Homework 13 - Solutions, Math 3210
Section 6.1: 12, 14
Section 6.2: 11, 12
Section 6.3: 7

6.1.12 The decimal expansion $.d_1d_2d_3\dots$ represents the series $\sum_{k=1}^{\infty} \frac{d_k}{10^k}$. Since $d_k/10^k \leq 9/10^k$ and $\sum_{k=1}^{\infty} \frac{9}{10^k}$ is a convergent geometric series, by the Comparison Test $\sum_{k=1}^{\infty} \frac{d_k}{10^k}$ converges.

6.1.14 Let $s_n = \sum_{k=1}^{n} a_k$ be the nth partial sum. Then $a_{k+1} = \frac{1-s_k}{3}$. We claim that $s_n < 1$. When $n = 1$, $s_1 = a_1 = 1/3 < 1$ by definition. Now assume that $s_n < 1$. Then

$$s_{n+1} = a_{n+1} + s_n = \frac{1-s_n}{3} + s_n = \frac{1+2s_n}{3} < \frac{1+2}{3} = 1$$

since $s_n < 1$. Therefore $s_n < 1$ for all n.

Since $s_n < 1$ we have $a_{n+1} = \frac{1-s_n}{3} > 0$. Therefore s_n is an increasing sequence. Since s_n is increasing and bounded it has a limit s. By Theorem 6.1.2 this implies that $a_n \to 0$. Taking the limit of both sides of $a_{n+1} = \frac{1-s_n}{3}$ we see that $0 = \frac{1-s}{3}$ and therefore $s = 1$.

6.2.11 Since $\{b_k\}$ is bounded there exists an $M > 0$ such that $|b_k| \leq M$. Therefore $|a_kb_k| \leq M|a_k|$. By the Comparison Test, since $\sum |a_k|$ converges, $\sum |a_kb_k|$ converges and the series $\sum a_kb_k$ converges absolutely.

6.2.12 We'll show that if $\sum a_k$ converges then $\sum b_k$ converges. Let $s_n \sum_{k=1}^{n} a_k$ and $r_n = \sum_{k=1}^{n} b_k$ be the partial sums. Since s_n converges it is Cauchy and for all $\epsilon > 0$ there exists an N_0 such that if $n > m > N_0$ then $|s_n - s_m| < \epsilon$. Since $a_k = b_k$ except for finitely many k, there exists and N_1 such that if $k > N_1$ then $a_k = b_k$. This implies that if $n, m > N_1$ then $s_n - s_m = r_n - r_m$.

Let $N = \max\{N_0, N_1\}$. Then if $n, m > N$ we have $|r_n - r_m| = |s_n - s_m| < \epsilon$. So r_n is Cauchy and $\sum b_k$ converges.

6.3.7 Let s_n, s_n^+ and s_n^- be the partial sums of the respective series. Note that $s_n = s_n^+ + s_n^-$. Since s_n converges if s_n^+ converges then $s_n^- = s_n - s_n^+$ will also converge. Similarly if s_n^- converges then s_n^+ converges. Therefore either neither series converges (and we're done) or they both do.

Now assume both series converge. Then the partial sums $\sum_{k=1}^{n} |a_k| = s_n^+ - s_n^-$ also converge so $\sum_{k=1}^{\infty} a_k$ converges absolutely.