Homework 8 - Solutions, Math 3210
Section 3.3: 4, 5, 6, 8
Section 3.4: 5, 12

3.3.4 Fix $\epsilon > 0$ and let $\delta = \epsilon$. If $x, y \in [0, \infty)$ and $|x - y| < \delta$ then

$$|f(x) - f(y)| = \left| \frac{x}{x+1} - \frac{y}{y+1} \right|$$

$$= \left| \frac{x(y+1) - y(x+1)}{(x+1)(y+1)} \right|$$

$$\leq |x - y| = \delta = \epsilon.$$ 

Therefore $f$ is uniformly continuous on $[0, \infty)$.

3.3.5 Since $\sqrt{x}$ is continuous on $[0, 1]$ this follows from Theorem 3.3.4. This is a sufficient answer but we also prove it directly from the definition.

Fix $\epsilon > 0$ and let $\delta = \sqrt{\epsilon}$. Assume that $x, y \in [0, 1]$ and $x \geq y$. Note that $\sqrt{x} - \sqrt{y} \leq \sqrt{x - y}$ since

$$(\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{xy} - y \leq x - y.$$ 

Therefore if $|x - y| < \delta$ we have

$$|\sqrt{x} - \sqrt{y}| = \sqrt{x} - \sqrt{y} \leq \sqrt{x - y} < \sqrt{\delta} = \epsilon$$

and $\sqrt{x}$ is uniformly continuous on $[0, 1]$.

3.3.6 Fix $\epsilon > 0$. Since $f$ is uniformly continuous on $I$ and $J$ there exists $\delta_0, \delta_1 > 0$ such that if $x, y \in I$ and $|x - y| < \delta_0$ then $|f(x) - f(y)| < \epsilon / 2$ and if $x, y \in J$ and $|x - y| < \delta_1$ then $|f(x) - f(y)| < \epsilon / 2$.

Let $\delta = \min\{\delta_0, \delta_1\}$ and assume that $x, y \in I \cup J$ with $|x - y| < \delta$. If $x, y \in I$ or $x, y \in J$ then by the above inequalities $|f(x) - f(y)| < \epsilon / 2 < \epsilon$. Without loss of generality we can assume that $x \in I$ and $y \in J$. Let $z \in I \cap J$ (which is non-empty by assumption). If $z \leq x$ then $[z, y] \subset J$ so $x \in J$ and we are done. If $z \geq y$ then $[x, z] \subset I$ so $y \in I$ and again we are done. If neither of these hold we must have $x \leq z \leq y$ so $|x - z| \leq |x - y| < \delta$ and $|y - z| \leq |x - y| < \delta$. Therefore

$$|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| < \epsilon / 2 + \epsilon / 2 = \epsilon.$$ 

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since \( x, z \in I \) and \( y, z \in J \).

3.3.8 Fix \( \epsilon > 0 \) and let \( \delta = \left( \frac{\epsilon}{K} \right)^{1/r} \). If \( x, y \in I \) and \( |x - y| < \delta \) then

\[
|f(x) - f(y)| \leq K|x - y|^r \\
\leq K\delta^r \\
\leq K \left( \frac{\epsilon}{K} \right) = \epsilon
\]

so \( f \) is uniformly continuous.

3.4.5 Let \( f_n(x) = x^n(1 - x) \). Since \( f_n'(x) = nx^{n-1} - (n + 1)x^n \) we have \( f_n'(x) = 0 \) if \( x = 0 \) or \( x = \frac{n}{n+1} \). Therefore the max of \( f_n \) on \([0, 1]\) will occur at \( x = 0, 1 \) or \( \frac{n}{n+1} \).

Since \( f_n \left( \frac{n}{n+1} \right) = \left( \frac{n}{n+1} \right)^n \left( \frac{1}{n+1} \right) 0 = f_n(0) = f_n(1) \) the maximum occurs at \( x = \frac{n}{n+1} \) and \( |f_n(x) - 0| = |f_n(x)| \leq \left( \frac{n}{n+1} \right)^n \left( \frac{1}{n+1} \right) \leq \frac{1}{n+1} \).

Fix \( \epsilon > 0 \) and choose \( N \) such that \( N + 1 > \frac{1}{\epsilon} \). Then for all \( x \in [0, 1] \) if \( n > N \) we have that

\[
|f_n(x) - 0| \leq \frac{1}{n+1} < \frac{1}{N+1} < \epsilon
\]

so \( f_n \to 0 \) uniformly on \([0, 1]\).

3.4.12 The functions \( s_n \) are continuous on \([-r, r] \) so if they converge uniformly the limit is continuous. If this holds for all \( 0 < r < 1 \) then the limit will be continuous on \((-1, 1) \) since \( x \in (-1, 1) \) then \( x \in [-r, r] \) for some \( r \in (0, 1) \).

We now show that \( s_n \) converges uniformly on \([-r, r] \). Fix \( \epsilon > 0 \) and chose \( N \) such that \( r^N < \frac{(1-r)\epsilon}{M} \) where \( M \geq |a_k| \) for all \( k \). If \( n \geq m > N \) we have

\[
|s_n(x) - s_m(x)| = \left| \sum_{k=m+1}^{n} a_k x^k \right| \\
\leq \sum_{k=m+1}^{\infty} |a_k x^k| \\
\leq Mr^{m+1} \sum_{k=0}^{\infty} r^k = \frac{Mr^{m+1}}{1-r} \\
\leq \frac{Mr^{N+1}}{1-r} < \epsilon.
\]

Therefore \( s_n \) is uniformly Cauchy, hence uniformly convergent on \([-r, r] \).