Midterm 1, Math 3210
February 2, 2018
Solutions

1. Use induction to prove that
$$2^n < 3^n$$
for all $n \in \mathbb{N}$. You can use all of the usual properties of addition, multiplication and order for the natural numbers (but not properties of exponents).

   Solution: Let P_n be the statement that $2^n < 3^n$. When $n = 1$, P_1 is true since $2 < 3$.

   Now assume that P_n is true. Therefore $2^n < 3^n$. Since $2 < 3$ we have $2 \cdot 2^n < 3 \cdot 3^n$. This is equivalent to $2^{n+1} < 3^{n+1}$ so P_{n+1} is true.

   Therefore by induction P_n is true for all $n \in \mathbb{N}$.

2. For the following you should assume that x, y and z are elements of a field F as defined in the book and notes.

   (a) Prove that if $xz = yz$ and $z \neq 0$ then $x = y$.

   (b) Prove that $xy = 0$ then either $x = 0$ or $y = 0$.

   In your proofs you can only use the properties of a field given in the notes along with the following two results we proved in class:

   (i) If $x + z = y + z$ then $z = y$.

   (ii) $x \cdot 0 = 0$.

   Solution (a): Since $z \neq 0$ there exists a $z^{-1} \in F$ with $z \cdot z^{-1}$ by M4. Multiplying both sides of the equation $xz = yz$ by z^{-1} we have $(xz)z^{-1} = (yz)z^{-1}$. By M2 this becomes $x(z \cdot z^{-1}) = y(z \cdot z^{-1})$ which simplifies to $x = y$.

   (b): Assume that $x \neq 0$. Then by M4 and M1 there exists an x^{-1} with $x \cdot x^{-1} = x^{-1} \cdot x = 1$. Multiplying on the left by x^{-1} we have $x^{-1}(xy) = x^{-1} \cdot 0 = 0$ where the last equality holds by (ii). Applying M2 we also have $x^{-1}(xy) = (x^{-1} \cdot x)y = 1 \cdot y = y$. Therefore $y = 0$.

   Since $xy = yx$ by M2, a similar argument shows that if $y \neq 0$ then $x = 0$ so if $xy = 0$ either $x = 0$ or $y = 0$.

3. If L is a Dedekind cut show that the set
$$K = \{x \in \mathbb{Q} | \exists y \in L \text{ with } x = y + 1\}$$

is a Dedekind cut.

Solution: Since L is a Dedekind cut by (a) there exists a $y \in L$ and therefore $y + 1 \in K$ and $K \neq \emptyset$.

Also by (a) there exists a $y' \notin L$. We claim that $y' + 1 \notin K$ for if it was there would be a $y'' \in L$ with $y' + 1 = y'' + 1$ which would imply that $y' = y''$, a contradiction. Therefore K satisfies (a).

Next we show that K has no largest element. If $x \in K$ there exists a $y \in L$ such that $x = y + 1$. By (b), L has no largest element and there exists a $y' \in L$ with $y < y'$. Then $y' + 1 \in K$ and $x = y + 1 < y' + 1$ so K satisfies (b).

Finally we show that if $x \in K$ and $x' < x$ then $x' \in K$. Since $x \in K$ there exists $y \in L$ with $x = y + 1$. Note that $x' = y' + 1$ for some $y' \in \mathbb{Q}$. Since $x' < x$ we have $x' - 1 = y' < y = x - 1$ and therefore by (c), $y' \in L$ and $x' = y' + 1 \in K$. This shows that K satisfies (c).

As K satisfies (a), (b) and (c) it is a Dedekind cut.