Notes and problems on infinite sets and countability

A set $X$ is *infinite* if there exists a map from $X$ to $X$ that is injective but not surjective.

**Theorem 1** If $X$ is infinite there is an injective map from $\mathbb{N}$ to $X$.

**Proof.** Let $\phi : X \to X$ be injective but not surjective. We inductively define an injective map $\psi : \mathbb{N} \to X$ as follows. Define $\psi(1)$ to be an element of $X \setminus \phi(X)$. Now assume $\psi$ has been defined on $\{1, \ldots, n\}$ and that $\psi(k) \in \phi^{k-1}(X) \setminus \phi^{k}(X)$ for $k \in \{1, \ldots, n\}$. Now define $\psi(n + 1)$ to be an element of $\phi^{n}(X) \setminus \phi^{n+1}(X)$.

This defines $\psi$ on all of $\mathbb{N}$. The map is injective since

$$(\phi^{n}(X) \setminus \phi^{n+1}(X)) \cap (\phi^{m}(X) \setminus \phi^{m+1}(X)) = \emptyset$$

if $n \neq m$. \qed

A set $X$ is *countable* if there exists a bijection from $\mathbb{N}$ to $X$.

**Problem 1** Show that:

- $\mathbb{Z}$ is countable.
- The union of two countable sets is countable.

**Theorem 2** The product of two countable sets is countable.

**Proof.** We just need to show that $\mathbb{N} \times \mathbb{N}$ is countable. We can write $\mathbb{N} \times \mathbb{N}$ in a list:

$$(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (4, 1), (2, 3), (3, 2), (4, 1), \ldots$$

\qed

**Problem 2** Explicitly write down a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$.

**Theorem 3** An infinite subset of a countable set is countable.
Proof. We can assume that the countable set is \( \mathbb{N} \). Let \( A \) be an infinite subset of \( \mathbb{N} \). Every subset of \( \mathbb{N} \) has a least element. We use this fact to inductively define a bijection \( \psi : \mathbb{N} \to A \).

Define \( \psi(1) \) to be the least element of \( A \) and let \( A_1 = A \setminus \{ \psi(1) \} \). Now assume we have defined \( \psi(k) \) and \( A_k \) for \( k \in \{1, \ldots, n\} \). Then we inductively define \( \psi(n+1) \) to be the least element of \( A_n \) and define \( A_{n+1} = A_n \setminus \{ \psi(n) \} \). This define an injective map \( \psi \) from \( \mathbb{N} \) to \( A \).

We need to show that \( \psi \) is surjective. We claim that \( \psi(n) \geq n \). We again use induction. Clearly \( \psi(1) \geq 1 \) since 1 is the least element of \( \mathbb{N} \) and \( \psi(1) \in A \subseteq \mathbb{N} \). Now assuming that \( \psi(n) \geq n \) we will show that \( \psi(n+1) \geq n + 1 \). Note that \( \psi(n) \) is strictly less than any element of \( A_n \) so \( \psi(n) < \psi(n+1) \) or \( \psi(n) + 1 \leq \psi(n+1) \). Since \( \psi(n) \geq n \) we have \( \psi(n+1) \geq n + 1 \) as desired.

Since \( \psi(n) \geq n \) for all \( n \in \mathbb{N} \) we have \( n \not\in A_m \) for \( n \leq m \). If \( n \in A \) and \( n \not\in A_n \) then we must have \( \psi(m) = n \) for some \( m < n \) proving that \( \psi \) is surjective.

Theorem 4 Let \( S(X) \) be the set of all subsets of a set \( X \). Then there is an injective map from \( X \) to \( S(X) \) but there is no surjective map from \( X \) to \( S(X) \). In particular there are infinite sets that are not countable.

Proof. The map \( x \mapsto \{ x \} \) is an injective map from \( X \) to \( S(X) \).

Now we see there is no surjective map. Let \( \psi : X \to S(X) \) be a map and define a susbset \( A \) by

\[
A = \{ x | x \not\in \psi(x) \}.
\]

We claim that \( A \) is not in the image of \( \psi \).

We work by contradiction and suppose there is an \( x \in X \) such that \( \psi(x) = A \). There are two cases.

Case 1: Suppose \( x \) is in \( A \). Then \( x \in \psi(X) = A \) so \( x \not\in A \) which is a contradiction.

Case 2: Suppose \( x \) is not in \( A \). Then \( x \not\in \psi(X) = A \) so \( x \in A \) which is again a contradiction.

Therefore there does not exist an \( x \in X \) with \( \psi(x) = A \) and \( \psi \) is not surjective.

We’d also like to prove that the real numbers are not countable. We first give a definition of a real numbers. Our definition is not the usual one but it is convenient for showing that \( \mathbb{R} \) is not countable.

A real number is a function \( f : \mathbb{Z} \to \{0, 1, \ldots, 9\} \) with the following properties:

1. There exits an \( N > 0 \) such \( f(n) = 0 \) if \( n > N \);
2. For every \( n \) such that \( f(n) = 9 \) there is an \( m < n \) such that \( f(m) \neq 9 \).
Here is an example. There real number 32.71 is represented by the function $f$ with $f(1) = 3$, $f(0) = 2$, $f(-1) = 7$, $f(-2) = 1$ and $f(n) = 0$ for $n \not\in \{0, -1, -2\}$. A more complicated example is the number $1/7$. This number is represented by a function $f$ with $f(-1) = 1$, $f(-2) = 4$, $f(-3) = 2$, $f(-4) = 8$, $f(-5) = 5$, $f(-6) = 7$, $f(n) = f(n + 6)$ if $n < -6$ and $f(n) = 0$ if $n \geq 0$.

**Theorem 5** $\mathbb{R}$ is uncountable.

**Proof.** Let $\phi$ be a map from $\mathbb{N}$ to $\mathbb{R}$ and let $f_n = \phi(n)$. We will show that $\phi$ is not surjective. Define $g \in \mathbb{R}$ by setting $g(n)$ to be some element of $\{0, 1, \ldots, 8\} \setminus \{f_n(n)\}$ if $n < 0$ and $g(n) = 0$ if $n \geq 0$. Then $g \neq f_n$ for any $n \in \mathbb{N}$ since $g(n) \neq f_n(n)$. Therefore $\phi$ is not surjective.

The number $f \in \mathbb{R}$ **eventually periodic** if there exists and $N \in \mathbb{Z}$ and a $k \in \mathbb{N}$ such that $f(n) = f(n - k)$ if $n < N$. The **period** of $f$ is $k$.

**Problem 3** Show that $f$ is rational if and only if $f$ is eventually periodic. (**Hint:** To show that and eventually periodic $f$ is rational show $10^k f - f$ is rational where $k$ is the period of $f$. It is harder to show that a rational number has a eventually periodic decimal expansion is harder.)

If $f$ and $g$ are real numbers we define $f > g$ if there exists an $n_0 \in \mathbb{Z}$ such that $f(n) = g(n)$ for all $n > n_0$ and $f(n_0) > g(n_0)$.

**Problem 4** Let $f_0$ and $f_1$ be real numbers. Show that there exists a rational number $g_0$ and an irrational number $g_1$ such that $f_0 < g_i < f_1$ for $i = 1, 2$. 

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