Midterm 3, Math 3210
April 6, 2018
You must write in complete sentences and justify all of your work. All 4 problems will be equally weighted.

1. Show that the following functions are uniformly continuous on the given domain or prove that they are not:

   (a) $f(x) = x^2$ with domain $[0, 1]$.
   Solution: As $f(x)$ is continuous on a closed and bounded interval it is uniformly continuous by Theorem 0.6 in the notes.

   (b) $g(x) = 1/x$ with domain $(0, 1]$.
   Solution: $g(x)$ is not uniformly continuous. To see this let $\epsilon = 1/2$. Then for all $\delta > 0$ we can find a $x, y \in (0, 1]$ such that $|x-y| < \delta$ but $|g(x)-g(y)| > 1/2$. For example if we choose $x = 1/n \in (0, \delta)$ and $y = 1/(n + 1)$ then $|x - y| < \delta$ and $|g(x) - g(y)| = |n - (n + 1)| = 1 > 1/2$.

2. Let $f\colon (-1, 1) \to \mathbb{R}$ be a continuous (but not necessarily differentiable) function with $f(0) = 1$ and let $g(x) = xf(x)$. Show that $g'(0) = 1$. (In particular show that $g'(0)$ exists.)

   Solution: We have $g'(0) = \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{xf(x)}{x} = \lim_{x\to 0} f(x) = 1$.

3. Let $f(x) = \sqrt{x+1}$ and show that $f'(x) \leq \frac{1}{2}$ if $x \geq 0$. Use this and the Mean Value Theorem to show that $\sqrt{x+1} \leq 1 + \frac{x}{2}$ if $x \geq 0$.

   Solution: We have $f'(x) = \frac{1}{2\sqrt{x+1}} \leq 1/2$ since $\sqrt{x+1} \geq 1$ is $x \geq 0$.

   For the second part we observe that by the Mean Value Theorem for all $x > 0$ there exists a $x \in (0, x)$ with

   $$\frac{\sqrt{x+1} - \sqrt{0+1}}{x - 0} = \frac{1}{2\sqrt{c+1}} \leq 1/2.$$

   Simplifying the expression on the left and rearranging this becomes $\sqrt{x+1} \leq 1 + x/2$.

4. Let $f(x) = 1/x$ with domain $[1, 2]$.

   (a) Let $P = \{1 < 3/2 < 2\}$ be a partition and calculate the upper and lower sums $U(f, P)$ and $L(f, P)$.

Solution: We have

$$L(f,P) = (\inf_{[1,3/2]} f(x))(3/2-1) + (\inf_{[3/2,2]} f(x))(2-3/2) = \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{7}{12}$$

and

$$U(f,P) = (\sup_{[1,3/2]} f(x))(3/2-1) + (\sup_{[3/2,2]} f(x))(2-3/2) = 1\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{2} = \frac{5}{6}$$

(b) Use the previous problem to give upper and lower bounds for the integral

$$\int_1^2 f(x)dx.$$

Solution: We have $L(f,P) \leq \int_1^2 f(x)dx \leq U(f,P)$ for all partitions so the integral is bounded below by $7/12$ and above by $5/6$.