Homework 9 - Solutions, Math 3210
Section 4.1: 2, 11
Section 4.2: 1, 7, 11,12

4.1.2 Since $\lim_{x\to 2} x = 2$, by (d) of the Main Limit Theorem $\lim_{x\to 2} x^2 = \left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2} x\right) = 4$. By (c) of the Main Limit Theorem

$$\begin{aligned}\lim_{x\to 2} x^2 + x - 2 &= \lim_{x\to 2} x^2 + \lim_{x\to 2} x + \lim_{x\to 2}(-2) \\ &= 4 + 2 - 2 = 4\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 2} x - 1 &= \lim_{x\to 2} x + \lim_{x\to 2}(-1) \\ &= 2 - 1 = 1.\end{aligned}$$

By (e) of the Main Limit Theorem

$$\lim_{x\to 2}\frac{x^2+x-2}{x-1} = \frac{\lim_{x\to 2} x^2 + x - 2}{\lim_{x\to 2} x - 1} = \frac{4}{1} = 4.$$

4.1.11 Let $L = \lim_{x\to u} f(x)$. Let $\epsilon = L$ and choose δ such that if $|x-u| < \delta$ then $|f(x) - L| < \epsilon = L$. Then $-L < f(x) - L < L$ so $f(x) > 0$.

4.2.1 We calculate

$$\begin{aligned}f'(x) = \lim_{h\to 0}\frac{\frac{1}{x_h} - \frac{1}{x}}{h} &= \lim_{h\to 0}\frac{x-(x+h)}{x(x+h)h} \\ &= \lim_{h\to 0}\frac{-h}{x(x+h)h} \\ &= \lim_{h\to 0}\frac{-1}{x(x+h)} = \frac{-1}{x^2}.\end{aligned}$$

4.2.7 Let $f(x) = \sqrt{x}$ and $g(x) = x^2$. Then $g'(x) = 2x$ and by Theorem 4.2.9

$$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

4.2.11 We apply the definition of the derivative:

$$\lim_{x\to 0}\frac{f(x)-f(x)}{x-0} = \lim_{x\to 0}\frac{x\sin 1/x}{x} = \lim_{x\to 0}\sin 1/x.$$

This limit does not exist so f is not differentiable at 0. To see this we look at the sequences $a_n = \frac{1}{2\pi n}$ and $b_n = \frac{1}{\pi/2+2\pi n}$. Both sequences converge to 0 so by Theorem 4.2.9

$$\lim_{x\to 0} \sin 1/x = \lim_{n\to\infty} \sin 1/a_n = \lim_{n\to\infty} \sin 1/b_n$$

but $\lim_{n\to\infty} \sin 1/a_n = \lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \sin 1/b_n = \lim_{n\to\infty} 1 = 1$. This is a contradiction so the limit doesn't exist and f is not differentiable at 0.

For the second function we similarly see that if $f'(0)$ exists it is:

$$\lim_{x\to 0} \frac{f(x)-f(x)}{x-0} = \lim_{x\to 0} \frac{x^2 \sin 1/x}{x} = \lim_{x\to 0} x \sin 1/x.$$

By Problem 3.1.12 this limit exists and is 0 so $f'(0) = 0$ and f is differentiable at 0.

4.2.12 As before we want to calculate $\lim_{x\to 0} \frac{f(x)-f(x)}{x-0}$. However, as the function is defined differently for $x > 0$ then for $x < 0$ we need to find the two one-sided limits. In particular, we have $\lim_{x\to 0^+} \frac{f(x)-f(x)}{x-0} = \lim_{x\to 0^+} \frac{x^2-0}{x-0} = \lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^-} \frac{f(x)-f(x)}{x-0} = \lim_{x\to 0^-} \frac{0-0}{x-0} = \lim_{x\to 0^-} 0 = 0$. As the left and right limits are both equal to 0, by Theorem 4.1.7 we have $f'(0) = \lim_{x\to 0} \frac{f(x)-f(x)}{x-0} = 0$ and f is differentiable at 0.