Homework 12 - Solutions, Math 3210
Section 5.3: 6, 8
Section 5.4: 4, 5, 12

5.3.6 Since f is differentiable it is continuous and therefore integrable so $f \cdot f'$ is integrable. Let $F(x) = \frac{1}{2}f(x)^2$. Then $F'(x) = f(x)f'(x)$ so by the 1st Fundamental Theorem of Calculus

$$\int_a^b f(x)f'(x)dx = \frac{1}{2}(f(b)^2 - f(a)^2).$$

5.3.8 We first use l'Hôpital's rule to show that for a positive integer n

$$\begin{aligned} \lim_{t\to 0^+} t^n \ln t &= \lim_{t\to 0^+} \frac{\ln t}{t^{-n}} \\ &= \lim_{t\to 0^+} \frac{1/t}{-nt^{-n-1}} \\ &= \lim_{t\to 0^+} \frac{-t^n}{n} = 0. \end{aligned}$$

Therefore the function $t^n \ln t$ extends to a continuous function on $[0, x]$. To take the integral we use integration by parts where $f(t) = \frac{t^{n+1}}{n+1}$ and $g(t) = \ln t$. We then have

$$\begin{aligned} \int_0^x t^n \ln t dt &= \frac{t^{n+1}\ln t}{n+1}|_0^x - \int_0^x \frac{t^{n+1}}{n+1}\frac{1}{t}dt \\ &= \frac{x^{n+1}\ln x}{n+1} - \left(\frac{t^{n+1}}{(n+1)^2}|_0^x\right) \\ &= \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} \end{aligned}$$

5.4.4 From the definition and Theorem 5.4.7 (a)

$$\begin{aligned} a^{x+y} &= \exp((x+y)\ln a) \\ &= \exp(x\ln a + y\ln a) \\ &= \exp(x\ln a)\exp(y\ln a) \\ &= a^x a^y. \end{aligned}$$

Similarly from the defintion and Theorem 5.4.7 (b)

$$\begin{aligned} a^{xy} &= \exp(xy\ln a) \\ &= (\exp(x\ln a))^y \\ &= (a^x)^y. \end{aligned}$$

5.4.5 We have

$$\begin{aligned} a^{x+y} &= \exp((x+y)\ln a) \\ &= \exp(x\ln a + y\ln a) \\ &= \exp(x\ln a)\exp(y\ln a) \text{ by 5.47 (a)} \\ &= a^x a^y \end{aligned}$$

and

$$\begin{aligned} a^{xy} &= \exp(xy\ln a) \\ &= (\exp(x\ln a))^y \text{ by 5.4.7 (b)} \\ &= (a^x)^y. \end{aligned}$$

5.4.12 As with Problem 5.3.8 we again use integration by parts to see that

$$\begin{aligned} \int_0^1 \ln x dx &= x\ln x|_0^1 - \int_0^1 \frac{x}{x} dx \\ &= \ln 1 - 0 - \left(x|_0^1\right) = -1. \end{aligned}$$

Note that to find $\lim_{x\to 0} x\ln x = 0$ we are using the calculation from Problem 5.3.8 with $n = 1$. Therefore we have shown that the the integral converges with value -1.