Name:

Midterm 2, Math 3210
February 28, 2018

You must write in complete sentences and justify all of your work. All 4 problems will be equally weighted.

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1. Let \( a_n = \frac{4n-1}{2n+1} \).
   
   (a) Fix \( \epsilon > 0 \) and show that if \( n > \frac{3}{2\epsilon} \) then \( |a_n - 2| < \epsilon \).

   (b) Directly using the definition of a limit show that \( \lim_{n \to \infty} a_n = 2 \).
2. Let \( \{a_n\} \) be a sequence. If \( b_n = na_n \) converges show that \( \lim_{n \to \infty} a_n = 0 \).
3. Let \( a_n = (-1)^n + 1/n. \)

(a) Find \( \limsup_{n \to \infty} a_n \) and \( \liminf_{n \to \infty} a_n. \)

(b) Is \( \{a_n\} \) a Cauchy sequence? Make sure to justify your answer.
4. Let $f: (0, \infty) \to \mathbb{R}$ be a function such that there exists an $M > 0$ such that $|f(x)| < M$ for all $x \in (0, \infty)$. Define $g: [0, \infty) \to \mathbb{R}$ by

$$g(x) = \begin{cases} 
xf(1/x) & \text{if } x > 0 \\
0 & \text{if } x = 0
\end{cases}$$

Show that $g$ is continuous at $x = 0$. 
Scratchwork
Notes

**Limits.** Let \( \{a_n\} \) be a sequence. Then

\[
\lim a_n = a
\]

if for all \( \epsilon > 0 \) there exists an \( N \) such that if \( n > N \) then \( |a_n - a| < \epsilon \). If no such \( a \) exists then the sequence is divergent. The sequence \( a_n \) is Cauchy if for all \( \epsilon > 0 \) there exists an \( N > 0 \) such that if \( n, m > N \) then \( |a_n - a_m| \leq \epsilon \).

**Theorem 0.1** A sequence is convergent if and only if it is Cauchy.

**Theorem 0.2** Every bounded sequence of real numbers has a convergent subsequence.

**Theorem 0.3** Suppose \( a_n \to a, b_n \to b \), \( c \) is a real number and \( k \) a natural number. Then

(a) \( ca_n \to ca \);
(b) \( a_n + b_n \to a + b \);
(c) \( a_n b_n \to ab \);
(d) \( a_n/b_n \to a/b \) if \( b \neq 0 \) and \( b_n \neq 0 \) for all \( n \);
(e) \( a_n^k \to a^k \);
(f) \( a_n^{1/k} \to a^{1/k} \) if \( a_n \geq 0 \) for all \( n \).

If \( A \) is a subset of \( \mathbb{R} \) the \( a = \text{sup} A \) if \( a \geq x \) for all \( x \in A \) and \( a' \geq x \) for all \( x \in A \) then \( x \leq y \). We define \( \text{inf} A \) be reversing the inequalities. If we allow \( +\infty \) and \( -\infty \) the \( \text{sup} A \) and \( \text{inf} A \) always exist.

Let \( \{a_n\} \) be a sequence and define \( i_n = \text{inf}\{a_k : k \geq n\} \) and \( s_n = \text{sup}\{a_k : k \geq n\} \). Then

\[
\lim \text{inf} a_n = \lim i_n
\]

and

\[
\lim \text{sup} a_n = \lim s_n.
\]

If \( x \neq 1 \) then

\[
\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.
\]

**Continuity.** Let \( f : D \to \mathbb{R} \) be a function defined on a domain \( D \subset \mathbb{R} \). Then

\[
\lim_{x \to a} f = b
\]

if for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if for all \( x \in D \) with \( 0 < |x - a| < \delta \) then \( |f(x) - b| < \epsilon \). The function \( f \) is continuous at \( a \) if

\[
\lim_{x \to a} f = f(a)
\]

There is a theorem similar to Theorem 0.3 for limits of functions.