Homework 10 - Solutions, Math 3210
Section 4.3: 3,4, 12, 13
Section 4.4: 1, 4, 5, 15
Section 5.1: 1, 5

4.3.3 We apply the Mean Value Theorem to the function $f(x) = \ln x$. Note that if $x = y$ the inequality is clearly true as both sides will be 0. So we can assume that $x < y$. Then there exists a $c \in (x, y)$ such that $\frac{\ln y - \ln x}{y-x} = f'(c) = 1/c$. If $r \leq x$ then $r \leq c$ and $1/r \geq 1/c$. Rearranging the equality from the Mean Value Theorem we have

$$\ln y - \ln x = \frac{y-x}{c} \leq \frac{y-x}{r}.$$

4.3.4 By the Mean Value Theorem there exists a $c \in (0, x)$ such that $\frac{f(x)-f(0)}{x-0} = f'(c)$ or $\left|\frac{f(x)}{x}\right| = |f'(c)|$. Since $|f'(c)| \leq M$ we can rearrange this to become $|f(x)| \leq Mx$.

4.3.12 By the Mean Value Theorem there exists a $x \in (x, y)$ such that $f'(c)\frac{f(y)-f(x)}{y-x}$ so

$$\left|\frac{f(y)-f(x)}{y-x}\right| = |f'(c)| \leq r$$

and $|f(x) - f(y) \leq r|x - y|$.

4.3.13 We'll give two proofs. The first applies in a much more general context and is extremely important tool.

We first show that the sequence is Cauchy. Let $C = |x_1 - x_2|$. We claim that $|x_n - x_{n+1}| \leq Cr^{n-1}$. We prove this by induction on n. By the choice of C this is true when $n = 1$. Now, assume the inequality holds for n and we'll prove it for $n+1$. By 4.3.12 we have that $|x_{n+1} - x_{n+2}| = |f(x_n) - f(x_{n+1})| \leq r|x_n - x_{n+1}|$. Since the inequality holds for n we have that $|x_n - x_{n+1}| \leq Cr^{n-1}$. Combining the two inequalities we have $|x_{n+2} - x_{n+1}| \leq r|x_n - x_{n+1}| \leq Cr^n$ completing the induction step.

To show that the sequence is Cauchy we follow the proof of 2.5.9 from Homework 6. Fix $\epsilon > 0$ and choose N such that $\frac{Cr^{-N}}{1-r} < \epsilon$. Then if $n, m > N$ and $n \geq m$ we have

$$\begin{aligned} |x_n - x_m| &\leq \sum_{k=m}^{n-1} |x_{k+1} - x_k| \\ &\leq \sum_{k=m}^{n-1} Cr^{-k} = Cr^{-m} \sum_{k=0}^{n-m-1} r^{-k} \\ &\leq \frac{Cr^{-m}}{1-r} \\ &\leq \frac{Cr^{-N}}{1-r} < \epsilon. \end{aligned}$$

Therefore x_n is a Cauchy sequence and converges.

Let $x \in \mathbb{R}$ such that $x_n \to x$. We claim that $f(x) = x$. Since f is continuous we have that $\lim_{n\to\infty} f(x_n) = f(x)$. However since $f(x_n) = x_{n+1}$ we also have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$. Therefore $f(x) = x$.

Here is another proof: Let $h(x) = x - f(x)$. Then $h'(x) = 1 - f'(x)$ so $h'(x) \geq 1 - r$. We'll find a $c \in \mathbb{R}$ such that $h(c) = 0$ which implies that $f(c) = c$.

If $h(0) = 0$ we are done. First assume that $h(0) < 0$ and choose $x \in \mathbb{R}$ such that $x > \frac{-h(0)}{1-r} > 0$. By the Mean Value Theorem there exists a $c \in (0, x)$ such that

$$\frac{h(x) - h(0)}{x - 0} = f'(c) \geq 1 - r.$$

Therefore $h(x) \geq x(1-r) + h(0)$. Since $x > \frac{-h(0)}{1-r}$ we get that $h(x) > \frac{-h(0)(1-r)}{1-r} + h(0) = 0$. By the Intermediate Value Theorem there exists a $c \in (0, x)$ such that $h(c) = 0$. This completes the proof if $h(0) < 0$.

The proof when $h(0) > 0$ is similar. Choose $x \in \mathbb{R}$ such that $x < \frac{h(0)}{1-r} < 0$. By the Mean Value Theorem there exists a $c \in (x, 0)$ such that

$$\frac{h(x) - h(0)}{x - 0} = f'(c) \geq 1 - r.$$

As $x < 0$ this becomes $h(x) \leq x(1-r) + h(0)$. As above substituting in the inequality for x this becomes $h(x) < \frac{h(0)(1-r)}{1-r} + h(0) = 0$. Again as above we use the Intermediate Value Theorem to find a $x \in (x, 0)$ with $h(c) = 0$ completing the proof in this case.

4.4.1 By the Cauchy Mean Value Theorem there exists $c \in (1, x)$ such that

$$\frac{\ln x - \ln 0}{x^r - 1^r} = \frac{\frac{1}{c}}{rc^{r-1}} = \frac{1}{rc^r} \leq \frac{1}{r}$$

since $c^r \geq 1$. Therefore $\ln x \leq \frac{x^r - 1}{r}$ if $x > 1$.

4.4.4 Applying the Cauchy Mean Value Theorem to $f(x)$ and $g(x) = x^n$ we find a $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x^n - 0^n} = \frac{f'(c)}{nc^{n-1}}$$

so

$$f(x) = \frac{f'(c)}{c^{n-1}} \frac{x^n}{n}$$

since $f(0) = 0$.

4.4.5 Assume that kth-derivative of f and g is zero at at $x = 0$ for $k = 0, \ldots, n-1$. We claim that for $k = 1, \ldots, n$ there exists $c_k \in (0, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f^{(k)}(c_k)}{g^{(k)}(c_k)}.$$

For $k = 1$ this is the Cauchy Mean Value theorem since $f(0) = g(0) = 0$. We now assume that the statement is true for k and prove it for $k+1$ by applying the Cauchy Mean Value Theorem to $f^{(k)}$ and $g^{(k)}$ on the interval $(0, c_k)$. In particular there exists a $c_{k+1} \in (0, c_k) \subset (0, x)$ such that

$$\frac{f^{(k)}(c_k) - f^{(k)}(0)}{g^{(k)}(c_k) - g^{(k)}(0)} = \frac{f^{(k+1)}(c_{k+1})}{g^{(k+1)}(c_{k+1})}.$$

Since $f^{(k)}(0) = g^{(k)}(0) = 0$ this becomes

$$\frac{f^{(k)}(c_k)}{g^{(k)}(c_k)} = \frac{f^{(k+1)}(c_{k+1})}{g^{(k+1)}(c_{k+1})}$$

and applying the formula for k we have

$$\frac{f(x)}{g(x)} = \frac{f^{(k+1)}(c_{k+1})}{g^{(k+1)}(c_{k+1})}$$

and the induction is complete.

If we let $g(x) = x^n$ and multiply both sides by x^n this becomes

$$f(x) = x^n \frac{f^{(n)}(c)}{n!} = f^{(n)}(c) \frac{x^n}{n!}$$

since $g^{(n)}(x) = n!$.

4.4.15 By L'Hôpital's Rule we have

$$\lim_{x\to\infty} \frac{e^{x/r} f(x)}{e^{x/r}} = \lim_{x\to\infty} \frac{\frac{1}{r} e^{x/r} f(x) + e^{x/r} f'(x)}{\frac{1}{r} e^{x/r}} = \lim_{x\to\infty} (f(x) + r f'(x)) = L.$$

On the other hand we have

$$\lim_{x\to\infty} \frac{e^{x/r} f(x)}{e^{x/r}} = \lim_{x\to\infty} f(x)$$

so we have $\lim_{x\to\infty} f(x) = L$. Next we see that

$$\begin{aligned} \lim_{x\to\infty} r f'(x) &= \lim_{x\to\infty} (-f(x) + f(x) + r f'(x)) \\ &= -\lim_{x\to\infty} f(x) + \lim_{x\to\infty} (f(x) + r f'(x)) \\ &= -L + L = 0. \end{aligned}$$

5.1.1 If we divide $[1,2]$ into 4 sub-intervals of equal length the partition is $P = \{1 < 5/4 < 3/2 < 7/8 < 2\}$. Then

$$\begin{aligned} U(f,P) &= \left(\sup_{[1,5/4]} \frac{1}{x} + \sup_{[5/4,3/2]} \frac{1}{x} + \sup_{[3/2,7/4]} \frac{1}{x} + \sup_{[7/4,2]} \frac{1}{x} \right) \frac{1}{4} \\ &= \left(1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right)\left(\frac{1}{4}\right) \end{aligned}$$

and

$$\begin{aligned} L(f,P) &= \left(\inf_{[1,5/4]} \frac{1}{x} + \inf_{[5/4,3/2]} \frac{1}{x} + \inf_{[3/2,7/4]} \frac{1}{x} + \inf_{[7/4,2]} \frac{1}{x} \right) \frac{1}{4} \\ &= \left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\right)\left(\frac{1}{4}\right). \end{aligned}$$

5.1.5 Let $P = \{x_0 = 0 < x_1 < \cdots < x_n = 1\}$ be a partition of $[0,1]$. Then $\sup_{[x_{k-1},x_k]} f(x) = 1$ since there is an irrational number between any two numbers. We also have $\inf_{[x_{k-1},x_k]} f(x) = 0$ since there is a rational number between any two numbers. Therefore

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} \big(\sup_{[x_{k-1},x_k]} f(x)\big)(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} (x_k - x_{k-1}) = x_n - x_0 = 1 \end{aligned}$$

and

$$L(f,P) = \sum_{k=1}^{n} \big(\inf_{[x_{k-1},x_k]} f(x)\big)(x_k - x_{k-1}) = 0.$$

Since P is an arbitrary partition this implies that the upper integral is 1 while the lower integral is 0 so f is not integrable.