Homework 11 - Solutions, Math 3210
Section 5.1: 8
Section 5.2: 1, 9, 10
Section 5.3: 4, 5

5.1.8 Let $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$. Then for the upper sum we have

$$U(f, P) \leq \sum_{k=1}^{n} M(x_k - x_{k-1}) = M(b - a)$$

and for the lower sum

$$L(f, P) \geq \sum_{k=1}^{n} m(x_k - x_{k-1}) = m(b - a).$$

As this is true for all partitions we have

$$m(b - a) \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \leq M(b - a).$$

If f is integrable we have

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$$

and

$$m(b - a) \leq \int_a^b f(x)dx \leq M(b - a).$$

5.2.1 By Theorem 5.2.1, g and h are integrable. By Theorem 5.2.3 (a), $-h$ is integrable and by 52.3 (b), $g + (-h) = g - h = f$ is integrable.

5.2.9 If $|f(x)| \leq M$ for all $x \in [a, b]$ then

$$\begin{aligned} |f^2(x) - f^2(y)| &= |f(x) + f(y)||f(x) - f(y)| \\ &\leq 2M|f(x) - f(y)| \end{aligned}$$

for $x, y \in [a, b]$.

Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition. If $x, y \in [x_{k-1}, x_k]$ then

$$|f(x) - f(y)| \leq \sup_{[x_{k-1}, x_k]} f - \inf_{[x_{k-1}, x_k]} f$$

since both $f(x)$ and $f(y)$ are at most $\sup\limits_{[x_{k-1},x_k]} f$ and at least $\inf\limits_{[x_{k-1},x_k]} f$. Therefore for all $x, y \in [x_{k-1}, x_k]$ we have

$$\begin{aligned} |f^2(x) - f^2(y)| &\leq 2M|f(x) - f(y)| \\ &\leq 2M \left| \sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f \right| \end{aligned}$$

where $|f(x)| \leq M$, for all $x \in [a, b]$.

This implies that

$$\sup_{[x_{k-1},x_k]} f^2 - \inf_{[x_{k-1},x_k]} f^2 \leq 2M \left(\sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f \right).$$

Fixe $\epsilon > 0$ and assume P has been chosen such that $U(f, P) - L(f, P) \leq \frac{\epsilon}{2M}$. Then

$$\begin{aligned} U(f^2, P) - L(f^2, P) &= \sum_{k=1}^{n} \left(\sup_{[x_{k-1},x_k]} f^2 - \inf_{[x_{k-1},x_k]} f^2 \right) (x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} 2M \left(\sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f \right) (x_k - x_{k-1}) \\ &= U(f, P) - L(f, P) \leq \frac{\epsilon}{2M}. \end{aligned}$$

Therefore f^2 is integrable.

5.2.10 By Theorem 5.2.3 (b), $f + g$ is integrable. By Problem 5.2.9, f^2 and g^2 are integrable. By Theorem 5.2.3 (a), $-f^2$ and $-g^2$ are integrable and by (b) $(f + g)^2 + (-f^2) + (-g^2) = (f + g)^2 - f^2 - g^2$ is integrable. Finally by 5.2.3 (b)

$$\frac{1}{2} \left[(f + g)^2 - f^2 - g^2 \right] = fg$$

is integrable.

5.3.4 Let $H(x) = \int_0^x e^{-t^2} dt$. Then

$$F(x) = \int_{1/x}^{x} e^{-t^2} dt = H(x) - H(1/x)$$

so by the 2nd Fundamental Theorem of Calculus and the chain rule

$$\begin{aligned} F'(x) &= H'(x) - H'(1/x) \left(\frac{-1}{x^2} \right) \\ &= e^{-x^2} + \frac{e^{-1/x^2}}{x^2}. \end{aligned}$$

5.3.5 The function $1/x$ is not defined at $x = 0$ so Theorem 5.3.1 doesn't apply. In fact, $1/x$ cannot not even be extended to a continuous, differentiable function at $x = 0$. There are elementary ways to prove this but one more sophisticated method is that if it did the integral of $1/x^2$ would be negative as shown in the problem.