Homework 8 - Solutions, Math 3210
Section 3.3: 4, 5, 6, 8
Section 3.4: 5, 12

3.3.4 Fix $\epsilon > 0$ and let $\delta = \epsilon$. If $x, y \in [0, \infty)$ and $|x - y| < \delta$ then

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{x}{x+1} - \frac{y}{y+1} \right| \\ &= \left| \frac{x(y+1) - y(x+1)}{(x+1)(y+1)} \right| \\ &\leq |x - y| = \delta = \epsilon. \end{aligned}$$

Therefore f is uniformly continuous on $[0, \infty)$.

3.3.5 Since $\sqrt{x}$ is continuous on $[0, 1]$ this follows from Theorem 3.3.4. This is a sufficient answer but we also prove it directly from the definition.

Fix $\epsilon > 0$ and let $\delta = \sqrt{\epsilon}$. Assume that $x, y \in [0, 1]$ and $x \geq y$. Note that $\sqrt{x} - \sqrt{y} \leq \sqrt{x - y}$ since

$$\begin{aligned} (\sqrt{x} - \sqrt{y})^2 &= x - 2\sqrt{xy} - y \\ &\leq x - y. \end{aligned}$$

Therefore if $|x - y| < \delta$ we have

$$\begin{aligned} |\sqrt{x} - \sqrt{y}| &= \sqrt{x} - \sqrt{y} \\ &\leq \sqrt{x - y} \\ &< \sqrt{\delta} = \epsilon \end{aligned}$$

and $\sqrt{x}$ is uniformly continuous on $[0, 1]$.

3.3.6 Fix $\epsilon > 0$. Since f is uniformly continuous on I and J there exists $\delta_0, \delta_1 > 0$ such that if $x, y \in I$ and $|x - y| < \delta_0$ then $|f(x) - f(y)| < \epsilon/2$ and if $x, y \in J$ and $|x - y| < \delta_1$ then $|f(x) - f(y)| < \epsilon/2$.

Let $\delta = \min\{\delta_0, \delta_1\}$ and assume that $x, y \in I \cup J$ with $|x - y| < \delta$. If $x, y \in I$ or $x, y \in J$ then by the above inequalities $|f(x) - f(y)| < \epsilon/2 < \epsilon$. Without loss of generality we can assume that $x \in I$ and $y \in J$. Let $z \in I \cap J$ (which is non-empty by assumption). If $z \leq x$ then $[z, y] \subset J$ so $x \in J$ and we are done. If $z \geq y$ then $[x, z] \subset I$ so $y \in I$ and again we are done. If neither of these hold we must have $x \leq z \leq y$ so $|x - z| \leq |x - y| < \delta$ and $|y - z| \leq |x - y| < \delta$. Therefore

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(z)| + |f(z) - f(y)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

since $x, z \in I$ and $y, z \in J$.

3.3.8 Fix $\epsilon > 0$ and let $\delta = \left(\frac{\epsilon}{K}\right)^{1/r}$. If $x, y \in I$ and $|x - y| < \delta$ then

$$\begin{aligned} |f(x) - f(y)| &\leq K|x-y|^r \\ &\leq K\delta^r \\ &\leq K\left(\frac{\epsilon}{K}\right) = \epsilon \end{aligned}$$

so f is uniformly continuous.

3.4.5 Let $f_n(x) = x^n(1-x)$. Since $f'(x) = nx^{n-1} - (n+1)x^n$ we have $f_n'(x) = 0$ if $x = 0$ or $x = \frac{n}{n+1}$. Therefore the max of f_n on $[0,1]$ will occur at $x = 0, 1$ or $\frac{n}{n+1}$. Since $f_n(\frac{n}{n+1}) = \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right) 0 = f_n(0) = f_n(1)$ the maximum occurs at $x = \frac{n}{n+1}$ and $|f_n(x) - 0| = |f_n(x)| \leq \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right) \leq \frac{1}{n+1}$.

Fix $\epsilon > 0$ and choose N such that $N + 1 > \frac{1}{\epsilon}$. Then for all $x \in [0,1]$ if $n > N$ we have that

$$|f_n(x) - 0| \leq \frac{1}{n+1} < \frac{1}{N+1} < \epsilon$$

so $f_n \to 0$ uniformly on $[0,1]$.

3.4.12 The functions s_n are continuous on $[-r, r]$ so if they converge uniformly the limit is continuous. If this holds for all $0 < r < 1$ then the limit will be continuous on $(-1,1)$ since $x \in (-1,1)$ then $x \in [-r,r]$ for some $r \in (0,1)$.

We now show that s_n converges uniformly on $[-r,r]$. Fixe $\epsilon > 0$ and chose N such that $r^N < \frac{(1-r)\epsilon}{Mr}$ where $M \geq |a_k|$ for all k. If $n \geq m > N$ we have

$$\begin{aligned} |s_n(x) - s_m(x)| &= \left|\sum_{k=m+1}^{n} a_k x^k\right| \\ &\leq \sum_{k=n+1}^{\infty} \left|a_k x^k\right| \\ &\leq Mr^{m+1}\sum_{k=0}^{\infty} r^k = \frac{Mr^{m+1}}{1-r} \\ &\leq \frac{Mr^{N+1}}{1-r} < \epsilon. \end{aligned}$$

Therefore s_n is uniformly Cauchy, hence uniformly convergent on $[-r, r]$.