Homework 3, Math 3210
January 30, 2018
Section 1.4: 4, 7, 10, 11
Section 1.5: 3, 4, 12

1.4.4 We have that $x^2 < 1 - x$ if and only if $x^2 - 1 + x < 0$. By the quadratic formula

$$x^2 + x - 1 = \left(x - \frac{1-\sqrt{5}}{2}\right)\left(x - \frac{-1+\sqrt{5}}{2}\right).$$

This is negative when one term is negative and the other is positive. This occurs exactly when $x \in \left(\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$ so

$$A = \left(\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$$

and the least upper bound is $\frac{-1+\sqrt{5}}{2}$.

1.4.7 By Example 1.4.9 we can find an $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$. Let $k \in \mathbb{N}$ be the largest natural number such that $\frac{k}{n} \leq x$. Then $x < \frac{k+1}{n}$. Since $\frac{1}{n} < y - x$ adding this inequality to $\frac{k}{n} \leq x$ we have $\frac{k+1}{n} < y$.

In the next two problems (a), (b) and (c) are the properties of a Dedekind cut given in Definition 1.4.1 in the book.

1.4.10 Since $L_x, L_y \neq \varnothing$ we have $r \in L_x$ and $s \in L_y$ by (a). Therefore $r + s \in L_x + L_y$ and $L_x + L_y \neq \varnothing$.

Also by (a) we have an $a \notin L_x$ and a $b \notin L_y$. If $x \in L_x + L_y$ then $z = r + s$ for some $r \in L_x$ and $s \in L_y$. We have that $r < a$ since otherwise by (c) we would have that $a \in L_x$. Similarly $s < b$. Therefore $z = r + s < a + b$ and since z was an arbitrary element of $L_x + L_y$ we have that $a + b \notin L_x + L_y$ and the set is not all of $\mathbb{Q}$.

We next show that $L_x + L_y$ has no largest element. If $z \in L_x + L_y$ then there exists $r \in L_x$ and $s \in L_y$ with $z = r + s$. By (b), L_x has not largest element so there exists $r' \in L_x$ with $r < r'$. Then $z = r + s < r' + s$ and $r' + s \in L_x + L_y$ so there is no largest element in $L_x + L_y$.

Finally we show that if $z \in L_x + L_y$ and $z' < z$ then $z' \in L_x + L_y$. As above $z = r + s$ with $r \in L_x$ and $s \in L_y$. Let $s' = z' - r < z - r = s$. Then $s' \in L_y$ so $z' = r + s' \in L_x + L_y$.

We have show that $L_x + L_y$ satisfies (a), (b) and (c) and is therefore a Dedekind cut.

1.4.11 Since K contains all negative rationals $K \neq \varnothing$

If $a \notin L_x$ and $b \notin L_y$ then they are both positive since L_x and L_y represent positive numbers and therefore contain all non-positive rational numbers. Therefore ab is positive.

If $ab \in K$ then there exits non-negative $r \in L_x$ and $s \in L_y$ such that $rs = ab$. But as in the previous problem $x < a$ and $y < b$ so $xy < ab$. Therefore $ab \notin K$ and $K \neq \mathbb{Q}$.

To see that K has no largest element we note if $z \in K$ with $z > 0$ then there exists $r \in L_x$ and $s \in L_y$ such that $z = rs$ with both r and s positive. Applying (b) to L_x we see that there is an $r' \in L_x$ with $r < r'$. Therefore $rs < r's$ and $r's \in K$.

If $z \leq 0$ then it is also not the largest element since K contains positive numbers. Together this implies that K has no largest element.

Finally we show that if $z \in K$ and $z' < z$ is rational then $z' \in K$. If $z' \leq 0$ then $z' \in K$ since K contains all non-positive rational numbers. If $z' > 0$ then we also have $z > 0$ so there exists $r \in L_x$ and $s \in L_y$ with $z = rs$. Let $s' = z'/r < z/r = s$. Then by (c) we have that $s' \in L_y$ and therefore $z' = rs' \in K$.

We have checked three properties so K is a Dedekind cut.

1.5.3 Since $\sup A - \frac{1}{n} < \sup A$ there exists an $a_n \in A$ with $\sup A - \frac{1}{n} < a_n$. But since $a_n \in A$ we also have $a_n \leq \sup A$.

1.5.4 Since $\sup A = \infty$, the set A does not have an upper bound and for every $n \in \mathbb{N}$ there exists $a_n \in A$ with $n < a_n$.

1.5.12 We first claim that $(f+g)(A) \subset f(A) + g(A)$. Let $y \in (f+g)(A)$. Then there exists an $x \in A$ such that $y = (f+g)(x) = f(x) + g(x)$. But then $f(x) \in f(A)$ and $g(x) \in g(A)$ so $y = f(x) + g(x) \in f(A) + g(A)$.

By Theorem 1.5.7 (e), if $E \subset F$ are non-empty subsets of $\mathbb{R}$ then $\sup E \leq \sup F$. Applying this to $(f+g)(A) \subset f(A) + g(A)$ we have

$$\begin{aligned} \sup_A (f+g) &= \sup (f+g)(A) \\ &\leq \sup f(A) + g(A) \\ &= \sup f(A) + \sup g(A) \quad \text{(by (c) of Theorem 1.5.7)} \\ &= \sup_A f + \sup_A g. \end{aligned}$$

To prove $\inf_A f + \inf_A g \leq \inf_A (f+g)$ we the above inequality to $-f$ and $-g$ to get

$$\sup_A (-f-g) \leq \sup_A -f + \sup_A -g$$

and then apply (b) of Theorem1.5.10 to get the desired statement.