Midterm 2, Math 3210
February 28, 2018
Solutions

1. Let $a_n = \frac{4n-1}{2n+1}$.

   (a) Fix $\epsilon > 0$ and show that if $n > \frac{3}{2\epsilon}$ then $|a_n - 2| < \epsilon$.
   If $n > \frac{3}{2\epsilon}$ then $\epsilon > \frac{3}{2n}$ and we have

$$\begin{aligned} |a_n - 2| &= \left| \frac{4n-1}{2n+1} - 2 \right| \\ &= \left| \frac{4n-1-2(2n+1)}{2n+1} \right| \\ &= \left| \frac{-3}{2n+1} \right| \\ &\leq \frac{3}{2n} \\ &< \epsilon. \end{aligned}$$

   (b) Directly using the definition of a limit show that $\lim_{n\to\infty} a_n = 2$.
   Fix $\epsilon > 0$ and let $N = \frac{3}{2\epsilon}$. By (a) if $n > N$ we have

$$|a_n - 2| < \epsilon$$

   and therefore $\lim_{n\to\infty} a_n = 2$.

2. Let $\{a_n\}$ be a sequence. If $b_n = na_n$ converges show that $\lim_{n\to\infty} a_n = 0$.
   Let $c_n = 1/n$. Then $\lim_{n\to\infty} c_n = 0$ and $a_n = b_n c_n$. By (c) of the Main Limit Theorem we have $\lim_{n\to\infty} a_n = 0 \cdot \lim_{n\to\infty} b_n = 0$.

3. Let $a_n = (-1)^n + 1/n$.

   (a) Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

   We have $s_n = \{3/2, 3/2, 5/4, 5/4, 7/6, 7/6, \ldots\}$ and $i_n = \{-1, -1, \ldots\}$ and therefore

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} s_n = 1$$

   and

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} i_n = -1.$$

(b) Is $\{a_n\}$ a Cauchy sequence? Make sure to justify your answer.

A sequence is Cauchy if and only if it converges. As $\limsup_{n\to\infty} a_n \neq \liminf_{n\to\infty} a_n$ the sequence doesn't converge and hence is not Cauchy.

More concretely for all $n \in \mathbb{N}$ we have

$$|a_{2n} - a_{2n_1}| = |(-1)^{2n} + \frac{1}{2n} - (-1)^{2n+1} - \frac{1}{2n+1}| = 2 + \frac{1}{2n(2n+1)} > 2$$

so for all $N > 0$ there exists $n, m > N$ such that

$$|a_n - a_m| > 2$$

and therefore $\{a_n\}$ is not Cauchy.

4. Let $f\colon (0,\infty) \to \mathbb{R}$ be a function such that there exists an $M > 0$ such that $|f(x)| < M$ for all $x \in (0,\infty)$. Define $g\colon [0,\infty) \to \mathbb{R}$ by

$$g(x) = \begin{cases} xf(1/x) & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Show that g is continuous at $x = 0$.

Fix $\epsilon > 0$ and let $\delta = \frac{\epsilon}{M}$. The if $x \in [0,\infty)$ and $|x - 0| < \delta$ we have

$$|g(x) - 0| = |xf(1/x)| = |x||f(1/x)| \leq \delta M = \epsilon$$

and therefore $g(x)$ is continuous at $x = 0$.