Homework 2, Math 3210
January 23, 2018
Section 1.3: 8, 9, 10, 12, 13, 14

1.3.8 By (O5) $0 \cdot y \leq x \cdot y$. Since $0 \cdot y = 0$ we have $0 \leq x \cdot y$. If $x \cdot y = 0$ then by Problem 7 $x = 0$ or $y = 0$. But by assumption $x > 0$ and $y > 0$ so $xy \neq 0$ and $xy > 0$.

1.3.9 By (b) of Example 1.3.8 $(x^{-1})^2 \geq 0$. By Problem 7 $x^{-1} \neq 0$ since $x \cdot x^{-1} = 1 \neq 0$. But (7) then also implies that $x^{-1} \cdot x^{-1} = (x^{-1})^2 \neq 0$ so $(x^{-1})^2 > 0$. Since $x > 0$ by (O5) we have $x(x^{-1})^2 > x \cdot 0 = 0$ and therefore $x^{-1} = (x \cdot x^{-1}) \cdot x^{-1} > 0$.

1.3.10 By Problem 9, $x^{-1} > 0$ and $y^{-1} > 0$ so by Problem 8 $x^{-1}y^{-1} > 0$. If $x < y$ by (O5) we have $x \cdot x^{-1}y^{-1} < y \cdot x^{-1}y^{-1}$. Using the commutativity of multiplication this becomes $y^{-1} < x^{-1}$.

1.3.12 Let $x \in \mathbb{Q}$ be a solution to the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Then $x = \frac{p}{q}$ where p and q are relatively prime integers and $q > 0$. After substituting $\frac{p}{q}$ into the equation and rearranging we have

$$\frac{p^n}{q^n} = -\left(a_{n-1}\frac{p^{n-1}}{q^{n-1}} + a_{n-2}\frac{p^{n-2}}{q^{n-2}} + \cdots + a_1\frac{p}{q} + a_0\right).$$

We next multiply both sides by q^n:

$$\begin{aligned} p^n &= -q^n\left(a_{n-1}\frac{p^{n-1}}{q^{n-1}} + a_{n-2}\frac{p^{n-2}}{q^{n-2}} + \cdots + a_1\frac{p}{q} + a_0\right) \\ &= -q\left(a_{n-1}p^{n-1} + a_{n-2}qp^{n-2} + \cdots + a_1 q^{n-1}p + a_0 q^n\right). \end{aligned}$$

Note that the term in parentheses on the right is an integer so $q \mid p^n$. In particular if $q' \neq 1$ is a prime factor of q then q' is a factor of p^n. We claim that this implies that q' is a factor of p. We use induction. This is true when $n = 1$. If we assume it is true for $n-1$ then for n we have that either q' divides p or p^{n-1} since $p^n = p \cdot p^{n-1}$. If it is the former we are done. If it is the later the $q' \mid p$ by the induction step.

However, this contradicts our assumption that p and q are relatively prime. Therefore we must have $q' = 1$ and $q = 1$. Hence, x is an integer.

1.3.13 Let $S = \{r \in \mathbb{N} \mid r = am + bm \text{ with } a, b \in \mathbb{Z}\}$. Let $am+bn$ be the smallest element of S (which exists by Problem 1.2.19). By the division algorithm (Problem 1.2.20) there exists $q, r \in \mathbb{N}$ with $0 \leq r < am + bn$ such that $m = (am + bn)q + r$. Rearranging we have

$$r = (1 - aq)m + (-bq)n.$$

If $r \neq 0$ then $r \in S$ and $r < am + bn$ which contradicts the choice of $am + bn$. Therefore $r = 0$ and $am + bn \mid m$. Similarly $am + bn \mid n$. Since m and n are relatively prime this implies that $am + bn = 1$.

1.3.14 Assume $p \nmid n$. By Problem 13 there exists $a, b \in \mathbb{Z}$ such that $ap + bn = 1$. Multiplying both sides by m we have $apm + bnm = m$. Since $p \mid nm$, on the left side of this equation both sides are divisible by p and therefore there sum is and $p \mid m$ as desired.