6.3.11 The series $\sum_{k=0}^{\infty} 2^{-k}$ converges absolutely so by the product formula

$$4 = \left( \sum_{k=0}^{\infty} 2^{-k} \right) \left( \sum_{k=0}^{\infty} 2^{-k} \right)$$

$$= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{2^{-k}2^{-(n-k)}}{2} \right)$$

$$= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} 2^{-n} \right)$$

$$= \sum_{n=0}^{\infty} (n + 1)2^{-n}.$$

6.4.10 For $x \in [0,1]$, $a_k x_k$ is non-negative and non-increasing so

$$f(x) = \sum_{k=0}^{\infty} (-1)^{k+1} a_k x^k$$

converges. Let $s_n(x) = \sum_{k=0}^{n} (-1)^{k+1} a_k x^k$. Fix $\epsilon > 0$. Since $a_k \to 0$ there exists an $N$ such that if $n > N$ then $a_n < \epsilon$ so if $n > N$ by the Alternating Series test $|f(x) - s_n(x)| \leq a_{n+1} x^{n+1} < \epsilon$ for all $x \in [0,1]$. Therefore $s_n(x)$ converge uniformly to $f(x)$ on $[0,1]$ and therefore $f$ is a continuous function.

6.4.11 From Example 6.4.11 in the book we know that $\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$ on the interval $(-1,1)$. However, the series on the right satisfies the conditions of the previous problem so we know that it converges to a continuous function $f(x)$ on $[0,1]$. Since $f(x) = \ln(1 + x)$ on $[0,1]$ and both functions extend to a continuous function on $[0,1]$ we have that $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = f(1) = \ln(1 + 1) = \ln 2$.

In Example 6.4.11 we only see that the series converge on $(-1,1)$ to $\ln(1 + x)$. Even though it is easy to show that the series continues to converge when $x = 1$ we still need Problem 6.4.10 to show that the limiting function is continuous.

6.4.12 We work by induction. By Theorem 6.4.12, $f(x)$ is differentiable on $(a - R, a + R)$ and $f'(x)$ is a power series centered at $a$ with radius of convergence $R$. Now assume that $f^{(n)}(x)$ is a power series centered at $a$ with radius of convergence $R$. Then
by Theorem 6.4.12, \( f^{(n)} \) is differentiable on \((a - R, a + R)\) and \( f^{(n+1)}(x) \) is a power series centered at \( a \) with radius of convergence \( R \). This completes the induction step and shows that \( f \) is infinitely differentiable on \((a - R, a + R)\).

**6.5.5** We first show that \( f^{(k)}(x) = (-1)^{k+1} \frac{(2k-3)!}{4^{k-1}((k-2)!)^2} (1 + x)^{1/2-k} \) when \( k \geq 2 \) via induction. This a direct calculation when \( k = 2 \). Now we assume the formula holds for \( k \) and prove it for \( k + 1 \). We have

\[
f^{(k+1)}(x) = (-1)^{k+1} \frac{(2k-3)!}{4^{k-1}((k-2)!)^2} (1/2 - k)(1 + x)^{1/2-(k+1)}
\]

\[
= (-1)^{k+1} (-1) \frac{(2k-3)!}{4^{k-1}((k-2)!)^2} \cdot 2 (1 + x)^{1/2-(k+1)}
\]

\[
= (-1)^{k+1} \frac{(2k-3)!}{4^{k-1}((k-2)!)^2} \cdot \frac{2(2k+1) - 4}{2(2k+1) - 4} (1 + x)^{1/2-(k+1)}
\]

\[
= (-1)^{(k+1)+1} \frac{(2k+1) - 3)!}{4^{k+1}(k+1)!((k+1) - 2)!} (1 + x)^{1/2-(k+1)}
\]

so the formula holds. We then have that the \( n \)th Taylor polynomial is

\[
1 + x/2 + \sum_{k=2}^{n} (-1)^{k+1} \frac{(2k-3)!}{4^{k-1}((k-2)!)^2} x^k
\]

and the remainder term is

\[
R_n(x) = (-1)^n \frac{(2(n+1) - 3)!}{4^n((n+1) - 2)!((n+1)!)^2} (1 + c)^{1/2-(k+1)} x^{n+1}
\]

\[
= (-1)^n \frac{(2n-1)!}{4^n((n-1)!)^2((n+1)!)} (1 + c)^{1/2-(k+1)} x^{n+1}.
\]

**6.5.9** If we view the expression on each side of the inequality as a function of \( x \) we see we have two lines. There are two cases if \( x \) depending on the sign of \( x \). If \( x > 0 \) then the lines are \( y = x \) and \( y = \frac{x - t}{1 + t} \). By setting \( x = \frac{x - t}{1 + t} \) we see that the two lines intersect when \( x = -1 \) and when \( x \geq -1 \) we have \( x \geq \frac{x - t}{1 + t} \). If \( x < 0 \) then the two lines are \( y = -x \) and \( y = \frac{t - x}{1 + t} \) (since \( x \leq t \leq 0 \) so \( t - x \) and \( 1 + t \) are positive). Again if we set \( -x = \frac{t - x}{1 + t} \) we see that these two lines intersect at \( x = -1 \) and \( -x \geq \frac{t - x}{1 + t} \) when \( x \geq -1 \). This proves the inequality in this case.
6.5.11 Fix \( x \in (a - r, a + r) \). We need to show that \( R_n(x) \to 0 \) as \( n \to \infty \). By Taylor’s formula we have

\[
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}
\]

for some \( c \) between \( a \) and \( r \). Since \( |x - a| < r \) (which implies \( \frac{|x-a|}{r} < 1 \)) and \( f^{(n+1)}(c) \leq K \frac{(n+1)!}{r^{n+1}} \) this becomes

\[
|R_n(x)| \leq K \frac{(n + 1)!}{r^{n+1}(n + 1)!}|x - a|^{n+1} = K \left( \frac{|x - a|}{r} \right)^{n+1} \to 0
\]

as \( n \to \infty \).