + b$ where $a \in A_i$ and $b \in B_i$.

17. Let $D_i = A_i \cap B_i$ and show that $D = \{ (D_i, \partial_i) \}$ is a subcomplex of $C$.

18. Show that $A \bigoplus B = \{ (A_i \bigoplus B_i, \partial_i \oplus \partial_i) \}$ is a chain complex.

19. Let $\iota_A$ and $\iota_B$ be the inclusion maps of $D$ in $A$ and $B$, respectively. Show that these are chain maps and the map $D \to A \bigoplus B$ given by $d \mapsto (\iota_A(d), -\iota_B(d))$ is a chain map.

20. Let $j_A$ and $j_B$ be the inclusion maps of $A$ and $B$ into $C$. Show that the map $(a, b) \mapsto j_A(a) + j_B(b)$ is a chain map from $A \bigoplus B \to C$.

21. Show that

$$0 \to D \xrightarrow{\iota_A - \iota_B} A \bigoplus B \xrightarrow{j_A + j_B} C \to 0$$

is a short exact sequence of chain complexes.

**Simplicial complexes**

Let $S$ be a set. Then $\mathbb{Z}(S)$ is the group of formal sums of $S$ with $\mathbb{Z}$-coefficients. That is an element of $n \in \mathbb{Z}(S)$ is an assignment to each $s \in S$ an integer $n_s$ such that at but finitely many of the coefficients in $n$ are zero. The group operation is then adding coefficients.

If $S$ is a finite set (as it will be for our examples) then the last condition automatically holds. However, there are many natural situations (often arising in topology) where $S$ can be an infinite set. One can also replace the group $\mathbb{Z}$ with an arbitrary group. Common examples are $\mathbb{R}$ or more generally an arbitrary field but we will stick to $\mathbb{Z}$.
1. Show that $\mathbb{Z}(S)$ is a group.

2. Let $\mathcal{R}$ be another set. Show that any map of $\mathbb{R}$ to $\mathbb{Z}(S)$ extends to a unique homomorphism from $\mathbb{Z}(\mathcal{R})$ to $\mathbb{Z}(S)$.

Now let $S$ be a finite ordered set with $n + 1$ elements. Let $S^{(k)}$ be the set of subsets of $S$ with $k + 1$ elements. Note that $S^{(k)}$ will have $\binom{n+1}{k+1}$ elements.

Let $\{v_0, \ldots, v_k\}$ be an element in $S^{(k)}$, where the indices indicate the order, and define $\partial_k: S^{(k)} \to \mathbb{Z}(S^{(k-1)})$ by

$$\partial_k \{v_0, \ldots, v_k\} = \sum_{i=0}^{k} (-1)^i \{v_0, \ldots, \hat{v_i}, \ldots, v_k\}.$$  

Here, the $\hat{v_i}$ indicates that $v_i$ has been removed from the set.

By (2) this extends to a homomorphism $\partial_k: \mathbb{Z}(S^{(k)}) \to \mathbb{Z}(S^{(k-1)})$.

3. Show that $\partial_{k-1} \circ \partial_k = 0$ so that $\{\mathbb{Z}(S^{(k)}), \partial_k\}$ is a chain complex.

Now let $X$ be a collection of subsets of $S$ with the property that if $A \in X$ and $B$ is a subset of $A$ then $B \in X$. Then $X$ is an abstract simplicial complex. We let $X^{(k)} = X \cap S^{(k)}$ be those subsets in $X$ that have $k + 1$ elements.

4. If $X$ is an abstract simplicial complex show that $\partial_k (\mathbb{Z}(X^{(k)})) \subset \mathbb{Z}(X^{(k-1)})$ and therefore $\{\mathbb{Z}(X^{(k)}), \partial_k\}$ is sub-chain complex of $\{\mathbb{Z}(S^{(k)}), \partial_k\}$.

A topological simplicial complex is a topological space $X$ that is a union of simplices such that the intersection of any two simplices is a single simplex.

5. Let $X$ be a topological simplicial complex. Let $X^{(0)}$ be the set of vertices of $X$ and give this set an order. Let $X^{(k)}$ be the subsets of $X^{(0)}$ with $k + 1$ elements that span a $k$-simplex in $X$. Show that $\cup X^{(k)}$ is an abstract simplicial complex.

6. Show that any abstract simplicial complex can be realized as a topological simplicial complex.