1. Use induction to prove that 
\[ 2^n < 3^n \]
for all \( n \in \mathbb{N} \). You can use all of the usual properties of addition, multiplication and order for the natural numbers (but not properties of exponents).

**Solution:** Let \( P_n \) be the statement that \( 2^n < 3^n \). When \( n = 1 \), \( P_1 \) is true since \( 2 < 3 \).

Now assume that \( P_n \) is true. Therefore \( 2^n < 3^n \). Since \( 2 < 3 \) we have \( 2 \cdot 2^n < 3 \cdot 3^n \).

This is equivalent to \( 2^{n+1} < 3^{n+1} \) so \( P_{n+1} \) is true.

Therefore by induction \( P_n \) is true for all \( n \in \mathbb{N} \).

2. For the following you should assume that \( x, y \) and \( z \) are elements of a field \( F \) as defined in the book and notes.

   (a) Prove that if \( xz = yz \) and \( z \neq 0 \) then \( x = y \).

   (b) Prove that \( xy = 0 \) then either \( x = 0 \) or \( y = 0 \).

In your proofs you can only use the properties of a field given in the notes along with the following two results we proved in class:

   (i) If \( x + z = y + z \) then \( z = y \).

   (ii) \( x \cdot 0 = 0 \).

**Solution (a):** Since \( z \neq 0 \) there exists a \( z^{-1} \in F \) with \( z \cdot z^{-1} \) by M4. Multiplying both sides of the equation \( xz = yz \) by \( z^{-1} \) we have \( (xz)z^{-1} = (yz)z^{-1} \). By M2 this becomes \( x(z \cdot z^{-1}) = y(z \cdot z^{-1}) \) which simplifies to \( x = y \).

**Solution (b):** Assume that \( x \neq 0 \). Then by M4 and M1 there exists an \( x^{-1} \) with \( x \cdot x^{-1} = x^{-1} \cdot x = 1 \). Multiplying on the left by \( x^{-1} \) we have \( x^{-1}(xy) = x^{-1} \cdot 0 = 0 \) where the last equality holds by (ii). Applying M2 we also have \( x^{-1}(xy) = (x^{-1} \cdot x)y = 1 \cdot y = y \). Therefore \( y = 0 \).

Since \( xy = yx \) by M2, a similar argument shows that if \( y \neq 0 \) then \( x = 0 \) so if \( xy = 0 \) either \( x = 0 \) or \( y = 0 \).

3. If \( L \) is a Dedekind cut show that the set 
\[ K = \{ x \in \mathbb{Q} \mid \exists y \in L \text{ with } x = y + 1 \} \]
is a Dedekind cut.

Solution: Since $L$ is a Dedekind cut by (a) there exists a $y \in L$ and therefore $y + 1 \in K$ and $K \neq \emptyset$.

Also by (a) there exists a $y' \notin L$. We claim that $y' + 1 \notin K$ for if it was there would be a $y'' \in L$ with $y' + 1 = y'' + 1$ which would imply that $y' = y''$, a contradiction. Therefore $K$ satisfies (a).

Next we show that $K$ has no largest element. If $x \in K$ there exists a $y \in L$ such that $x = y + 1$. By (b), $L$ has no largest element and there exists a $y' \in L$ with $y < y'$. Then $y' + 1 \in K$ and $x = y + 1 < y' + 1$ so $K$ satisfies (b).

Finally we show that if $x \in K$ and $x' < x$ then $x' \in K$. Since $x \in K$ there exists $y \in L$ with $x = y + 1$. Note that $x' = y' + 1$ for some $y' \in \mathbb{Q}$. Since $x' < x$ we have $x' - 1 = y' < y = x - 1$ and therefore by (c), $y' \in L$ and $x' = y' + 1 \in K$. This shows that $K$ satisfies (c).

As $K$ satisfies (a), (b) and (c) it is a Dedekind cut.