Homework 1, Math 3210
January 16, 2018
Section 1.1: 2,4,5,9,11
Section 1.2: 8,10

1.1.2 If $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in B \cup C$. If $x \in B \cup C$ then $x \in B$ or $x \in C$. Therefore $x \in A$ and $x \in B$ or we have $x \in A$ and $x \in C$. In particular $x \in A \cap B$ or $x \in A \cap C$ which implies that $x \in (A \cap B) \cup (A \cap C)$ which then implies that

$$A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C).$$

If $x \in (A \cap B) \cup (A \cap C)$ then $x \in A \cap B$ or $x \in A \cap C$. in the first case $x \in B$ and in the second case $x \in C$ and therefore $x \in B \cup C$. In both cases $x \in A$. Therefore $x \in \cap(B \cup C)$ which implies that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

Together this implies that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

1.1.4 Let

$$A = \bigcap_{(a,b) \supset [0,1]} (a, b).$$

We'll show that $A = [0, 1]$.

If $x > 1$ then the open interval $(-1, x)$ contains $[0, 1]$ so $A \subset (-1, x)$. However $x \notin (-1, x)$ and therefore $x \notin A$. Similarly if $x < 0$ then $(x, 2)$ contains A but doesn't contain x so $x \notin A$. Together this implies that $A \subset [0, 1]$.

If $x \in [0, 1]$ then $x \in (a, b)$ for all open intervals (a, b) that contain $[0, 1]$. Therefore $A \supset [0, 1]$ and with the previous inclusion this implies that $A = [0, 1]$.

1.1.5 Let

$$A = \bigcap_{[a,b] \supset (0,1)} [a, b].$$

We'll show that $A = [0, 1]$.

Since $[0, 1] \supset (0, 1)$ we have $A \subset [0, 1]$.

If $[a, b] \supset (0, 1)$ then $a \leq 0$ and $b \geq 1$ so if $x \in [0, 1]$ and $[a, b] \supset (0, 1)$ then $x \in [a, b]$. Therefore $x \in A$ and $[0, 1] \supset A$.

These two inclusions imply that $A = [0, 1]$.

1.1.9 If $x \in f^{-1}(E \cap F)$ then $f(x) \in E \cap F$ and $f(x) \in E$ and $f(x) \in F$. Therefore $x \in f^{-1}(E)$ and $x \in f^{-1}(F)$ so $x \in f^{-1}(E) \cap f^{-1}(F)$. This implies that

$$f^{-1}(E) \cap f^{-1}(F) \supset f^{-1}(E \cap F).$$

If $x \in f^{-1}(E) \cap f^{-1}(F)$ then $f(x) \in E$ and $f(x) \in F$ so $f(x) \in E \cap F$. Therefore $x \in f^{-1}(E \cap F)$ and

$$f^{-1}(E) \cap f^{-1}(F) \subset f^{-1}(E \cap F).$$

The two inclusions then imply that

$$f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F).$$

1.1.11 If $y \in f(E \cup F)$ there exists $x \in E \cup F$ such that $f(x) = y$. Since $x \in E$ or $x \in F$ we have that $y = f(x) \in f(E)$ or $y = f(x) \in f(F)$. This implies that $y \in f(E) \cup f(F)$ and

$$f(E \cup F) \subset f(E) \cup f(F).$$

If $y \in f(E) \cup f(F)$ then either there exists an $x \in E$ with $f(x) = y$ or an $x \in F$ with $f(x) = y$. Therefore there exists an $x \in E \cup F$ such that $f(x) = y$ which implies that $x \in f(E \cup F)$ and

$$f(E \cup F) \supset f(E) \cup f(F).$$

The two inclusions then imply that

$$f(E \cup F) = f(E) \cup f(F).$$

1.2.8 Let P_n be the proposition that $7^n - 2^n$ is divisible by 5. P_1 is true since $7^1 - 2^1 = 5$. Now assume that P_n is true. Adding and subtracting $7 \cdot 2^n$ to $7^{n+1} - 2^{n+1}$ and then factoring gives

$$\begin{aligned} 7^{n+1} - 2^{n+1} &= 7^{n+1} - 7 \cdot 2^n + 7 \cdot 2^n - 2^{n+1} \\ &= 7(7^n - 2^n) + 2^n(7-2). \end{aligned}$$

Since P_n is true we have that $7(7^n - 2^n)$ is divisible by 5. We also have that $2^n(7-2) = 2^n \cdot 5$ is divisible by 5. Therefore the sum $7(7^n - 2^n) + 2^n(7-2) = 7^{n+1} - 2^{n+1}$ is divisible by 5 and P_{n+1} is true.

By induction P_n is true for all $n \in \mathbb{N}$.

1.2.10 Let P_n be the statement that

$$\sum_{k=1}^{n}(2k-1) = n^2..$$

For $n = 1$ we have

$$\sum_{k=1}^{1}(2k-1) = 2 \cdot 1 - 1 = 1 = 1^2.$$

Therefore P_1 is true.

Now assume that P_n is true. Then

$$\begin{aligned}
\sum_{k=1}^{n+1}(2k-1) &= \sum_{k=1}^{n}(2k-1) + 2(n+1) - 1 \\
&= n^2 + 2n + 1 \\
&= (n+1)^2.
\end{aligned}$$

Therefore P_{n+1} is true and by induction P_n is true for all $n \in \mathbb{N}$.