Homework 7 - Solutions, Math 3210
Section 3.1: 4, 10, 11, 12
Section 3.2: 3, 8, 10

3.1.4 Fix $\epsilon > 0$ and let $\delta = \epsilon$. By the triangle inequality $|x - a| + |a| \geq |x|$ so $|x - a| \geq |x| - |a|$ and similarly $|x - a| = |a - x| \geq |a| - |x|$. Together this implies that $|x - a| \geq ||x| - |a||$.

Therefore if $|x - a| < \delta = \epsilon$ we have $|f(x) - f(a)| = ||x| - |a|| < \epsilon$ and f is continuous at a.

3.1.10 Since f is continuous at a, given $\epsilon/2 > 0$ we can find $\delta > 0$ such that if $x \in D$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon/2$. In particular if $x, y \in D \cap (a - \delta, a + \delta)$ then $x, y \in D$, $|x - a| < \delta$ and $|y - a| < \delta$ so $|f(x) - f(a)| < \epsilon/2$ and $|f(y) - f(a)| < \epsilon/2$. Therefore

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(a)| + |f(y) - f(a)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

as required.

3.1.11 Let $\epsilon = 1/2$. Then for all $\delta > 0$ there exists an n such that $\frac{2}{(2n+1)\pi} < \delta$ so $\left|\frac{2}{(2n+1)\pi} - 0\right| < \delta$ but $\left|f\left(\frac{2}{(2n+1)\pi}\right) - f(0)\right| = 1 > 1/2$. Therefore there does not exist a $\delta > 0$ such that $|x - 0| < \delta$ then $|f(x) - f(0)| < 1/2$ and f is not continuous at 0.

3.1.12 Fix $\epsilon > 0$ and let $\delta = \epsilon$. Then if $|x - 0| < \delta$ we have that

$$\begin{aligned} |f(x) - f(0)| &= |x \sin 1/x| \\ &\leq |x| < \delta = \epsilon \end{aligned}$$

and therefore f is continuous at 0.

3.2.3 Let $d(x)$ be the distance between (x_0, y_0) and $(x, f(x))$. Then

$$d(x) = \sqrt{(x - x_0)^2 + (f(x) - y_0)^2}.$$

As $d(x)$ is formed by composing and taking sums of continuous functions, $d(x)$ is continuous on the closed bounded interval $[a, b]$. Therefore there exists a $c \in [a, b]$ such that $d(x) \leq d(x)$ for all $x \in [a, b]$ by Theorem 3.2.1. In particular, $(c, f(c))$ is the closest point on the graph of f to (x_0, y_0).

3.2.8 Let $h(x) = f(x) - g(x)$. The h is continuous on $[a, b]$ since it is the difference of continuous functions. Also $h(a) = f(a) - g(a) < 0$ and $h(b) = f(b) - g(b) > 0$ so $h(a) < 0 < h(b)$ and 0 lies between $h(a)$ and $h(b)$. By the Intermediate Value Theorem there exists a $x \in [a, b]$ such that $h(c) = 0$. Note that $c \neq a$ and $c \neq b$. so $c \in (a, b)$. Finally $h(c) = 0$ if and only if $f(c) = g(c)$ so c is the desired value.

3.2.10 Let $f(x) = x^n$. For all $a > 0$ there exists a $b > 0$ such that $b^n > a$ Since f is continuous on $[0, b]$ and a is between $f(0) = 0$ and $f(b) = b^n$ by the Intermediate Value Theorem there exists $c \in [a, b]$ such that $f(c) = c^n = a$. The $c > 0$ is an nth root of a.