Let \( \{C_i\} \) be a collection of abelian groups indexed by the non-negative integers and \( \partial_i: C_i \rightarrow C_{i-1} \) homomorphisms with \( \partial_i \circ \partial_{i+1} = 0 \). For convenience we assume that \( \partial_0 = 0 \). Then \( C = \{(C_i, \partial_i)\} \) is a chain complex. For each \( i \) we define subgroups of \( C_i \) by

\[
Z_i(C) = \ker \partial_i \quad \text{and} \quad B_i(C) = \im \partial_{i+1}.
\]

The \( Z_i \) is the subgroup of cycles and \( B_i \) is the subgroup of boundaries. An element of \( C_i \) is a chain. The condition that \( \partial_i \circ \partial_{i+1} = 0 \) implies that \( B_i(C) \subset Z_i(C) \). We then define the homology groups by

\[
H_i(C) = Z_i(C) / B_i(C).
\]

Two cycles \( z_0, z_1 \in Z_i(C) \) are homologous if they differ by a boundary; that is there exists a \( b \in B_i(C) \) such that \( b = z_0 - z_1 \) or, equivalently, there exists a chain \( c \in C_{i+1} \) such that \( \partial_{i+1} c = z_0 - z_1 \). If \( z \in Z_i(C) \) is a cycle then \([z]\) will represent the homology class in \( H_i(C) \).

Let's begin with some examples. In what follows assume that \( C_i = 0 \) for \( i > 2 \) and \( i = 0 \) and \( C_1 = C_2 = \mathbb{Z} \). Then we must have \( \partial_i = 0 \) if \( i \neq 1 \).

1. Calculate \( H_2(C) \), \( H_1(C) \) and \( H_0(C) \) if
   (a) \( \partial_1 \) is an isomorphism;
   (b) \( \partial_1 \) is the zero map;
   (c) \( \partial_1 \) is multiplication by \( n \).

A chain map \( \phi: A \rightarrow C \) be chain complexes \( A \) and \( C \) is a collection of homomorphisms \( \phi_i: A_i \rightarrow C_i \) such that \( \phi_i \circ \partial_{i+1} = \partial_i \circ \phi_i \).

2. Show that \( \phi_i(Z_i(A)) \subset Z_i(C) \).

3. Show that if \( z_0, z_1 \in Z_i(A) \) are homologous then \( \phi_i(z_0) \) and \( \phi_i(z_1) \) are homologous.

4. Show that there is a well defined homomorphism \( (\phi_i)_*: H_i(A) \rightarrow H_i(C) \) given by \( (\phi_i)_*[z] = [\phi_i(z)] \).

Now let \( A_i \) be family of abelian groups and \( \phi_i: A_i \rightarrow A_{i-1} \) homomorphisms. This sequence is exact if \( \im \phi_i = \ker \phi_{i+1} \). A sequence that is indexed by non-negative integers is typically called a long exact sequence. A sequence of length five where the starting and
ending groups are trivial is a short exact sequence. If \( A, B \) and \( C \) are chain complexes and \( \phi: A \to B \) and \( \psi: B \to C \) are chain maps then

\[
0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0
\]

is a short exact sequence of chain complexes if for each \( i \) we have that

\[
0 \to A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i \to 0
\]

is a short exact sequence.

A fundamental result is that a short exact sequence of chain complexes determines a long exact sequence of homology groups. This is called the “snake lemma” and the proof follows from “diagram chasing”.

5. Show that \( \text{im}(\phi_i)_* \subset \ker(\psi_i)_* \).

6. Show that if \( \beta \in B_i \) is a cycle and \( \psi_i(\beta) = 0 \) then there exists a cycle \( \alpha \in A_i \) with \( \phi_i(\alpha) = \beta \). Conclude that \( \text{im}(\phi_i)_* = \ker(\psi_i)_* \).

7. Given a cycle \( \gamma \in C_i \) show that there exists a chain \( \beta \in B_i \) with \( \psi_i(\beta) = \gamma \) and a \( \alpha \in A_{i-1} \) such that \( \phi_{i-1}(\alpha) = \partial_i \beta \).

For the below problems assume that \( \alpha \in A_{i-1}, \beta \in B_i \) and \( \gamma \in C_i \) with \( \phi_{i-1}(\alpha) = \partial_i \beta \) and \( \psi_i(\beta) = \gamma \).

8. Show that if \( \gamma = 0 \) then \( \alpha \) is a boundary. Conclude that if \( \beta_0, \beta_1 \in B_i \) with \( \psi_i(\beta_j) = \gamma \) and \( \alpha_0, \alpha_1 \in A_{i-1} \) with \( \phi_{i-1}(\alpha_j) = \partial_i \beta_j \) then \( \alpha_0 \) and \( \alpha_1 \) are homologous.

9. If \( \gamma \) is a boundary show that \( \beta \) can be chosen to be a boundary. Use this and (8) to show that if \( \gamma_0, \gamma_1 \in C_i \) are homologous, \( \beta_0, \beta_1 \in B_i \) with \( \psi_i(\beta_j) = \gamma_j \) and \( \alpha_0, \alpha_1 \in A_{i-1} \) with \( \phi_{i-1}(\alpha_j) = \partial_i \beta_j \) then \( \alpha_0 \) and \( \alpha_1 \) are homologous.

10. Conclude that there is a well defined homomorphism \( \delta_i: H_i(C) \to H_{i-1}(A) \) given by \( \delta_i([\gamma]) = [\alpha] \).

11. If \( \beta \) is a cycle show that \( \alpha = 0 \) and conclude that \( \text{im}(\psi_i)_* \subset \ker \delta_i \).

12. If \( \alpha \) is a boundary show that \( \beta \) can be chosen to be a cycle. Conclude that \( \ker \delta_i \subset \text{im}(\psi_i)_* \) and therefore \( \ker \delta_i = \text{im}(\psi_i)_* \).

13. By the definition of \( \alpha \) we have that \( \psi_{i-1}(\alpha) = \partial_i \beta \) is a boundary. Conclude that \( \text{im} \delta_i \subset \ker(\psi_{i-1})_* \).
14. Given a cycle $\alpha' \in A_{i-1}$ such that $\phi_{i-1}(\alpha')$ is a boundary show that there exists a cycle $\beta' \in B_i$ and a cycle $\gamma' \in C_i$ with $\psi_i(\beta') = \gamma'$ and $\phi_{i-1}(\alpha') = \partial_i \beta'$. Conclude that $\ker(\psi_{i-1})_* \subset \im \delta_i$ and therefore $\ker(\psi_{i-1})_* = \im \delta_i$.

Congratulations! You have proved the snake lemma!
What does this have to do with topology? We’ll see this next....