Notes and problems on the topology of $\mathbb{R}^n$

Let $X$ be a set and $d : X \times X \to [0, \infty)$ a function with:
1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) + d(y, z) \geq d(x, z)$.

Then $d$ is a metric on $X$ and the pair $(X, d)$ is a metric space. Property (3) is the triangle inequality.

Define a $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by setting

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

**Problem 1** Show that $d$ is a metric on $\mathbb{R}^n$.

The open ball of radius $r$ centered at $x$ is the set

$$B_r(x) = \{y | d(x, y) < r\}.$$ 

The triangle inequality implies that if $r_0 < r_1$ then $B_{r_0}(x) \subset B_{r_1}(x)$.

A subset $U \subset \mathbb{R}^n$ is open if for every $x \in U$ there is an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$.

**Theorem 1** The open subsets of $\mathbb{R}^n$ satisfy the following properties:

1. $\mathbb{R}^n$ and $\emptyset$ are open.
2. If $\{U_\alpha\}$ is a collection of open sets then $\bigcup U_\alpha$ is open.
3. If $U_1, \ldots, U_n$ are open then $\bigcap U_i$ is open.

**Proof of 1.** Obvious.

2. If $x \in \bigcup U_\alpha$ then $x \in U_\alpha$ for some $\alpha$. Since $U_\alpha$ is open there exists an $\epsilon$ such that $B_\epsilon(x) \subset U_\alpha$. But $U_\alpha$ is contained in $\bigcup U_\alpha$ so we also have $B_\epsilon(x) \subset \bigcup U_\alpha$ and $\bigcup U_\alpha$ is open.

3. If $x \in \bigcap U_i$ then $x \in U_i$ for all $i = 1, \ldots, n$ so there exists $\epsilon_i$ with $B_{\epsilon_i}(x) \subset U_i$. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Since $B_\epsilon(x) \subset B_{\epsilon_i}(x)$ for all $i = 1, \ldots, n$ we have $B_\epsilon(x) \subset U_i$ for all $i$. Therefore $B_\epsilon(x) \subset \bigcap U_i$ and $\bigcap U_i$ is open. 

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A subset $U$ of $\mathbb{R}^n$ is closed if $U^c = \mathbb{R}^n \setminus U$ is open.

**Problem 2** Prove that the closed subsets of $\mathbb{R}^n$ satisfy the following properties:

1. $\mathbb{R}^n$ and $\emptyset$ are closed.
2. If $\{U_\alpha\}$ is a collection of closed sets then $\bigcap U_\alpha$ is closed.
3. If $U_1, \ldots, U_n$ are closed then $\bigcup U_i$ is closed.

Here is another characterization of a closed set.

**Theorem 2** A set $U$ is closed if and only if for every sequence $\{x_i\}$ in $U$ with $x_i$ converging to some $x \in \mathbb{R}^n$ then $x \in U$.

The interior of a set $U$, denoted $\text{int}U$, is the union of all open set contained in $U$.

**Problem 3** Show that $\text{int}U = \{x \in U \mid \text{there exists } \epsilon > 0 \text{ with } B_\epsilon(x) \subset U\}$.

The closure of $U$, denoted $\bar{U}$, is the intersection of all closed sets that contain $U$. Let $A$ be a subset of $B$. Then $A$ is dense in $B$ if $A \supset B$.

**Problem 4** Show that $\mathbb{Q}$ is dense in $\mathbb{R}$. More generally show that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

Let $B_Q$ be the collection of balls $B_r(x)$ with $x \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$.

**Problem 5** Show that $B_Q$ is countable.

**Theorem 3** If $U$ is an open set define

$$U_Q = \bigcup_{B \in B_Q \text{ and } B \subset U} B.$$ 

Then $U = U_Q$.

**Proof.** Clearly $U_Q \subset U$ so we only need to show that $U \subset U_Q$. If $x \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}^n$ there exists $y \in \mathbb{Q}^n \cap B_{\epsilon/3}(x)$. Again using the density of $\mathbb{Q}$ in $\mathbb{R}$ we can find an $r \in (\epsilon/3, \epsilon/2) \cap \mathbb{Q}$. Then $B_r(y) \subset B_Q$. Since $d(x, y) \leq \epsilon/3$ we also have $x \in B_r(y)$. Furthermore if $z \in B_r(y)$ then by the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z) \leq \epsilon/3 + r \leq \epsilon/3 + \epsilon/2 < \epsilon$$

and therefore $B_r(y) \subset B_\epsilon(x) \subset U$. Hence $x \in U_Q$ and $U \subset U_Q$ as desired.