Notes and problems on compactness

Let $\mathcal{O}$ be a collection of open sets in $\mathbb{R}^n$. Then $\mathcal{O}$ is an open cover of a set $A \subset \mathbb{R}^n$ if $A \subset \bigcup_{U \in \mathcal{O}} U$.

A set $K$ is compact if every open cover has a finite subcover. That is $K$ is compact if for every open cover $\mathcal{O}$ there are sets $U_1, \ldots, U_k \in \mathcal{O}$ such that

$$K \subset \bigcup_{i=1}^k U_i.$$  

**Theorem 1** A compact set is closed.

**Proof.** We will prove the contrapositive. Assume that $A$ is not closed. We will construct an open cover that has no finite subcover. Since $A$ is not closed there exists a sequence $\{x_i\}$ in $A$ that converges to some $x \not\in A$. Note that

$$\left( \bigcup_{i=1}^{\infty} \{x_i\} \right) \cup \{x\}$$

is closed set so its complement, which we denote $U$, is open. Let $\mathcal{O}$ be the collection of balls $B_{d(x_i,x)/2}(x_i)$ and the set $U$. Then $\mathcal{O}$ is an open cover of $A$. We will show that $\mathcal{O}$ has no finite subcover.

Let $\mathcal{O}'$ be a finite subcollection of the open sets in $\mathcal{O}$. Since $\mathcal{O}'$ contains only finite many sets there exists an $N$ such that if $i > N$ then $B_{d(x_i,x)/2}(x_i)$ is not in $\mathcal{O}'$. Let $\epsilon = \min\{d(x_1,x)/2, \ldots, d(x_N,x)/2\}$. Since $x_i \to x$ there exists an $n_0$ such that $d(x_{n_0},x) < \epsilon$. By the triangle inequality $d(x_i,x) \leq d(x_i,x_{n_0}) + d(x_{n_0},x)$ and after rearranging this becomes $d(x_i,x_{n_0}) \geq d(x_i,x) - d(x_{n_0},x)$. If $i \leq N$ then $d(x_i,x) \geq 2\epsilon$ so we have $d(x_i,x_{n_0}) > 2\epsilon - \epsilon = \epsilon$. In particular $x_{n_0} \not\in B_{d(x_i,x)/2}(x_i)$. Since $x_{n_0}$ is also not in $U$ the open sets in $\mathcal{O}'$ cannot cover $A$ and $\mathcal{O}$ has no finite subcover. \qed

**Theorem 2** If $A$ is a subset of $K$, $A$ is closed and $K$ is compact then $A$ is compact.

**Proof.** Let $\mathcal{O}$ be an open cover of $A$. Let $\mathcal{O}'$ be all of the open sets in $\mathcal{O}$ and the open set $A^c$. Then $\mathcal{O}'$ is an open cover of $K$ and therefore there are finitely many open sets $U_1, \ldots, U_n$ each in $\mathcal{O}'$, that cover $K$. If $A^c$ is not one of the $U_i$ then all of the $U_i$ are in $\mathcal{O}$ and they are a finite subcover. If $A^c$ is one of the $U_i$, say $U_n$, then $U_1, \ldots, U_{n-1}$ are all in $\mathcal{O}$. But $U_1, \ldots, U_{n-1}$ are also a finite subcover of $A$ because if $x \in A \subset K$ then $x \in U_i$ for some $i$ since the $U_i$ cover $K$. Since $x \not\in U_n = A^c$ we must have $x \in U_i$ for some $i \leq n - 1$ and therefore the $U_1, \ldots, U_{n-1}$ cover $A$. \qed
Theorem 3 Let $K_i$ be non-empty compact sets with $K_{i+1} \subset K_i$. Then
\[ \bigcap_{i=1}^{\infty} K_i \neq \emptyset. \]

Proof. We assume the intersection is empty and we will obtain a contradiction. The sets $K_i$ are closed and hence compact so the sets $U_i = K_i^c$ are open. Since
\[ \bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} K_i^c = \left( \bigcap_{i=1}^{\infty} K_i \right)^c = \emptyset^c = \mathbb{R}^n \supset K_1, \]
the collection $\{U_i\}$ is an open cover of $K_1$. Since
\[ \bigcup_{i=1}^{n} U_i = U_n = (K_n)^c \]
no finite subcollection of the $U_i$ covers $K_1$. This contradicts the compactness of $K_1$ so the intersection must be non-empty. \qed

Theorem 4 Let $I_n = [a_n, b_n]$ be a sequence of nested intervals, i.e. $I_{n+1} \subset I_n$ for all $n$. Show that
\[ \bigcap_{n=1}^{\infty} I_n \neq \emptyset. \]

Proof. Let $n$ and $m$ be positive integers with $n \leq m$. Then $I_n \subset I_m$ so $a_n \leq a_m \leq b_m \leq b_n$. In particular $a_i < b_j$ for all $i$ and $j$. This implies that
\[ a = \sup\{a_i\} \leq b_i \]
for all $i$. By the definition of the supremum we also have $a \geq a_i$ for all $i$ so $a \in I_i$ for all $i$ and the intersection is non-empty. \qed

A closed $n$-cell is a product of closed intervals. That is
\[ Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \]
is a closed $n$-cell.

Problem 1 Show that a nested family of closed $n$-cells has a non-empty intersection.
Theorem 5 A closed n-cell Q is compact.

Proof. We will assume Q is not compact. Then there exists an open cover, $\mathcal{O}$, of Q that contains no finite subcover. We will construct a sequence of nested, closed n-cells $Q_0 \supset Q_1 \supset Q_2 \ldots$ with the property that for each $Q_i$ the collection $\mathcal{O}$ is a cover with no finite subcover and such that diam$(Q_i) \to 0$.$^1$

Assuming we have constructed the $Q_i$, we can finish the proof. By Problem 1 the intersection

$$Q_\infty = \bigcap_{i=0}^{\infty} Q_i$$

is non-empty. We claim that $Q_\infty$ contains only one point. Let $x$ and $y$ be points in $Q_\infty$. Since diam$(Q_i) \to 0$ for $\epsilon > 0$ there exists an $k$ such that diam$(Q_k) < \epsilon$. Both $x$ and $y$ are in $Q_k$ so $d(x, y) < \epsilon$ and as $\epsilon$ is arbitrary we must have $d(x, y) = 0$. Therefore $x = y$ and $Q_\infty$ contains only one point which we label $q$.

Let $U$ be an open set in the collection $\mathcal{O}$ with $q \in U$. Since $U$ is open there exists a $\delta > 0$ such that $B_\delta(q) \subset U$. Again using the fact that diam$(Q_i) \to 0$ we can find an $m$ such that diam$(Q_m) < \delta$. By the definition of diameter, if $A$ is a set with $d >$ diam$(A)$ and $x \in A$ then $A \subset B_d(x)$. In particular, $Q_m \subset B_\delta(q) \subset U$. This gives us a contradiction since $\{U\}$ is a finite subcover of $Q_m$.

Now we need to construct the $Q_i$. We will do so inductively. We begin by setting $Q_0 = Q$. By assumption $\mathcal{O}$ has no finite subcover of $Q_0$. The n-cell $Q_0$ is the product of n-intervals. We can assume the longest interval has length $\ell$.

Now assume we have constructed nested, closed n-cells $Q_0 \subset Q_1 \subset \cdots \subset Q_{k-1}$ such that $\mathcal{O}$ has no finite subcover on any of the $Q_i$ and the length of the longest side of $Q_i$ is $2^{-i}\ell$. To choose $Q_n$ we subdivide $Q_{k-1}$ into $2^n$ closed n-cells which we label $Q_{k,1}, \ldots, Q_{k,2^n}$. The $Q_{k,i}$ are of the following form. The n-cell $Q_{n-1}$ is the product of $n$ intervals, $[a_1, b_1], \ldots, [a_n, b_n]$. Let $c_i$ be the midpoint of $[a_i, b_i]$. Then each $Q_{k,i}$ is a product $I_1 \times \cdots \times I_n$ with each $I_j$ either the interval $[a_j, c_j]$ or the interval $[c_j, b_j]$. For each $I_j$ there are two choices of intervals and there are $n$ intervals $I_j$ so there are exactly $2^n$ possible $Q_{k,i}$. Note that $Q_{k-1} = \bigcup Q_{k,i}$ so if $\mathcal{O}$ has a finite subcover for each the $Q_{k,i}$ then $\mathcal{O}$ has a finite subcover on $Q_{k-1}$. Since we are assuming this is not true there is some $Q_{k,i_k}$ such that $\mathcal{O}$ doesn’t have a finite subcover on $Q_{k,i_k}$. Let $Q_k = Q_{k,i_k}$.

To finish the construct of the $Q_i$ we need to calculate the length of the longest interval in product $Q_k$. This is easy to do since the length of the intervals in the product that forms $Q_k$ are exactly half the length of the intervals in $Q_{k-1}$. Therefore the length of the longest interval is $2^{-1} \times 2^{-(k-1)}\ell = 2^{-k}\ell$ and we have inductively found nested, closed n-cells $Q_i$ with the length of the longest interval in each $Q_i$ exactly $2^{-i}\ell$. An application

\footnote{The diameter of a set $A$ is defined to be diam$(A) = \inf\{d|f(x, y) \in A \text{ then } d(x, y) \leq d\}$.
of the triangle inequality shows that \(\text{diam}(Q_i) \leq n2^{-i}\ell\) so \(\text{diam}(Q_i) \to 0\) as \(i \to \infty\).

**Problem 2** Let \(x_n\) be a sequence with no convergent subsequence. Show that the set \(\{x_1, x_2, \ldots\}\) is closed.

**Problem 3** A point \(x\) is isolated in a set \(A \subset \mathbb{R}^n\) if there exists an \(\epsilon > 0\) such that \(B_\epsilon(x) \cap A = \{x\}\). Show that \(x\) is isolated if and only if there doesn’t exist a sequence of distinct points \(x_i \in A\) with \(x_i \to x\).

**Theorem 6** Let \(K\) be a subset of \(\mathbb{R}^n\). The following are equivalent:

1. \(K\) is closed and bounded;
2. \(K\) is compact;
3. Every sequence in \(K\) has a subsequence that converges in \(K\).

**Proof.** (1 \(\Rightarrow\) 2) A bounded set is contained in some closed \(n\)-cell \(Q\). By Theorem 5, Since \(K\) is a closed subset of a compact set \(K\) is compact by Theorem 2.

(2 \(\Rightarrow\) 3) Let \(x_n\) be a sequence in \(K\). If the sequence has a convergent subsequence then the limit is in \(K\) since \(K\) is compact and therefore closed. In this case we are done.

Now we assume the sequence has no convergent subsequence and we will obtain a contradiction. Then by Problem 2 the set \(C = \{x_1, x_2, \ldots\}\) is closed. By Theorem 2, \(C\) is also compact. Problem 3 implies that every point in \(C\) is isolated. In particular, for each \(x_i\) there is an \(\epsilon_i\) such that \(B_{\epsilon_i}(x_i) \cap C = \{x_i\}\). The collection \(\mathcal{O} = \{B_{\epsilon_i}(x_1), B_{\epsilon_2}(x_2), \ldots\}\) is an open cover of \(C\). However if we remove any of the \(B_{\epsilon_i}(x_i)\) from \(\mathcal{O}\) we no longer have an open cover since \(x_i\) is not in any of the open subsets. Therefore \(\mathcal{O}\) has no finite subcover, contradicting the compactness of \(C\).

(3 \(\Rightarrow\) 1) We will prove the contrapositive. If \(K\) is not closed there exists a sequence \(\{x_i\}\) in \(K\) such that \(x_i \to x\) but \(x \notin K\). Every subsequence \(\{x_i\}\) will then also converge to \(x\) so \(\{x_i\}\) has no subsequence that converges in \(K\).

If \(K\) is not bounded, for each \(i\) we can find an \(x_i \in K\) such that \(d(x_i, 0) > i\). Given an \(i\) choose \(j\) such that \(j_0 > d(x_i, 0) + 1\). Then for all \(j > j_0\), \(d(x_i, x_j) \geq d(x_j, 0) - d(x_i, 0) > j_0 - d(x_i, 0) > 1\). This implies that \(\{x_i\}\) has no Cauchy, and therefore no convergent, subsequence.
We now define the Cantor set, $C$, in a way somewhat different than was done in class. Define

$$C = \left\{ x \in [0,1] | x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \text{ where } a_i \in \{0,2\} \right\}.$$

Some examples of points in $C$ are $2/3$ and $2/9$. It is less obvious, but $1/3$ is also in $C$ since $1/3 = \sum_{i=2}^{\infty} 2/3^i$.

**Problem 4** Show that the Cantor set is:

1. closed;
2. has no interior;
3. has no isolated points;
4. is uncountable.