Homework 4, Math 3210
February 6, 2018
Section 2.1: 2,5,6,8,10

2.1.2 If $|x-1| < 1/2$ and $|x-2| < 1/2$ then

$$\begin{aligned} 1 = 1/2 + 1/2 &> |x-1| + |x-2| \\ &= |x-1| + |x-2| \\ &\geq |x-1+2-1| \\ &= 1. \end{aligned}$$

Since $1 > 1$ is a contradiction we can't have that $|x-1| < 1/2$ and $|x-2| < 1/2$.

2.1.5 We claim that

$$\lim_{n\to\infty} \frac{2n-1}{3n+1} = \frac{2}{3}.$$

Fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ such $N > \frac{5}{9\epsilon}$. Then if $n \geq N$ we have

$$\left|\frac{2n-1}{3n+1} - \frac{2}{3}\right| = \frac{5}{3(3n+1)} < \frac{5}{9n} < \epsilon$$

since if $n > N > \frac{5}{9\epsilon}$ then $\frac{5}{9n} < \epsilon$. Therefore

$$\frac{2n-1}{3n+1} \to \frac{2}{3}.$$

2.1.6 We claim that $\lim\limits_{n\to\infty} \frac{(-1)^n}{n} = 0$. Fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Then if $n \geq N$ we have

$$\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} < \epsilon$$

since if $n > N > \frac{1}{\epsilon}$ then $\frac{1}{n} < \epsilon$. Therefore $\frac{(-1)^n}{n} \to 0$.

2.1.8 We claim that $\lim\limits_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$. Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $N > \frac{1}{4\epsilon^2}$. Note that if $n > N$ then $n + 1 > \frac{1}{4\epsilon^2}$ which implies that $\epsilon > \frac{1}{2\sqrt{n+1}}$. Therefore

$$\begin{aligned} \left|\sqrt{n+1} - \sqrt{n} - 0\right| &= \left|\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}\right| \\ &= \left|\frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}\right| \leq \frac{1}{2\sqrt{n+1}} \\ &< \epsilon \end{aligned}$$

and $\sqrt{n+1} - \sqrt{n} \to 0$.

2.1.10 We first show that $2^n \geq n$ by induction. When $n = 1$ we have that $2^1 = 2 \geq 1$. Now we assume that $2^n \geq n$ and show that $2^{n+1} \geq n+1$. For this we see that

$$\begin{aligned} 2^{n+1} &= 2^n + 2^n \\ &\geq n + n \\ &\geq n + 1. \end{aligned}$$

We have proved the induction step and it follows that $2^n \geq n$ for all n.

Now fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Then if $n > N$ we have

$$|2^{-n} - 0| = 1/2^n \leq 1/n \leq 1/N < \epsilon$$

and $2^{-n} \to 0$.